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G-Compact Spaces Characterized by the Intersection of Countable Neighborhoods

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Abstract. In this research, we introduce and analyze the concepts of G-compactness, G-Lindelöfness, and G-countably compactness within the framework of topological spaces, which are characterized by more rigorous conditions compared to those governing compact and Lindelöf spaces. We formulate a series of theorems and present a variety of examples to clarify the relationships among G-compactness, G-Lindelöfness, compactness, and Lindelöfness. Additionally, we define the G-separation axioms utilizing G_δ sets and explore the interrelations among these concepts.

2020 Mathematics Subject Classifications: 54D30, 54E99, 54D10

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1. Introduction and Preliminary

The notion of compactness in mathematics pertains to a characteristic of topological spaces that generalizes the idea of closed and bounded subsets found in Euclidean space.

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A topological space is deemed compact if every open cover of that space admits a finite subcover, indicating that from any collection of open sets that collectively cover the space, a finite selection of these sets can also serve as a cover. This concept is rooted in the Heine-Borel theorem, which defines compact subsets of Euclidean space as those that are both closed and bounded. The formalization of compactness within topology emerged alongside the establishment of topology as a separate mathematical discipline during the late 19th and early 20th centuries. This era was marked by significant contributions from mathematicians such as Henri Poincaré, who laid down essential principles of topology, and Felix Hausdorff, who formulated the axiomatic basis for general topological spaces. Compactness is integral to numerous branches of mathematics, including analysis and functional analysis. Notably, continuous functions defined on compact spaces exhibit significant properties, such as the ability to achieve maximum and minimum values and the potential for approximation by polynomial or Fourier series, as articulated in the Stone-Weierstrass theorem. The exploration and generalization of this concept continue to be a focal point in contemporary mathematical research, encompassing fields such as infinite-dimensional topology and set-theoretic topology see [1, 2].

Metric spaces play a crucial role in the field of topology, as they offer a framework for establishing a topology on a given set. A topology consists of a collection of open sets that adhere to specific axioms. In the context of a metric space, a subset is deemed open if, for every point within that subset, there exists a radius such that all points located within that radius are also included in the subset. This relationship enables metric spaces to exemplify topological spaces, facilitating the investigation of concepts such as continuity and convergence see [3]. For more on metric theory we refer [4–14] and references therein.

Separation axioms in topology represent a collection of criteria utilized to differentiate among various types of topological spaces, particularly in terms of the ability to separate distinct points and sets through neighborhoods. The emergence of these axioms coincided with the maturation of topology as a specialized area of mathematics, significantly influenced by the work of mathematicians such as Felix Hausdorff. In his seminal 1914 publication, "Grundzüge der Mengenlehre," Hausdorff articulated the separation property, which became a cornerstone of contemporary topology by establishing axioms applicable to general spaces. This property, referred to as the Hausdorff condition, stipulates that for any two distinct points within a space, there exist disjoint open sets that separate them, thereby classifying the space as a Hausdorff space, or (T_2) space. The development of separation axioms is closely linked to the overall progression of topology, which originated in the 19th century, with key contributions from mathematicians such as Henri Poincaré, who played a pivotal role in defining topology as a distinct discipline through his 1895 work, "Analysis Situs." The advancement of topology involved the investigation of various spatial properties, including compactness and convergence, which were systematically formalized by mathematicians like Maurice Fréchet and Pavel Alexandrov through axiomatic methods. Today, separation axioms are integral to topological research, offering a structured approach to understanding the interactions between points and sets within a topological space. They are essential for differentiating among various types of spaces and for establishing numerous theorems within the field of topology see [1, 2].

Various forms of compactness are examined in numerous research articles, as illustrated in references [15–18]. In [19] the authors study topological spaces on symbolic m-plithogenic intervals, while [20] extends this to neutrosophic and refined neutrosophic real intervals, both using partial order relations.

In this document, we represent the set of natural numbers as and the set of real numbers as .

Definition 1 (Metric Space [3]). *A metric space (X, d) consists of a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying:*

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

Metric spaces provide a framework for establishing topology on a given set, where we consider a subset open if, for every point within that subset, there exists a radius such that all points within that radius also belong to the subset.

Definition 2 (Separation Axioms [1], [2]). *Separation axioms in topology serve as criteria differentiating various topological spaces, particularly regarding the ability to separate distinct points and sets through neighborhoods. The main separation axioms include:*

(i) T_0 (Kolmogorov): For any two distinct points, at least one has a neighborhood not containing the other.

(ii) T_1 (Fréchet): For any two distinct points, each has a neighborhood not containing the other.

(iii) T_2 (Hausdorff): For any two distinct points, disjoint neighborhoods exist containing them separately.

(iv) T_3 (Regular): A T_1 space where for any point and any closed set not containing it, disjoint neighborhoods exist containing them separately.

(v) T_4 (Normal): A T_1 space where for any two disjoint closed sets, disjoint neighborhoods exist containing them separately.

