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Some Properties of the Non-central Stirling Numbers of the Second Kind with Complex Parameters

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Abstract. This work extends the non-central Stirling numbers of the second kind to complex arguments. This extension is achieved through an integral representation employing a Hankel contour. Furthermore, we investigate the extent to which key properties, including the recurrence relations, of the classical Stirling numbers are preserved under this complex generalization.

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Key Words and Phrases: Classical Stirling numbers, Non-central Stirling numbers, Complex numbers

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1. Introduction

Stirling numbers have long been a fundamental concept in combinatorics, originally defined for non-negative integers. These numbers play a crucial role in counting permutations, partitions, and other combinatorial structures. However, their application has typically been limited to discrete contexts. Extending Stirling numbers to real and complex arguments not only generalizes many well-known combinatorial identities but also significantly broadens their applicability across a wide array of mathematical fields.

In [1], Koutras introduced the non-central Stirling numbers of the second kind by a natural extension of the definition of the classical Stirling numbers of the second kind, $S(n,k)$. The non-central Stirling numbers of the second kind, denoted by $S_a(n,k)$ are defined as the coefficients of the following expansion with parameter a,

$$(t-a)^n = \sum_{k=0}^{n} S_a(n,k)(t)_n$$

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where $(t)_n$ is the falling factorial of t of order n given by

$$(t)_n = \prod_{i=1}^{n}(t-i+1) = t(t-1)(t-2)\cdots(t-n+1),$$

for $n \geq 1$; $(t)_0 = 1$ and $S_a(0,0) = 1$, $S_a(n,0) = (-a)^n$ and $S_a(0,k) = 0$, $n, k \neq 0$.

Note that if $a = 0$, the classical Stirling numbers, $S(n,k)$, are obtained. This definition leads to an alternative definition in terms of exponential generating functions which allows for further extensions to generalized non-central Stirling numbers.

Motivated by the problem of Graham, Knuth, and Patashnik, in [2], that is, to generalize stirling numbers of the second kind $S(n,k)$ by extending the range values of parameters n and k to complex numbers, Flajolet and Prodinger, in [3], defined an extension of the classical Stirling numbers $S(n,k)$ of the second kind to complex arguments. They established an alternative and more natural extension of Stirling numbers of complex arguments for which most classical identities are still satisfied. Some of these properties are listed here.

i) The Stirling numbers $S(x,y)$ of the second kind for complex arguments satisfy the following recurrence relation

$$S(x,y) = S(x-1,y-1) + yS(x-1,y).$$

ii) For any complex x and $n \in \mathbb{N}$, we have the following formula

$$S(x,k) = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^x.$$

Thus, following the work of Flajolet and Prodinger, it is interesting if we could extend non-central Stirling numbers of the second kind to complex arguments.

2. Preliminaries

This section covers fundamental concepts and results in complex analysis and combinatorics, including the binomial theorem, which will be applied in the later section. These results can be found in [4], [5], [6], [7], and [8].

Lemma 1. *[4] For any integer $n \geq 0$, we have*

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r.$$

The number $\binom{n}{r}$ is called a binomial coefficient and satisfies the following recurrence relation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

The following is the generalized binomial theorem due to Newton.

Lemma 2. *[4] For complex numbers x, y, and s, we have*

$$(x+y)^s = \sum_{r=0}^{\infty} \binom{s}{r} x^{s-r} y^r,$$

where

$$\binom{s}{r} = \begin{cases} 1, & r = 0 \\ \dfrac{s(s-1)(s-2)\cdots(s-r+1)}{r!}, & r > 0 \end{cases}.$$

Moreover, the Gamma function is being considered. This function is known to be related to Stirling-type numbers. The Gamma function [7], denoted by Γ, is an extension of the factorial function to real and complex numbers defined via improper integral that converges only for complex numbers with a positive real part. That is,

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt,$$

where $Re(z) > 0$. When $z = 1$,

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1.$$

When $z = n \in \mathbb{Z}^+$, we use integraton by parts and get

$$\Gamma(n) = (n-1) \int_0^{\infty} t^{n-2} e^{-t} dt.$$

Consequently, after applying integration by parts $(n-1)$ times, we obtain

$$\Gamma(n) = (n-1)(n-2)\cdots(1) \int_0^{\infty} e^{-t} dt = (n-1)!.$$

This shows that $\Gamma(z)$ is indeed an extension of the factorial function with its argument shifted down by 1.

The following theorem is a property of the Gamma function.

Lemma 3. *[9] The Gamma function $\Gamma(z)$ has an integral representation*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\gamma} e^t t^{-z} dt,$$

where γ is a Hankel contour.

The preceding formula is known as *Hankel's contour integral.* The path of integration γ starts at $-\infty - i0$ on the real axis, goes to $-\epsilon - i0$, circles the origin in the counter-clockwise direction with the radius ϵ to the point $-\epsilon + i0$ and returns to the point $-\infty + i0$.

Before presenting the main result, we introduce the following lemma, an original contribution by the author, which plays an important role in establishing several fundamental properties of non-central Stirling numbers with complex indices.

