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On the Diophantine equation $4(7^x) - p^y = z^2$

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Abstract. This paper determines all non-negative integer solutions to the Diophantine equation $4(7^x) - p^y = z^2$, where p is a prime. Using modular arithmetic and congruence arguments, we classify all solutions as follows: a unique solution for $p = 2$, an infinite family of solutions for $p = 3$, no solutions for $5 \leq p \leq 17$, and–for $p \geq 19$–the existence of solutions requires that $p \equiv 19 \pmod{24}$ subject to specific modular constraints. Computational results support the conjecture that no further solutions exist beyond those identified. This work illustrates how modular techniques can fully resolve an exponential Diophantine equation and offers a framework for analyzing similar equations involving mixed exponential and polynomial terms.

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1. Introduction

The study of Diophantine equations, named after the ancient Greek mathematician Diophantus of Alexandria (ca. 250 AD), forms a foundational aspect of number theory. Diophantus is often regarded as the "father of algebra" due to his pioneering work in using symbolic methods to represent equations in his famous treatise *Arithmetica*. Although originally consisting of thirteen books, only six have survived and contain around 130 problems with solutions. These problems demonstrate early algebraic reasoning and have inspired generations of mathematicians.

A well-known anecdote regarding Diophantus' life is his purported age at death, deduced from a riddle composed by the poet Metrodorus. The riddle, when interpreted algebraically, leads to the equation

$$x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4,$$

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whose solution is $x = 84$, suggesting that Diophantus lived to be 84 years old. This kind of problem serves as an early example of a linear Diophantine equation, which is an equation of the form

$$ax + by = c$$

where $a, b, c \in \mathbb{Z}$, and the goal is to find integer solutions for x and y. A necessary and sufficient condition for such an equation to have solutions is that $\gcd(a, b)|c$.

Over the centuries, linear Diophantine equations have evolved into more intricate forms, including exponential and nonlinear variants. Table 1 provides a brief historical overview of several notable Diophantine equations, ranging from Pell's Equation to Fermat's Last Theorem.

Table 1: Historical Diophantine Equations.

Equation discovered in	Reference	Equations	Equation Names
1768	[1]	$x^2 - ny^2 = \pm 1$	Pell's Equation
1918	[2]	$w^3 + x^3 = y^3 + z^3$	Hardy-Ramanujan Number
1948	[3]	$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$	Erdös-Straus Conjecture
1988	[4]	$x^4 + y^4 + z^4 = w^4$	Euler's Conjecture (disproved)
1995	[5]	$x^n + y^n = z^n, n \geq 3$	Fermat's Last Theorem

Among these developments, Catalan's conjecture—proposed by Eugène Catalan in 1844 and proved by P. Mihăilescu [6] in 2004 — stands out as a landmark result. It asserts that the equation $a^x - b^y = 1$ has a unique solution in natural numbers when $a, b, x, y \geq 2$, namely $(a, b, x, y) = (3, 2, 2, 3)$. This result has had significant implications in the field of exponential Diophantine equations.

In 2007, Acu [7] utilized Catalan's theorem to analyze the equation $2^x + 5^y = z^2$, which has inspired the investigation of equations involving exponential terms equated to perfect squares. Over the past decade, many researchers have investigated equations of the general type

$$a(p^x) \pm b(q^y) = z^2,$$

where $a, b \in \mathbb{Z}^+$, and p, q are primes. Table 2 summarizes several recent contributions on exponential Diophantine equations of the form $a(p^x) \pm b(q^y) = z^2$, where p, q are primes. Some of these studies demonstrate the effectiveness of modular arithmetic and parametric forms in analyzing such equations, and they inform the approach adopted in this article.

Motivated by previous studies on exponential Diophantine equations, we continue this line of inquiry by examining the equation

$$4(7^x) - p^y = z^2,$$

where p is a prime and $x, y, z \in \mathbb{Z}_0$. By employing modular arithmetic and a detailed analysis of congruence conditions, we classify all possible solutions and propose a conjecture on the nonexistence of further solutions beyond those explicitly identified.