These axioms form a hierarchy where $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Definition 3 (Compactness [1]). *A topological space (X, τ) demonstrates compactness if every open cover has a finite subcover. Equivalently, a space exhibits compactness if every collection of closed sets with the finite intersection property has a non-empty intersection. A space shows countable compactness if every countable open cover has a finite subcover.*

Definition 4 (Lindelöf Space [2]). *A topological space (X, τ) qualifies as Lindelöf if every open cover has a countable subcover. This property ranks weaker than compactness but stronger than separability for many spaces.*

Definition 5 (G_δ Set [1])**.** *In a topological space (X, τ), a G_δ set is any set expressible as a countable intersection of open sets. That is, $G \subseteq X$ constitutes a G_δ set if and only if*

$$G = \bigcap_{n=1}^{\infty} U_n$$

where each U_n belongs to τ.

Definition 6 (F_σ Set [2])**.** *In a topological space (X, τ), an F_σ set is any set expressible as a countable union of closed sets. That is, $F \subseteq X$ constitutes an F_σ set if and only if*

$$F = \bigcup_{n=1}^{\infty} C_n$$

where each C_n is closed in X.

Definition 7 (Paracompact Space [1])**.** *A topological space (X, τ) achieves paracompactness if every open cover of X has a locally finite open refinement. This property generalizes compactness and holds particular importance in differential geometry and analysis.*

Definition 8 (Metacompact Space [17])**.** *A topological space (X, τ) demonstrates metacompactness if every open cover of X has a point-finite open refinement. This property ranks weaker than paracompactness but stronger than the Lindelöf property in many cases.*

Topologists have examined various compactness forms, including sequential compactness, countable compactness, and pseudo-compactness. Each notion captures different aspects of the intuitive "boundedness" idea in topological spaces, as illustrated in references [15], [16], [17], and [18].

The interplay between countability, compactness, and separation properties forms a rich research area in topology. Particularly, the study of G_δ sets and F_σ sets provides insight into the fine structure of topological spaces and has applications in descriptive set theory and analysis [3].

2. G-compact and G-Lindelöf spaces

Definition 9. *Let $(X,\ \tau)$ be a topological space. Then the collection $\tilde{} = \{G_\alpha :\ \alpha \in \Delta\}$ is called a G-cover of X provided that $X = \bigcup_{\alpha \in \Delta} G_\alpha$ where, G_α is a G_δ set in X for all $\alpha \in \Delta$.*

Definition 10. *The topologica space $(X,\ \tau)$ is called G-compact space if every G -cover has a finite subcover.*

Theorem 1. *Every G-compact space is compact.*

Proof. Let $(X,\ \tau)$ be a G-compact space and let $\widetilde{U} = \{U_\alpha :\ \alpha \in \Delta\}$ be an open cover of X. Since every open set U_α is countable intersection of itself, U_α is a G_δ set. Hence $\widetilde{U}$ is a G-cover of X. But X is G-compact space, so there is a finite subcover of $\widetilde{U}$ that covers X and therefore, X is compact.

Here, it should be mentioned that the converse of Theorem 1 need not be true. To show this we have the following example:

Example 1. *The cofinite topology on , $(\mathbb{R},\ \tau_{cof})$ is compact but not G-compact space.*

Proof. It is known that $(,\tau_{cof})$ is compact. However, $(\mathbb{R},\ \tau_{cof})$ is not G-compact space. To show that let$\widetilde{\ }= \{G_k :\ k\in \mathbb{N}\}$, where

$$G_k = \bigcap_{i=k}^{\infty} (\mathbb{R}\setminus \{i\}) = (\mathbb{R}\setminus\mathbb{N}) \cup \{1,\ 2,\ \ldots k-1\}\,.$$

Then$\widetilde{\ }$is a G-cover of $\mathbb{R}$ which has no finite subcover. Assume by contrary that $\mathbb{R} = \bigcup_{j=1}^{n} G_{k_j}$, where $k_{j-1} < k_j$ for $j = 2, 3, \cdots, n$. Then

$$\bigcup_{j=1}^{n} G_{k_j} = \bigcup_{j=1}^{n} \bigcap_{i=k_j}^{\infty} (\mathbb{R}\setminus \{i\}) = (\mathbb{R}\setminus\mathbb{N}) \cup \{1,\ 2,\ \ldots, k_n - 1\} = G_{k_n}$$

which is a contradiction.

Theorem 2. *Every finite topological space is G-compact space.*

Proof.

Let $(X,\ \tau)$ be a topological space such that X is finite set, we can write X as $X = \{x_1,\ x_2,\ \ldots x_n\}$. Let$\widetilde{\ }= \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of X, that means

$$X = \bigcup_{i=1}^{n} \{x_i\} = \bigcup_{\alpha\in\Delta} G_\alpha.$$

So, for all $1 \leq i \leq n$, $x_i \in G_{\alpha i}$, for some $\alpha i \in \Delta$. Hence, $\{G_{\alpha 1}, G_{\alpha 2},\ \ldots,\ \ G_{\alpha n}\}$ is a finite subcover of$\widetilde{\ }$for X. Therefore, $(X,\ \tau)$ is G-compact space.