Lemma 4. *For complex numbers x and a with $Re(a) < 0$, we have*

$$\frac{x!}{2\pi i}\int_{\mathcal{H}} e^{(j-a)u}\frac{du}{u^{x+1}} = (j-a)^x.$$

Proof. Let $s = x + 1$ and $t = (j-a)u$. Then $u = (j-a)^{-1}t$ and $du = (j-a)^{-1}dt$.

Hence,

$$\begin{aligned}\frac{x!}{2\pi i}\int_{\mathcal{H}} e^{(j-a)u}\frac{du}{u^{x+1}} &= (j-a)^{s-1}\frac{(s-1)!}{2\pi i}\int_{\mathcal{H}} t^{-s}e^t dt\\ &= (j-a)^{s-1}\frac{\Gamma(s)}{2\pi i}\int_{\mathcal{H}} t^{-s}e^t dt.\end{aligned}$$

However, by Lemma 3,

$$\frac{\Gamma(s)}{2\pi i}\int_{\mathcal{H}} t^{-s}e^t dt = 1.$$

Thus,

$$\frac{x!}{2\pi i}\int_{\mathcal{H}} e^{(j-a)u}\frac{du}{u^{x+1}} = (j-a)^x.$$

In the following section, we begin by presenting the definition of the non-central Stirling numbers of the second kind for complex arguments and establish several key properties. This section presents the central result of our work, which we hope will serve as a foundation for further analysis and potential applications.

3. The Non-central Stirling Numbers with Complex Indices

The exponential generating funcion for $S_a(n,k)$ is given by

$$\sum_{n=k}^{\infty} S_a(n,k)\frac{u^n}{n!} = e^{-au}\frac{1}{k!}(e^u-1)^k.$$

That is,

$$\sum_{n\geq 0} S_a(n,k)\frac{k!}{n!}u^n = e^{-au}(e^u-1)^k.$$

Hence, by Cauchy Integral Formula, we have

$$S_a(n,k) = \frac{1}{2\pi i}\frac{n!}{k!}\int_{\gamma}\frac{e^{-au}(e^u-1)^k du}{u^{n+1}},$$

where γ is a small contour encircling the origin.

We can deform γ into a Hankel contour which starts from $-\infty$ below the negative $x-axis$ surrounding the origin couterclockwise and returns to $-\infty$ above the negative $x-axis$. This suggest the following definition. Here, we assume that $\mathcal{H}$ is at a distance ≤ 1 from the real axis.

Definition 3.1. *The non-central Stirling numbers of the second kind with complex arguments x and y, denoted by $S_a(x,y)$, are defined by*

$$S_a(x,y) = \frac{1}{2\pi i}\frac{x!}{y!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^y du}{u^{x+1}},$$

where a is a complex number with $Re(a) < 0$, $x! = \Gamma(x+1)$ and the logarithm involved in the functions $(e^u-1)^y$ and u^{x+1} is taken to be the principal branch.

Following this definition, we explore a fundamental relationship that is essential to the theory of Stirling-type numbers. The first theorem we present establishes a relation commonly employed to define these numbers.

Theorem 3.1. *For complex numbers x and a with $Re(a) < 0$, the non-central Stirling numbers for complex arguments satisfy the following relation:*

$$(t-a)^x = \sum_{k=0}^{\infty} S_a(x,k)(t)_k.$$

Proof. First, observe that

$$(t)_k = [t(t-1)(t-2)\cdots(t-k+1)]\frac{k!}{k!}\frac{(t-k)!}{(t-k)!} = k!\binom{t}{k}.$$

Now, applying Definition 3.1, we get

$$\begin{aligned}\sum_{k=0}^{\infty} S_a(x,k)(t)_k &= \sum_{k=0}^{\infty}\left\{\frac{1}{2\pi i}\frac{x!}{k!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^k du}{u^{x+1}}\right\}(t)_k \\ &= \frac{x!}{2\pi i}\int_{\mathcal{H}} e^{-au}\left\{\sum_{k=0}^{\infty}\frac{(e^u-1)^k}{k!}(t)_k\right\}\frac{du}{u^{x+1}}.\end{aligned}$$

Then by the generalized binomial theorem due to Newton,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(e^u-1)^k}{k!}(t)_k &= \sum_{k=0}^{\infty}\binom{t}{k}(e^u-1)^k \\ &= [(e^u-1)+1]^t = e^{ut}.\end{aligned}$$

Therefore, by Lemma 4,

$$\sum_{k=0}^{\infty} S_a(x,k)(t)_k = \frac{x!}{2\pi i}\int_{\mathcal{H}} e^{-au}e^{ut}\frac{du}{u^{x+1}} = (t-a)^x.$$

Theorem 3.2. *For complex numbers x and a with $Re(a) < 0$, $Re(x) > 0$, the non-central Stirling numbers $S_a(x,y)$ satisfy the following recurrence relation:*

$$S_a(x,y) = S_a(x-1,y-1) + (y-a)S_a(x-1,y).$$

Proof. By Definition 3.1,

$$S_a(x,y) = \frac{1}{2\pi i}\frac{x!}{y!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^y du}{u^{x+1}}.$$