Table 2: Examples of exponential Diophantine equations of the forms $a(p^x) \pm b(q^y) = z^2$ where a, b are positive integers and p, q are primes.

Year	Authors	Equation
2018	J.F.T. Rabago [8]	$4^x - p^y = 3z^2$
2019	K. Laipaporn, et al. [9]	$3^x + p(5^y) = z^2$
2020	A. Elshahed and H. Kamarulhaili [10]	$(4^n)^x - p^y = z^2$
2021	S. Thongnak, et al. [11]	$7^x - 5^y = z^2$
2022	W. Tangjai, et al. [12]	$7^x + 5(p^y) = z^2$
2022	W. Orosram and A. Unchai [13]	$2^{2nx} - p^y = z^2$
2022	M. Buosi, et al. [14]	$p^x - 2^y = z^2$
2024	S. Thongnak, et al. [15]	$11^x - 17^y = z^2$
2024	K. Laipaporn, et al. [16]	$a^x \pm a^y = z^n$
2024	Y. Li, et al. [17]	$2^x \pm (2^k p)^y = z^2$ and $-2^x + (2^k 3)^y = z^2$
2024	J. Zhang and Y. Li [18]	$(-1)^\alpha p^x + (-1)^\beta (2^k(2p-1))^y = z^2$
2025	K. Laipaporn, et al. [19]	$p^x + q^{2y} = z^{2n}$

2. Main theorem

We begin by analyzing the structure of the equation and presenting the main classification result. To support the classification, we first introduce an auxiliary lemma to understand the behavior of power of p modulo 9.

Lemma 1. *Let $p \equiv 6r + 1 \pmod 9$ for some integer $r \geq 0$. Then for all integers $x, y \geq 0$, the congruence class of $4(7^x) - p^y \pmod 9$ depends on the values of $x \pmod 3$ and $y \pmod 3$ as follows:*

$$4(7^x) - p^y \equiv \begin{cases} 1 \pmod 9 & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ 4 - p \pmod 9 & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ 2 + p \pmod 9 & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 2 \pmod 3, \\ 0 \pmod 9 & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ 1 - p \pmod 9 & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ p - 1 \pmod 9 & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 2 \pmod 3, \\ 6 \pmod 9 & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ 7 - p \pmod 9 & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ 5 + p \pmod 9 & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 2 \pmod 3. \end{cases}$$

Proof. Since $p \equiv 6r + 1 \pmod 9$, we derive $p^3 \equiv 216r^3 + 108r^2 + 18r + 1 \equiv 1 \pmod 9$. Therefore, for all integers $t, y \geq 0$, $p^y \equiv p^t \pmod 9$ where $y \equiv t \pmod 3$. It follows that

$$p^y \equiv \begin{cases} 1 \pmod 9 & \text{if } y \equiv 0 \pmod 3, \\ p \pmod 9 & \text{if } y \equiv 1 \pmod 3, \\ p^2 \pmod 9 & \text{if } y \equiv 2 \pmod 3. \end{cases}$$

Next, we compute $p^2 \pmod 9$ using the given congruence for p: $p^2 \equiv 36r^2 + 12r + 1 \equiv -6r + 1 \equiv 2 - p \pmod 9$. Therefore,

$$p^y \equiv \begin{cases} 1 \pmod 9 & \text{if } y \equiv 0 \pmod 3, \\ p \pmod 9 & \text{if } y \equiv 1 \pmod 3, \\ 2-p \pmod 9 & \text{if } y \equiv 2 \pmod 3. \end{cases}$$

Hence,

$$4(7^x) \equiv \begin{cases} 4(1) \equiv 4 \pmod 9 & \text{if } x \equiv 0 \pmod 3, \\ 4(7) \equiv 1 \pmod 9 & \text{if } x \equiv 1 \pmod 3, \\ 4(4) \equiv 7 \pmod 9 & \text{if } x \equiv 2 \pmod 3. \end{cases}$$

Combining the congruences for $4(7^x)$ and p^y, we obtain the desired result.