Theorem 3. *If (X, τ) represents a G-compact space and (Y, τ') demonstrates homeomorphism with X, then Y exhibits G-compactness.*

Proof. Let $f : X \to Y$ establish a homeomorphism, and consider a G-cover$\widetilde{\ }= \{G_\alpha : \alpha \in \Delta\}$ of Y. For each G_α, we express $G_\alpha = \bigcap_{n=1}^{\infty} U_{\alpha,n}$ where each $U_{\alpha,n}$ belongs to τ'.

Since f creates a homeomorphism, $f^{-1}(G_\alpha) = f^{-1}(\bigcap_{n=1}^{\infty} U_{\alpha,n}) = \bigcap_{n=1}^{\infty} f^{-1}(U_{\alpha,n})$. As each $f^{-1}(U_{\alpha,n})$ belongs to τ, $f^{-1}(G_\alpha)$ forms a G_δ set in X.

Therefore, $\{f^{-1}(G_\alpha) : \alpha \in \Delta\}$ constitutes a G-cover of X. Since X demonstrates G-compactness, we find a finite subcover $\{f^{-1}(G_{\alpha_1}), f^{-1}(G_{\alpha_2}), \ldots, f^{-1}(G_{\alpha_n})\}$.

It follows that $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ forms a finite subcover of $\tilde{}$ for Y, proving that Y exhibits G-compactness.

Definition 11. *The topological space $(X,\ \tau)$ is called G-Lindelöf space if every G -cover has a countable subcover.*

Theorem 4. *Every G-compact space is G-Lindelöf space.*

Proof.
The proof is obvious, because every finite subcover is countable.

Here, it should be mentioned that the converse of Theorem 4 need not be true. To show that we have the following example:

Example 2. *Consider the cofinite topology on , $(\mathbb{N},\ \tau_{cof})$. To show that $(\mathbb{N},\ \tau_{cof})$ is a G-Lindelöf space, let $\tilde{} = \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of $\mathbb{N}$. Then for any $n \in \mathbb{N}$, there is $G_{\alpha_n} \in G$ such that $n \in G_{\alpha_n}$. Hence $\{G_{\alpha_n} : n \in \mathbb{N}\}$ is a countable subcover of $\tilde{}$ for $\mathbb{N}$. Therefore, $(\mathbb{N},\ \tau_{cof})$ is a G-Lindelöf space.*

Now, to show that $(\mathbb{N},\ \tau_{cof})$ is not G-compact space, let $n \in \mathbb{N}$. Then

$$\{n\} = \bigcap_{k=1, k \neq n}^{\infty} (\mathbb{N} \setminus \{k\})$$

is a G_δ set containing n. Hence, $\{\{n\} : n \in \mathbb{N}\}$ forms a G -cover of $\mathbb{N}$ that has no finite subcover. Therefore, $(\mathbb{N},\ \tau_{cof})$ is not G-compact space.

Definition 12. *A space X is called G-countably compact if every countable G-cover has a finite subcover.*

Remark 1. *Every G-countably compact space is countably compact, but the reverse is not true in general (see Example 2).*

Theorem 5. *Every G-compact space exhibits G-countably compactness, but the converse generally fails.*

Proof. Let (X, τ) represent a G-compact space, and consider a countable G-cover $\tilde{} = \{G_n : n \in \mathbb{N}\}$ of X. Since X demonstrates G-compactness, every G-cover (including countable ones) admits a finite subcover. Therefore, X exhibits G-countably compactness.

To show the converse fails generally, we examine the space (ω_1, τ), where ω_1 represents the first uncountable ordinal with the order topology. This space lacks G-compactness because the collection $\{\{\alpha\} : \alpha < \omega_1\}$ creates a G-cover with no finite subcover, as each singleton forms a G_δ set. However, (ω_1, τ) exhibits G-countable compactness because any countable cover of ω_1 must include a set containing points arbitrarily close to ω_1, and by the order topology's nature, such a set would include a tail of the ordinals.

Theorem 6. *Every G-Lindelöf space is Lindelöf space.*

Proof.
Let $(X,\ \tau)$ be a G-Lindelöf space and let $\widetilde{U} = \{U_\alpha :\ \alpha \in \Delta\}$ be an open cover of X. Since every open set U_α is countable intersection of itself, U_α is a G_δ set. Hence $\widetilde{U}$ is a G –cover of X. But X is G-Lindelöf space, so there is a countable subcover of $\widetilde{U}$ that covers X and therefore, X is Lindelöf.

Here, it should be mentioned that the converse of Theorem 6 need not be true. To show that we have the following example:

Example 3. *Let $(\mathbb{R},\ \tau_u)$ be the real numbers with the usual topology. Then $(\mathbb{R},\ \tau_u)$ is Lindelöf but not G- Lindelöf space.*

Proof. Since $(\mathbb{R},\ \tau_u)$ is second countable, it is Lindelöf.
Now, to show that $(\mathbb{R},\ \tau_u)$ is not G-Lindelöf space, let $\widetilde{G} = \{\{x\} : x\in \mathbb{R}\}$ be a G-cover of $\mathbb{R}$, one can verify that for any $x\in \mathbb{R}$, $\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n},\ x + \frac{1}{n}\right)$.
However, it is clear that $\widetilde{G}$ has no countable subcover.

Theorem 7. *If X represents a G-Lindelöf space and Y constitutes a continuous image of X, then Y exhibits G-Lindelöfness.*

Proof. Let $f : X \to Y$ establish a continuous surjection, and consider a G-cover $\widetilde{\ } = \{G_\alpha : \alpha \in \Delta\}$ of Y. For each G_α, we have $G_\alpha = \bigcap_{n=1}^{\infty} U_{\alpha,n}$ where each $U_{\alpha,n}$ belongs to the topology of Y.
Since f maintains continuity, each $f^{-1}(U_{\alpha,n})$ belongs to the topology of X, and $f^{-1}(G_\alpha) = \bigcap_{n=1}^{\infty} f^{-1}(U_{\alpha,n})$ forms a G_δ set in X. Thus, $\{f^{-1}(G_\alpha) : \alpha \in \Delta\}$ creates a G-cover of X.
Since X demonstrates G-Lindelöfness, we find a countable subcover $\{f^{-1}(G_{\alpha_i}) : i \in \mathbb{N}\}$. As f achieves surjectivity, $\{G_{\alpha_i} : i \in \mathbb{N}\}$ forms a countable subcover of $\widetilde{\ }$ for Y, proving that Y exhibits G-Lindelöfness.

Remark 2. *Compact spaces need not be G- Lindelöf spaces.*

To show that we have the following example:

Example 4. *The closed interval $[0, 1]$ with the usual topology exhibits compactness but lacks G-Lindelöfness.*

Proof. We know that (by the Heine-Borel Theorem) $I = [0, 1]$ demonstrates compactness because it fulfills closedness and boundedness, as stated in [3]. However, I lacks G-Lindelöfness. We establish this by considering points $x \in I$ in three cases:

Case 1: If $x = 0$, then $\{0\} = \bigcap_{n=1}^{\infty} \left[0,\ \frac{1}{n}\right)$.

Case 2: If $x = 1$ then $\{1\} = \bigcap_{n=1}^{\infty} \left(\frac{n+1}{n+2},\ 1 \right]$.

Case 3: If $x \in (0,\ 1)$, then we find an ε-neighborhood containing x with $(x-\varepsilon, x+\varepsilon) \subseteq (0,\ 1)$. By the Archimedean property, we find $M_x \in \mathbb{N}$ such that $\frac{1}{M_x} < \varepsilon$, establishing $\left(x - \frac{1}{n},\ x + \frac{1}{n}\right) \subseteq (x-\varepsilon, x+\varepsilon) \subseteq (0,\ 1)$ for all $n \geq M_x$. Also,

$$\{x\} = \bigcap_{n=M_x}^{\infty} \left(x - \frac{1}{n},\ x + \frac{1}{n} \right).$$

In each case, $\{x\}$ forms a G_δ set for any $x \in I$. Therefore, $\widetilde{G} = \{\{x\} : x \in I\}$ creates a G-cover of I with no countable subcover since I demonstrates uncountability. This proves that I lacks G-Lindelöfness.

Remark 3. *G- Lindelöf spaces need not be compact spaces.*

To show that we have the following example:

Example 5. *The nested interval topology on $(0, 1)$ exhibits G-Lindelöfness but lacks compactness.*

Proof.
On the open interval $X = (0,\ 1)$ The nested interval topology τ is defined by declaring all open sets of the form $V_n = \left(0,\ 1 - \frac{1}{n}\right)$, for $n = 2,\ 3,\ 4,\ \ldots$, together with $\varnothing$ and X. This topological space is G- Lindelöf but not compact space. To see this, let $\widetilde{G} = \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of X, but any G_δ set in X belongs to $\{\varnothing,\ X,\ V_n :\ n = 2,\ 3,\ 4,\ \ldots\}$. Hence, $\widetilde{G}$ is countable. So, we can choose $\widetilde{G}$ itself as a countable subcover for X.

Theorem 8. *Every F_σ subspace of a G-compact space is G-compact.*

Proof.
Let $(X,\ \tau)$ be a G-compact space and let F be an F_σ subspace of X. To show that F is G-compact, let $\widetilde{G} = \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of F. Then $\widetilde{G} \cup \{X \backslash F\}$ is a G -cover of X. As X is G-compact, there exists a finite subcover of $\widetilde{G} \cup \{X \backslash F\}$, say $\mathcal{A} = \{G_{\alpha 1}, G_{\alpha 2},\ \ldots,\ G_{\alpha n}\}$. This covers F by the fact that it covers X.

Suppose $X \backslash F$ is an element of $\mathcal{A}$. Then $X \backslash F$ may be removed from $\mathcal{A}$, and the rest of $\mathcal{A}$ still covers F. Thus we have a finite subcover of $\widetilde{G}$ which covers F. Hence F is G-compact.

Theorem 9. *If (X, τ) represents a G-compact space and U constitutes a G_δ set in X, then U with the subspace topology exhibits G-Lindelöfness.*

Proof. Let U form a G_δ set in the G-compact space (X, τ), and consider a G-cover $\widetilde{\ } = \{G_\alpha \cap U : \alpha \in \Delta\}$ of U, where each G_α represents a G_δ set in X.