We now proceed to a complete classification of the solutions based on the value of p:

- Small primes $p = 2, 3$, where elementary computations can be used.
- Intermediate primes $5 \leq p \leq 17$, where contradiction via modular arguments can be established,
- Large prime $p \geq 19$, where structural congruence restrictions guide the solution forms.

Theorem 1. *Let p be any prime number. Then the solutions to the Diophantine equation $4(7^x) - p^y = z^2$ where x, y and z are non-negative integers satisfy the following:*

(i) For the prime $p = 2$, the unique solution is $(x, y, z, p) = (0, 2, 0, 2)$.

(ii) For the prime $p = 3$, if a solution exists, then it must be of the following set

$$\begin{aligned} &(x, y, z, p) \in \{(0,1,1,3), (1,1,5,3), (1,3,1,3), (2,3,13,3), (3,1,37,3)\} \\ &\cup \{(2k+1, 4l+1, 16m+n, 3) | \text{ for any integers } k \geq 2, l, m \geq 0 \text{ and } n = 3, 5, 11, 13\} \\ &\cup \{(2k+1, 4l+3, 16m+n, 3) | \text{ for any integers } k \geq 2, l, m \geq 0 \text{ and } n = 1, 7, 9, 15\}. \end{aligned}$$

(iii) There is no solution for any prime p with $5 \leq p \leq 17$.

(iv) For the prime $p \geq 19$, a necessary condition for the existence of solutions is that $p \equiv 19 \pmod{24}$. In such cases, all solutions must satisfy: $(x, y, z, p) \in \{(2k+1, 2l+1, 24m+n, 24r+19) | \text{ for any integers } k, l, m, r \geq 0 \text{ and } n = 3, 9, 15, 21\}$.

Proof. We begin by examining the degenerate cases $x = 0$ and $y \geq 0$. In this case, direct computation shows that $(x, y, z, p) = (0, 1, 1, 3)$ and $(0, 2, 0, 2)$ are valid solutions, as the expression $4(1) - p^y = z^2$ yields a non-negative perfect square. Next, suppose $x \geq 1$ and $y = 0$. Then the equation becomes $4(7^x) - 1 = z^2$, which is impossible modulo 4 since $z^2 \equiv 0$ or $1 \pmod 4$, but $4(7^x) - 1 \equiv 0 - 1 \equiv 3 \pmod 4$. Hence, no solution exists in this case. We now focus on the remaining case where both x and y are positive.

To proceed, we consider three cases based on the value of the prime p:

Case 1 $p = 2$.

$$\text{Then } 4(7^x) - 2^y \equiv \begin{cases} 0 - 1 \equiv 6 \pmod 7 \text{ if } y \equiv 0 \pmod 3, \\ 0 - 2 \equiv 5 \pmod 7 \text{ if } y \equiv 1 \pmod 3, \\ 0 - 4 \equiv 3 \pmod 7 \text{ if } y \equiv 2 \pmod 3. \end{cases}$$

However, for any integer $z, z^2 \equiv 0, 1, 2, 4 \pmod 7$ so no solution is possible in this case.

Case 2 $p = 3$.

Since z is odd and not divisible by 3, it follows that $z^2 \equiv 1 \pmod{24}$. Observe that for any $x > 0$, we have $4(7^x) - 3^y \equiv \begin{cases} 4 - 3 \equiv 1 \pmod{24} \text{ if } y \text{ is odd,} \\ 4 - 9 \equiv 19 \pmod{24} \text{ if } y \text{ is even.} \end{cases}$

This contradicts the fact $z^2 \equiv 1 \pmod{24}$ when y is even, so we only consider the case when y is an odd number.

Subcase 2.1 $x = 2k$ for some $k \geq 1$.