Since U forms a G_δ set, we write $U = \bigcap_{n=1}^{\infty} V_n$ where each V_n belongs to τ. For each $\alpha \in \Delta$, we express the set $G_\alpha = \bigcap_{n=1}^{\infty} W_{\alpha,n}$ where each $W_{\alpha,n}$ belongs to τ.

Consider the collection $\{G_\alpha \cup (X \setminus U) : \alpha \in \Delta\}$. Each set in this collection creates a G_δ set in X since:

$$G_\alpha \cup (X \setminus U) = G_\alpha \cup (X \setminus \bigcap_{n=1}^{\infty} V_n) = G_\alpha \cup \bigcup_{n=1}^{\infty} (X \setminus V_n)$$

This collection forms a G-cover of X. By G-compactness, it admits a finite subcover $\{G_{\alpha_1} \cup (X \setminus U), G_{\alpha_2} \cup (X \setminus U), ..., G_{\alpha_k} \cup (X \setminus U)\}$.

Therefore, $\{G_{\alpha_1} \cap U, G_{\alpha_2} \cap U, ..., G_{\alpha_k} \cap U\}$ constitutes a finite (and hence countable) subcover of ˜for U, demonstrating that U exhibits G-Lindelöfness.

Theorem 10. *Let $(X,\ \tau)$ be a topological space. The space X is G-compact if and only if each family $\mathcal{F}$ of F_σ subsets of X with the finite intersection property has nonempty intersection.*

Proof. First suppose that $(X,\ \tau)$ is G-compact. If $\{F_\alpha :\ \alpha \in \Delta\}$ is a family of F_σ sets of X having empty intersection, then $\{X \backslash F_\alpha :\ \alpha \in \Delta\}$ is a G -cover of X. By G-compactness, there is a finite subcover $\{X \backslash F_{\alpha_1}, X \backslash F_{\alpha_2},\ \ldots,\ X \backslash F_{\alpha_n}\ \}$ and then $\bigcap\limits_{i=1}^{n} F_{\alpha_i} = \varnothing$. So $\{F_\alpha :\ \alpha \in \Delta\}$ does not have the finite intersection property which is a contradiction.

Conversely, suppose that any family of F_σ sets of X with the finite intersection property has nonempty intersection and let $\widetilde{G} = \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of X with no finite subcover. Then $X \backslash (G_{\alpha 1} \cup G_{\alpha 2} \cup\ \ldots \cup G_{\alpha n}) \neq \varnothing$ for each finite collection $\{G_{\alpha 1}, G_{\alpha 2},\ \ldots,\ \ G_{\alpha n}\}$ from $\widetilde{G}$, in other words $\bigcap\limits_{i=1}^{n} (X \backslash G_{\alpha i}) \neq \varnothing$. Hence, we conclude that the collection $\{X \backslash G_{\alpha} :\ \alpha \in \Delta\}$ is a family of F_σ sets of X that has the finite intersection property. Therefore, we get $\bigcap\limits_{\alpha \in \Delta} (X \backslash G_\alpha) \neq \varnothing$, and hence, $\widetilde{G}$ is not a cover for X which is a contradiction.

Theorem 11. *The continuous image of a G-compact space is G-compact.*

Proof.

Suppose X is G-compact space and h is a continuous map of X onto Y. If $\widetilde{G} = \{G_\alpha :\ \alpha \in \Delta\}$ be a G -cover of Y, then $\left\{h^{-1}(G_\alpha) :\ \alpha \in \Delta\right\}$ is a G -cover of X and by G-compactness, a finite subcover exists, say $\left\{h^{-1}(G_{\alpha_1}),\ h^{-1}(G_{\alpha_2}),\ \ldots,\ h^{-1}(G_{\alpha_n})\right\}$. Then, since h is onto, the sets $G_{\alpha_1},\ G_{\alpha_2},\ \ldots,\ G_{\alpha_n}$ covers Y. Thus Y is G-compact space.

Remark 4. *Paracompact spaces need not be G-compact spaces.*

To show that we have the following example:

Example 6. *Consider the Sorgenfrey line $\mathbb{R}_\ell$ (the real line with the lower limit topology). This space demonstrates paracompactness but lacks G-compactness.*

Proof. Researchers have established that $\mathbb{R}_\ell$ exhibits paracompactness, as shown in [2]. To prove it lacks G-compactness, we observe that for any $x \in \mathbb{R}$, the singleton $\{x\}$ forms a G_δ set since

$$\{x\} = \bigcap_{n=1}^{\infty} [x, x + \frac{1}{n})$$

Hence, the collection $\{\{x\} : x \in \mathbb{R}\}$ creates a G-cover of $\mathbb{R}_\ell$ that has no finite subcover. Therefore, $\mathbb{R}_\ell$ lacks G-compactness.

Theorem 12. *Let $\{X_\alpha : \alpha \in \Lambda\}$ represent a family of topological spaces. If the product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ exhibits G-compactness, then each X_α demonstrates G-compactness.*

Proof. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ exhibit G-compactness and fix $\beta \in \Lambda$. Consider a G-cover $\widetilde{\mathcal{G}} = \{G_i : i \in I\}$ of X_β. For each G_i, define $\widetilde{G}_i = \pi_\beta^{-1}(G_i)$, where π_β denotes the projection map from X onto X_β. Since π_β maintains continuity and G_i forms a G_δ set in X_β, $\widetilde{G}_i$ constitutes a G_δ set in X.

The collection $\{\widetilde{G}_i : i \in I\}$ creates a G-cover of X. Since X exhibits G-compactness, we find a finite subcover $\{\widetilde{G}_{i_1}, \widetilde{G}_{i_2}, \dots, \widetilde{G}_{i_n}\}$. Then $\{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$ forms a finite subcover of $\widetilde{\mathcal{G}}$ for X_β, proving that X_β exhibits G-compactness.

Theorem 13. *Let (X, τ) represent a topological space. If X demonstrates metrizability and separability, then X exhibits G-Lindelöfness if and only if X demonstrates second countability.*

Proof. Suppose X constitutes a metrizable, separable space exhibiting G-Lindelöfness. Let $\{x_n : n \in \mathbb{N}\}$ form a countable dense subset of X. For each pair of rational numbers p, q with $p > 0$, and each $n \in \mathbb{N}$, define the open ball $B(x_n, p) = \{x \in X : d(x, x_n) < p\}$. The collection $\mathcal{B} = \{B(x_n, p) : n \in \mathbb{N}, p \in \mathbb{Q}^+\}$ demonstrates countability.

We claim that $\mathcal{B}$ forms a base for τ. Consider any open set $U \in \tau$ and point $x \in U$. By the metric topology's definition, we find $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Since $\{x_n : n \in \mathbb{N}\}$ demonstrates density, some x_m exists such that $d(x, x_m) < \epsilon/3$. Choose a rational number p such that $d(x, x_m) < p < \epsilon/3$. Then $x \in B(x_m, p) \subset B(x, \epsilon) \subset U$. This proves that $\mathcal{B}$ constitutes a countable base for X, establishing X's second countability.

Conversely, suppose X demonstrates metrizability and second countability. Let $\mathcal{B}$ form a countable base for τ. Every G_δ set G in X can be expressed as $G = \bigcap_{n=1}^{\infty} U_n$ where each $U_n \in \tau$. But we can express each U_n as a union of elements from the base $\mathcal{B}$. Since X demonstrates second countability, this means we find at most countably many different G_δ sets in X. Therefore, any G-cover of X contains at most countably many distinct elements, automatically making X G-Lindelöf.

3. G- Separation Axioms

Definition 13. *A space X is called T_{δ_0} space if whenever x and y are distinct points in X, there is a G_δ set containing one and not the other.*

Definition 14. *A space X is called T_{δ_1} space if whenever x and y are distinct points in X, there is a G_δ sets G_1 and G_2 such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$.*

Evidently, every T_{δ_1} space is T_{δ_0}.

Theorem 14. *A space X is T_{δ_0} if and only if it is T_0. Also, a space X is T_{δ_1} if and only if it is T_1.*

Proof.
It is easy to show that a space X is T_{δ_0} if and only if it is T_0. Now, to show that a space X is T_{δ_1} if and only if it is T_1 it is enough to show that if X is T_{δ_1}, then it is T_1. Hence, let X be a T_{δ_1} space and let $x, y \in X$ be two distinct elements, since X is T_{δ_1} there are two G_δ sets G_1 and G_2 such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. By the definition of G_δ sets assume $G_1 = \bigcap_{n=1}^{\infty} G_{1_n}$, $G_2 = \bigcap_{n=1}^{\infty} G_{2_n}$ where G_{1_n} and G_{2_n} are open in X for $n = 1, 2, \cdots$. Since $x \notin G_2$, so there is G_{2_i} for some i such that $x \notin G_{2_i}$. Similarly, Since $y \notin G_1$, so there is G_{1_k} for some k such that $y \notin G_{1_k}$ but $x \in G_{1_k}$ and $y \in G_{2_i}$. Therefore X is T_1 space.

Definition 15. *A space X is called T_{δ_2} space if whenever x and y are distinct points in X, there are disjoint G_δ sets G_1 and G_2 such that $x \in G_1$ and $y \in G_2$.*

Evidently, every T_{δ_2} space is T_{δ_1}.

Remark 5. *Clearly every T_2 space is T_{δ_2} space but the converse is not true in general.*

To show that we give the following example.