In this case, the equation $4(7^x) - p^y = z^2$ becomes $3^y = (2(7^k) - z)(2(7^k) + z)$, which is factorization of 3^y into two positive integers. Let $3^u = 2(7^k) - z$ and $3^{y-u} = 2(7^k) + z$ for some $0 \leq u < y$. Adding these two expressions yields $4(7^k) = 3(3^{u-1} + 3^{y-u-1})$. If $0 < u < y$, then the right-hand side is divisible by 3 while the left-hand side is not, giving a contradiction. Hence, the only possibility is $u = 0$, leading to $1 = 2(7^k) - z$ and $3^y = 2(7^k) + z$. Substituting the first equation into the second gives $3^y = 4(7^k) - 1$. We now consider whether k is even or odd. If $k = 2l$ for some $l \geq 1$. Then we obtain $3^y = 4(7^{2l}) - 1 = (2(7^l) - 1)(2(7^l) + 1)$. Since $l \neq 0$, both factors are greater than 1 and differ by 2, so both must be divisible by 3, which is impossible since 3 does not divide 2. Therefore, no solution exists in the case k is even. Next, we suppose $k = 2h + 1$ for some $h \geq 0$. Then we have $3^y = 4(7^{4h+2}) - 1 = (2(7^{2h+1}) - 1)(2(7^{2h+1}) + 1)$, which again leads to a contradiction unless $h = 0$, i.e., $x = 2$. Hence, the only solution in this case $(x, y, z) = (2, 3, 13)$.

Subcase 2.2 $x = 2k + 1$ for some $k \geq 0$.

For $k = 0$ or 1, direct computation shows that $(x, y, z) = (1, 1, 5), (1, 3, 1)$ and $(3, 1, 37)$ are valid solutions, as the expression $4(7^x) - 3^y = z^2$ yields a non-negative perfect square. For $k \geq 2$, consider the equation with modulo 16, we have

$$z^2 \equiv 4(7^x) - 3^y \equiv \begin{cases} 4(7) - 3 \equiv 9 \pmod{16} & \text{if } y \equiv 1 \pmod 4, \\ 4(7) - 11 \equiv 1 \pmod{16} & \text{if } y \equiv 3 \pmod 4. \end{cases}$$

Since

$$z \equiv \begin{cases} 3, 5, 11, 13 \pmod{16} & \text{if } z^2 \equiv 9 \pmod{16}, \\ 1, 7, 9, 15 \pmod{16} & \text{if } z^2 \equiv 1 \pmod{16}, \end{cases}$$

it follows that any solution (x, y, z) with $x = 2k + 1$, $k \geq 2$ and $y \equiv 1$ or $3 \pmod 4$ must satisfy: $(x, y, z) \in \{(2k+1, 4l+1, 16m+n)|$ for any integers $k \geq 2, l, m \geq 0$ and $n = 3, 5, 11, 13\} \cup \{(2k+1, 4l+3, 16m+n)|$ for any integers $k \geq 2, l, m \geq 0$ and $n = 1, 7, 9, 15\}$.

Case 3 $p \geq 5$.

Subcase 3.1 $x = 2k$ for some $k \geq 1$.

Since z is odd, the quantity $d = \gcd(2(7^k) - z, 2(7^k) + z)$ is also odd. Note that the product $p^y = (2(7^k)-z)(2(7^k)+z)$ consists of two factors whose sum is $4(7^k)$, implying $d|4(7^k)$. Therefore, $d = 1$ or $7|d$. If $d = 1$ then $2(7^k) - z = 1$, which gives $p^y = 4(7^k) - 1$. Reducing both sides modulo 3 yields $p^y \equiv 0 \pmod 3$, so $p = 3$, contradicting the assumption $p \geq 5$. Hence, we must have $7|d$, which implies $p = 7$. In this case, the equation becomes $7^y = (2(7^k) - z)(2(7^k) + z)$, where the two factors are coprime and their product is a power of 7. Therefore, we can write $7^u = 2(7^k) - z, 7^{y-u} = 2(7^k) + z$ for some $1 \leq u < y$. Adding these two equations yields $4(7^k) = 7^u + 7^{y-u}$. Reducing modulo 6, we find $7^u + 7^{y-u} \equiv 2 \pmod 6$, while $4(7^k) \equiv 4 \pmod 6$. Since both sides are not congruent modulo 6, the equality cannot hold. Hence there is no solution when x is even and $p \geq 5$.

Subcase 3.2 $x = 2k + 1$ for some $k \geq 0$.

Note that $7^{2k+1} \equiv (49^k)7 \equiv 7 \pmod{24}$. Since p is an odd prime, we have $p \equiv \pm1, \pm5, \pm7, \pm11$ or $13 \pmod{24}$, so $p^y \equiv \begin{cases} 1 \pmod{24} \text{ if } y \text{ is even,} \\ p \pmod{24} \text{ if } y \text{ is odd.} \end{cases}$

Therefore, $4(7^x) - p^y \equiv \begin{cases} 4(7) - 1 \equiv 3 \pmod{24} & \text{if } y \text{ is even,} \\ 4(7) - p \equiv 4 - p \pmod{24} & \text{if } y \text{ is odd.} \end{cases}$

Since z is odd, it square must satisfy $z^2 \equiv 1$ or $9 \pmod{24}$. But $4(7^x) - p^y \equiv 3 \pmod{24}$ where y is even, which is not a quadratic residue modulo 24.

Now, suppose y is odd. Then $4 - p \equiv 4(7^x) - p^y \equiv z^2 \equiv 1$ or $9 \pmod{24}$. This implies $p \equiv 3$ or $19 \pmod{24}$. However, $\gcd(p, 3) = 1$ so $p \not\equiv 3 \pmod{24}$. Therefore, we must have $p \equiv 19 \pmod{24}$, and hence $z^2 \equiv 9 \pmod{24}$.

Finally, we can conclude that if $4(7^x) - p^y = z^2$ has a solution with prime $p \geq 19$ then both x and y must be odd, $z \equiv 3, 9, 15$ or $21 \pmod{24}$, and $p \equiv 19 \pmod{24}$.

Based on Theorem 1(iv), we refine the possible values of the prime p by analyzing the divisibility properties of z, particularly with respect to modulo 3. This leads to the following corollary.

Corollary 1. *Let $p \geq 19$ be a prime. Then all non-negative integer solutions (x, y, z, p) to the equation $4(7^x) - p^y = z^2$ are of the following form:*

$$(x, y, z, p) \in \{(6k + 1, 6l + 3, 24m + n, 24r + 19)|k, l, m, r \in \mathbb{Z}_0^+ \text{ and } n = 3, 9, 15, 21\}$$

$$\cup \{(6k+1, 6l+1, 24m+n, 72r+19) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}$$
$$\cup \{(6k+1, 6l+5, 24m+n, 72r+19) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}$$
$$\cup \{(6k+3, 6l+5, 24m+n, 72r+43) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}$$
$$\cup \{(6k+5, 6l+1, 24m+n, 72r+43) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}$$
$$\cup \{(6k+3, 6l+1, 24m+n, 72r+67) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}$$
$$\cup \{(6k+5, 6l+5, 24m+n, 72r+67) | k, l, m, r \in \mathbb{Z}_0^+ \textit{ and } n = 3, 9, 15, 21\}.$$

Remark 1. *This corollary implies that for any prime $p \geq 19$, the equation $4(7^x) - p^y = z^2$ has no solution if $x \equiv 3$ or $5 \pmod 6$ and $y \equiv 3 \pmod 6$.*

Proof. From Theorem 1(*iv*), we know that all solutions are of the form $(x, y, z, p) \in \{(2k+1, 2l+1, 24m+n, 24r+19) | k, l, m, r \in \mathbb{Z}_0^+ \text{ and } n = 3, 9, 15, 21\}$. We first consider $z = 24m + n$ where $m \geq 0$ and $n \in \{3, 9, 15, 21\}$. Since every n is divisible by 3, it follows that $z^2 \equiv 0 \pmod 9$. Next, we observe that $p = 24r + 19$ for some non-negative integer r, which implies $p \equiv 6r + 1 \pmod 9$. By Lemma 1 we have that

$$p^y \equiv \begin{cases} 1 \pmod 9 \text{ if } y \equiv 0 \pmod 3, \\ p \pmod 9 \text{ if } y \equiv 1 \pmod 3, \\ 2 - p \pmod 9 \text{ if } y \equiv 2 \pmod 3. \end{cases}$$