Example 7. *Consider the cofinite topology on , $(\mathbb{N}, \tau_{cof})$. Then $(\mathbb{N}, \tau_{cof})$ is not T_2 space because is uncountable. To show that $(\mathbb{N}, \tau_{cof})$ is T_{δ_2}, let n, m be two distinct points in . Then, $\{m\}$, and $\{n\}$ are two disjoint G_δ-sets. In fact*

$$\{m\} = \bigcap_{i=1}^{\infty}((-\{i\}) \cup \{m\}),$$

$$\{n\} = \bigcap_{i=1}^{\infty}((-\{i\}) \cup \{n\}).$$

Remark 6. *Every $T_{\delta 2}$ space is $T_{\delta 1}$ but the converse is not true in general.*

Example 8. *Consider with the cofinite topology. Then, $(, \tau_{cof})$ is $T_{\delta 1}$ which is not $T_{\delta 2}$. Clearly, $(, \tau_{cof})$ is T_1 and hence, $T_{\delta 1}$.*
Now, assume to the contrary that $(, \tau_{cof})$ is $T_{\delta 2}$. Then for any $a, b \in$ there are two G_δ sets say $G_1 = \bigcap_{i=1}^{\infty} U_i$ and $G_2 = \bigcap_{i=1}^{\infty} V_i$, such that $a \in G_1,\ b \in G_2$ and $G_1 \cap G_2 = \phi$.
So,

$$\left(\bigcap_{i=1}^{\infty} U_i\right) \bigcap \left(\bigcap_{i=1}^{\infty} V_i\right) = \phi.$$

$\Longrightarrow$

$$\bigcap_{i=1}^{\infty} \left(U_i \bigcap V_i\right) = \phi.$$

$\Longrightarrow$

$$\left(\bigcap_{i=1}^{\infty} \left(U_i \bigcap V_i\right)\right)^c = .$$

$\Longrightarrow$

$$\bigcup_{i=1}^{\infty} \left(U_i \bigcap V_i\right)^c = .$$

$\Longrightarrow$

$$\bigcup_{i=1}^{\infty} \left({U_i}^c \bigcup {V_i}^c\right) = .$$

Since ${U_i}^c$ and ${V_i}^c$ are finite for each i, it follows that is countable, which a contradiction.

Theorem 15. *A space X exhibits T_{δ_2} if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ forms a G_δ set in the product space $X \times X$.*

Proof. Suppose X represents a T_{δ_2} space. For each pair of distinct points $x, y \in X$, we find disjoint G_δ sets G_x and G_y containing x and y respectively. This means $(x, y) \in G_x \times G_y$ and $(G_x \times G_y) \cap \Delta = \emptyset$.

Let $U_{x,y} = G_x \times G_y$, which constitutes a G_δ set in $X \times X$. Then $(X \times X) \setminus \Delta = \bigcup_{x \neq y} U_{x,y}$, making $(X \times X) \setminus \Delta$ an F_σ set in $X \times X$. Therefore, Δ forms a G_δ set in $X \times X$.

Conversely, suppose Δ constitutes a G_δ set in $X \times X$. Then $(X \times X) \setminus \Delta$ represents an F_σ set, which we can write as $\bigcup_{n=1}^{\infty} F_n$ where each F_n is closed in $X \times X$.

For any distinct points $x, y \in X$, the pair $(x, y) \in (X \times X) \setminus \Delta$, so $(x, y) \in F_n$ for some n. Since F_n is closed, (x, y) has an open neighborhood $U \times V$ contained in F_n. Since $(x, x) \notin F_n$, we must have $x \notin V$ or $y \notin U$. Without loss of generality, assume $x \notin V$.

Let $G_x = U$ and $G_y = V$. Then G_x and G_y form open sets containing x and y respectively, and $G_x \cap G_y = \emptyset$. By taking countable intersections of such open sets for different n, we can construct disjoint G_δ sets containing x and y, establishing that X exhibits T_{δ_2}.

4. Connections with Other Compactness Notions

In this section, we explore relationships between G-compactness and other well-established compactness notions in topology.

Theorem 16. *Every G-compact space exhibits metacompactness.*

Proof. Let (X, τ) represent a G-compact space and consider an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ of X. Since every open set U_α equals a countable intersection of itself, it constitutes a G_δ set. Hence, $\mathcal{U}$ forms a G-cover of X. Since X demonstrates G-compactness, we find a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ of $\mathcal{U}$. This finite collection qualifies as point-finite, establishing that X exhibits metacompactness.

Theorem 17. *A space X demonstrates G-compactness if and only if every filter base on X consisting of F_σ sets has a cluster point.*

Proof. Suppose X exhibits G-compactness and consider a filter base $\mathcal{F}$ consisting of F_σ sets. If $\mathcal{F}$ lacks any cluster point, then for each $x \in X$, we find $F_x \in \mathcal{F}$ such that $x \notin \overline{F_x}$. This means $x \in X \setminus \overline{F_x}$, which forms an open set.

Since F_x constitutes an F_σ set, its closure $\overline{F_x}$ also forms an F_σ set (as the closure of an F_σ set remains an F_σ set in a regular space). Therefore, $X \setminus \overline{F_x}$ constitutes a G_δ set.

The collection $\{X \setminus \overline{F_x} : x \in X\}$ creates a G-cover of X. By G-compactness, we find a finite subcover $\{X \setminus \overline{F_{x_1}}, X \setminus \overline{F_{x_2}}, \ldots, X \setminus \overline{F_{x_n}}\}$.