Next, we examine the behavior $4(7^x)$ with mod 9 and we have

$$4(7^x) \equiv \begin{cases} 4(1) \equiv 4 \pmod 9 \text{ if } x \equiv 0 \pmod 3, \\ 4(7) \equiv 1 \pmod 9 \text{ if } x \equiv 1 \pmod 3, \\ 4(7) \equiv 7 \pmod 9 \text{ if } x \equiv 2 \pmod 3. \end{cases}$$

Then we analyze the equation $4(7^x) - p^y \equiv z^2 \equiv 0 \pmod 9$. The resulting congruence modulo 9 for each pair $(x \bmod 3, y \bmod 3)$ yields the following:

$$\begin{aligned} \text{no solution} \quad & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ p \equiv 4 \pmod 9 \quad & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ p \equiv 7 \pmod 9 \quad & \text{if } x \equiv 0 \pmod 3 \text{ and } y \equiv 2 \pmod 3, \\ \text{no additional information} \quad & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ p \equiv 1 \pmod 9 \quad & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ p \equiv 1 \pmod 9 \quad & \text{if } x \equiv 1 \pmod 3 \text{ and } y \equiv 2 \pmod 3, \\ \text{no solution} \quad & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 0 \pmod 3, \\ p \equiv 7 \pmod 9 \quad & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 1 \pmod 3, \\ p \equiv 4 \pmod 9 \quad & \text{if } x \equiv 2 \pmod 3 \text{ and } y \equiv 2 \pmod 3. \end{aligned}$$

Combining these observations with Theorem 1(*iv*) and applying the Chinese Remainder Theorem to parameters x, y and p, we derive the following refined conditions for a prime $p \geq 19$:

(i) If $x \equiv 3$ or $5 \pmod 6$ and $y \equiv 3 \pmod 6$ then the equation has no solution.

(ii) Under the assumption that a solution exists:

  (a) If ($x \equiv 3 \pmod 6$ and $y \equiv 1 \pmod 6$) or ($x \equiv 5 \pmod 6$ and $y \equiv 5 \pmod 6$) then $p \equiv 67 \pmod{72}$.

  (b) If ($x \equiv 3 \pmod 6$ and $y \equiv 5 \pmod 6$) or ($x \equiv 5 \pmod 6$ and $y \equiv 1 \pmod 6$) then $p \equiv 43 \pmod{72}$.

  (c) If $x \equiv 1 \pmod 6$ and $y \equiv 1$ or $5 \pmod 6$ then $p \equiv 19 \pmod{72}$.

  (d) If $x \equiv 1 \pmod 6$ and $y \equiv 3 \pmod 6$ then $p \equiv 19 \pmod{24}$.

3. Conclusion

In this article, we have determined all non-negative integer solutions to the exponential Diophantine equation $4(7^x) - p^y = z^2$, where p is a prime number. Through a careful case-by-case analysis based on the value of the prime p, we find:

- A unique solution for $p = 2$, namely $(x, y, z, p) = (0, 2, 0, 2)$.
- A parametrized family of possible solutions satisfying specific congruence conditions for $p = 3$, where all solutions require x and y to be odd, except for the cases $(x, y, z, p) = (0, 1, 1, 3)$ and $(2, 3, 13, 3)$, in which x is even.
- No solutions exist for $5 \leq p \leq 17$.
- For $p \geq 19$, a necessary condition for the existence of the solutions is that $p \equiv 19 \pmod{24}$, and the values of x, y, z and p must satisfy precise congruence conditions derived using quadratic residues and the Chinese Remainder Theorem. Moreover, there is no solution when $x \equiv 3$ or $5 \pmod 6$ and $y \equiv 3 \pmod 6$.

In contrast to the previous studies summarized in Table 2, our analysis addresses a distinct class of exponential Diophantine equations in which the term p^y is subtracted from a fixed exponential base 7^x, rather than involving variable coefficients or dual exponential terms. For instance, Rabago [8] analyzed $4^x - p^y = 3z^2$, while Laipaporn et al. [9] examined $3^x + p(5^y) = z^2$, which is additive in nature. In contrast, our equation exhibits a more intricate balance between exponential growth and quadratic structure, requiring careful refinement through modular analysis.

Interestingly, the Diophantine equation $4(7^x) - p^y = z^2$ was also examined computationally using Algorithm 1, which internally calls two functions based on Algorithm 2, and Algorithm 3. The complete procedure was implemented in Python code (accessible at https://github.com/kadisak/EJPAM6066.git).

For the case $p = 3$, the computational search was performed over the range $0 \leq x \leq 100,000$ and $1 \leq y \leq \lfloor \log_p(4(7^x)) \rfloor$. Within this domain, exactly five solutions were obtained, namely $(x, y, z) = (0, 1, 1), (1, 1, 5), (1, 3, 1), (2, 3, 13)$, and $(3, 1, 37)$, see Table 3.

Table 3: Valid solutions to the equation $4(7^x) - p^y = z^2$ for $p = 3$, $0 \leq x \leq 100,000$, and $1 \leq y \leq \lfloor \log_p (4(7^x)) \rfloor$.

p	x	y	z
3	0	1	1
3	1	1	5
3	1	3	1
3	2	3	13
3	3	1	37

For prime numbers p satisfying $2 \leq p \leq 100,000$ with $p \neq 3$, and under the parameter constraints $1 \leq x \leq 10,000$ and $1 \leq y \leq \lfloor \log_p (4(7^x)) \rfloor$, a total of 31 solutions were found when $p \equiv 19 \pmod{24}$, summarized in Table 4.

Table 4: All computed solutions to the equation $4(7^x) - p^y = z^2$. The search was conducted over the range $2 \leq p \leq 100,000$, $1 \leq x \leq 10,000$, and $1 \leq y \leq \lfloor \log_p (4(7^x)) \rfloor$, for primes $p \equiv 19 \pmod{24}$, as related to Corollary 1.

Congruence class	p	x	y	z
$p \equiv 19 \pmod{72}$	19	1	1	3
	86491	7	1	1791
$p \equiv 67 \pmod{72}$	283	3	1	33
	643	3	1	27
	1291	3	1	9
$p \equiv 43 \pmod{72}$	2203	5	1	255
	5227	5	1	249
	8179	5	1	243
	11059	5	1	237
	16603	5	1	225
	19267	5	1	219
	21859	5	1	213
	24379	5	1	207
	33739	5	1	183
	35899	5	1	177
	37987	5	1	171
$p \equiv 43 \pmod{72}$	41947	5	1	159
	49003	5	1	135
	50587	5	1	129
	54907	5	1	111
	57427	5	1	99
	58579	5	1	93
	59659	5	1	87
	61603	5	1	75
	62467	5	1	69
	64627	5	1	51
	65203	5	1	45
	65707	5	1	39
	66499	5	1	27
	67003	5	1	15
	67219	5	1	3

Based on both our computational findings and Theorem 1, we propose the following conjectures:

- If $z \not\equiv 1, 5$ and $13 \pmod{16}$, then the equation $4(7^x) - 3^y = z^2$ has no solution.
- For any prime $p \equiv 19 \pmod{24}$ and $y \geq 3$, our computational results suggest that no solutions exist within the tested parameter range. This leads us to conjecture that such solutions are either extremely rare or do not exist at all.