This implies that $X = \bigcup_{i=1}^{n}(X \setminus \overline{F_{x_i}})$, or equivalently, $\bigcap_{i=1}^{n} \overline{F_{x_i}} = \emptyset$. But since $\mathcal{F}$ forms a filter base, the sets $F_{x_1}, F_{x_2}, \ldots, F_{x_n}$ have non-empty intersection, which means $\bigcap_{i=1}^{n} \overline{F_{x_i}} \neq \emptyset$, creating a contradiction.

Conversely, suppose every filter base consisting of F_σ sets has a cluster point, and consider a G-cover $\widetilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of X with no finite subcover. For any finite collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ from $\widetilde{G}$, we have $\bigcup_{i=1}^{n} G_{\alpha_i} \neq X$, which means $\bigcap_{i=1}^{n}(X \setminus G_{\alpha_i}) \neq \emptyset$.

Since each G_{α_i} constitutes a G_δ set, each $X \setminus G_{\alpha_i}$ forms an F_σ set. The collection $\{X \setminus G_\alpha : \alpha \in \Delta\}$ has the finite intersection property and consists of F_σ sets, so it generates a filter base of F_σ sets. By our assumption, this filter base has a cluster point $x \in X$.

But x must belong to some G_β from the original G-cover, which means $x \in X \setminus (X \setminus G_\beta)$. This contradicts x being a cluster point for the filter base, as x has a neighborhood disjoint from $X \setminus G_\beta$. Therefore, the original G-cover must have a finite subcover, establishing that X exhibits G-compactness.

Theorem 18. *Every paracompact Hausdorff space in which every closed set forms a G_δ set exhibits G-Lindelöfness.*

Proof. Let (X, τ) represent a paracompact Hausdorff space in which every closed set forms a G_δ set. Consider a G-cover $\widetilde{\mathcal{G}} = \{G_\alpha : \alpha \in \Delta\}$ of X. Since X demonstrates paracompactness, we find a locally finite open refinement $\mathcal{V} = \{V_\beta : \beta \in \Gamma\}$ of $\widetilde{\mathcal{G}}$.

For each $x \in X$, we find a neighborhood N_x of x that intersects only finitely many members of $\mathcal{V}$, say $V_{\beta_1}, V_{\beta_2}, \ldots, V_{\beta_{n_x}}$. For each V_{β_i}, we find a G_{α_i} in $\widetilde{\mathcal{G}}$ such that $V_{\beta_i} \subseteq G_{\alpha_i}$.

Let $\mathcal{G}' = \{G_{\alpha_i} : i = 1, 2, \ldots, n_x, x \in X\}$. Since $\mathcal{V}$ demonstrates local finiteness in a paracompact space, $\mathcal{G}'$ demonstrates countability. Therefore, $\mathcal{G}'$ forms a countable subcover of $\widetilde{\mathcal{G}}$, establishing that X exhibits G-Lindelöfness.

Future research

We encourage scholars to explore G-compactness within metric spaces and investigate its behavior under various topological constructions, extending work in [3] and [4]. Further research could examine G-compactness in relation to filter convergence, nets, and ultrafilters, building on concepts from [15] and [16]. The interaction between G-properties and topological dimension theory also presents an interesting investigation avenue, as suggested by [18].

Another promising direction involves studying G-compactness in function spaces, particularly regarding continuous functions, uniform convergence, and equicontinuity. The relationship between G-compactness and completeness in metric spaces could yield new insights into topological spaces' structure.

Additionally, developing more refined G-separation axioms based on G_δ sets could generate new set-theoretic topology insights, following approaches in [9] and [17]. The connections between G-compactness and descriptive set theory, particularly the classification of Borel sets and analytic sets, present rich exploration opportunities.

Conclusion

In this paper, we have established a new framework for studying topological spaces through G_δ sets. The introduced concepts of G-compactness, G-Lindelöfness, and G-countably compactness impose stronger requirements than their classical counterparts, creating finer distinctions among topological spaces. Our results demonstrate that while these G-properties imply their corresponding classical properties, the converse relationships generally fail, as our carefully constructed counterexamples illustrate.

The G-separation axioms we've developed form a hierarchy paralleling the classical separation axioms, but with distinctive characteristics allowing classification of spaces indistinguishable under traditional separation properties. We've shown that certain spaces can exhibit T_{δ_2} without demonstrating T_2, highlighting these new axioms utility.

The relationships we've established with other compactness-like properties (metacompactness, paracompactness) further integrate our G-properties into the broader topological theory landscape. Furthermore, our results on G-compactness behavior under various topological operations, such as continuous maps, product spaces, and subspaces, provide tools for recognizing and applying these properties in diverse contexts.

Our characterization of G-compactness in terms of filter bases consisting of F_σ sets connects this concept with classical filter-theoretic approaches to compactness. Similarly, the relationship between G-separation axioms and the diagonal in product spaces provides a geometric perspective on these properties.

This research contributes to the ongoing refinement of topological classification schemes and opens avenues for exploring topological spaces' structure through increasingly sophisticated properties. The G-framework introduced here can potentially lead to new insights in functional analysis, set-theoretic topology, and related mathematical disciplines.

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