Overall, this work contributes to the classification program of exponential Diophantine equations of the form $a(p^x) - b(q^y) = z^2$, especially when one exponential base is fixed. By combining classical number-theoretic tools with modern computation, we not only resolve the given equation but also establish a pathway for studying similar equations involving asymmetrical exponential and quadratic forms. We hope these results will inspire further theoretical generalizations and computational techniques for analyzing Diophantine equations of this type.

Algorithm 1 Pseudocode for solution search and verification.

```
Input: p_min: minimum prime considered; p_max: maximum prime considered; x_max: maximum value investigated for x.
Output: Verification Results: Diophantine Equation Solutions.
for p ← p_min to p_max do
    for x ← 1 to x_max do
        y_max ← ⌊log_p 4(7^x)⌋;        // the maximum value of y for each pair (p, x)
        for y ← 1 to y_max do
            if 4(7^x) − p^y is perfect square then
                z ← √(4(7^x) − p^y)
                if (x, y, z, p) satisfies Theorem 1 (verified by Algorithm 2) then
                    Output: "The solution (x, y, z, p) satisfies the Theorem 1"
                else
                    Output: "The solution (x, y, z, p) does not satisfy the Theorem 1"
                end
                if p ≥ 19 then
                    if (x, y, z, p) satisfies Corollary 1 (verified by Algorithm 3) then
                        Output: "The solution (x, y, z, p) satisfies the Corollary 1"
                    else
                        Output: "The solution (x, y, z, p) does not satisfy the Corollary 1"
                    end
                end
            end
            Output: "(x, y, z, p) is not a solution to the equation"
        end
    end
end
```

Algorithm 2 Pseudocode for verifying a solution (x, y, z, p) to the Theorem 1.

Input: (x, y, z, p): solution of the Diophantine equation
Output: 1: the solution satisfies the Theorem, 0: the solution does not satisfy Theorem

```
function satisfyMainTheorem(x,y,z,p)
    if (x, y, z, p) ∈ {(0, 2, 0, 2)} then
        return 1
    else if p = 3 then
        if (x ≥ 5 and x mod 2 = 1 and y mod 4 = 1 and z mod 16 ∈ {3, 5, 11, 13})
          or (x ≥ 5 and x mod 2 = 1 and y mod 4 = 3 and z mod 16 ∈ {1, 7, 9, 15})
          or ((x, y, z) ∈ {(0, 1, 1), (1, 1, 5), (1, 3, 1), (2, 3, 13), (3, 1, 37)}) then
            return 1
        end
    else if p ≥ 19 and x mod 2 = 1 and y mod 2 = 1 and z mod 24 ∈ {3, 9, 15, 21} then
        return 1
    return 0
```

Algorithm 3 Pseudocode for verifying a solution (x, y, z, p) to the Corollary 1.

Input: (x, y, z, p): solution of the Diophantine equation
Output: 1: the solution satisfies the Corollary, 0: the solution does not satisfy Corollary

```
function satisfyCorollary(x,y,z,p)
    if z mod 24 ∈ {3, 9, 15, 21} then
        if x mod 6 = 1 then
            if y mod 6 = 3 and p mod 24 = 19 then
                return 1 ;                // Congruence class: p ≡ 19 (mod 24)
            else if y mod 6 = 1 and p mod 72 = 19 then
                return 1 ;                // Congruence class: p ≡ 19 (mod 72)
            else if y mod 6 = 5 and p mod 72 = 19 then
                return 1 ;                // Congruence class: p ≡ 19 (mod 72)
        else if x mod 6 = 3 then
            if y mod 6 = 5 and p mod 72 = 43 then
                return 1 ;                // Congruence class: p ≡ 43 (mod 72)
            else if y mod 6 = 1 and p mod 72 = 67 then
                return 1 ;                // Congruence class: p ≡ 67 (mod 72)
        else if x mod 6 = 5 then
            if y mod 6 = 5 and p mod 72 = 67 then
                return 1 ;                // Congruence class: p ≡ 67 (mod 72)
            else if y mod 6 = 1 and p mod 72 = 43 then
                return 1 ;                // Congruence class: p ≡ 43 (mod 72)
    end
    return 0
```

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