Now let $s = e^{-au}(e^u-1)^y$ and $dt = \frac{du}{u^{x+1}}$. Then,

$$ds = \left(e^u e^{-au}y(e^u-1)^{y-1} - ae^{-au}(e^u-1)^y\right)du \quad \text{and} \quad t = \frac{1}{-xu^x}.$$

Thus, integration by parts yields

$$S_a(x,y) = \frac{1}{2\pi i}\frac{x!}{y!}\left\{\left[\frac{e^{-au}(e^u-1)^y}{-xu^x}\right]_{\mathcal{H}} - \int_{\mathcal{H}}\frac{1}{-xu^x}\left(e^u e^{-au}y(e^u-1)^{y-1} - ae^{-au}(e^u-1)^y\right)du\right\}.$$

However, as $u \to -\infty$, $\frac{1}{-xu^x} \to 0$. Hence,

$$\begin{aligned}S_a(x,y) &= \frac{1}{2\pi i}\frac{x!}{y!}\left[-\int_{\mathcal{H}}\frac{1}{-xu^x}e^{-au}(e^u-1)^{y-1}\left(e^u y - a(e^u-1)\right)du\right] \\ &= \frac{1}{2\pi i}\frac{(x-1)!}{y!}\int_{\mathcal{H}}\left(\frac{ye^{-au}(e^u-1)^{y-1}}{u^x} + \frac{(y-a)e^{-au}(e^u-1)^y}{u^x}\right)du \\ &= \frac{1}{2\pi i}\frac{(x-1)!}{(y-1)!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^{y-1}du}{u^{(x-1)+1}} + (y-a)\frac{1}{2\pi i}\frac{(x-1)!}{y!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^y du}{u^{(x-1)+1}}.\end{aligned}$$

Thus, by Definition 3.1,

$$S_a(x,y) = S_a(x-1,y-1) + (y-a)S_a(x-1,y).$$

The next result is an explicit formula for $S_a(x,k)$.

Theorem 3.3. *For complex numbers x and a with $Re(a) < 0$, and a nonnegative integer k, we have*

$$S_a(x,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j-a)^x.$$

Proof. By Definition 3.1, we have

$$S_a(x,k) = \frac{1}{2\pi i}\frac{x!}{k!}\int_{\mathcal{H}}\frac{e^{-au}(e^u-1)^k du}{u^{x+1}}.$$

Applying the binomial theorem,

$$\begin{aligned}(e^u-1)^k &= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(e^u)^j\\ &= \sum_{j=0}^{k}(-1)^{k-j}(e)^{uj}\binom{k}{j}.\end{aligned}$$

Thus,

$$\begin{aligned}S_a(x,k) &= \frac{1}{2\pi i}\frac{x!}{k!}\int_{\mathcal{H}}e^{-au}\left(\sum_{j=0}^{k}(-1)^{k-j}e^{uj}\binom{k}{j}\right)\frac{du}{u^{x+1}}\\ &= \frac{1}{k!}\sum_{j=0}^{k}\left[(-1)^{k-j}\binom{k}{j}\cdot\frac{x!}{2\pi i}\int_{\mathcal{H}}e^{-au}e^{uj}\frac{du}{u^{x+1}}\right].\end{aligned}$$

Moreover, by Lemma 4,

$$S_a(x,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(j-a)^x.$$

To illustrate this, let us consider the case where $a = -1$. Using the explicit formula above, we compute the values of $S_{-1}(i,0)$, $S_{-1}(i,1)$, and $S_{-1}(1+i,1)$ as follows:

$$\begin{aligned}S_{-1}(i,0) &= \frac{1}{0!}\sum_{j=0}^{0}(-1)^{0-j}\binom{0}{j}(j+1)^i\\ &= 1(-1)^0\binom{0}{0}(0+1)^i = 1^i = 1;\end{aligned}$$

$$
\begin{aligned}
S_{-1}\big(i,1\big) &= \frac{1}{1!}\sum_{j=0}^{1}(-1)^{1-j}\binom{1}{j}(j+1)^{i}\\
&= 1\left[(-1)^{1}\binom{1}{0}(0+1)^{i} \;+\; (-1)^{0}\binom{1}{1}(1+1)^{i}\right]\\
&= -1+\cos{(\ln 2)}+i\sin{(\ln 2)};
\end{aligned}
$$

$$
\begin{aligned}
S_{-1}\big(1+i,1\big) &= \frac{1}{1!}\sum_{j=0}^{1}(-1)^{1-j}\binom{1}{j}(j+1)^{1+i}\\
&= 1\left[(-1)^{1}\binom{1}{0}(0+1)^{1+i} \;+\; (-1)^{0}\binom{1}{1}(1+1)^{1+i}\right]\\
&= -1+2\cos{(\ln 2)}+2i\sin{(\ln 2)}.
\end{aligned}
$$

The same value is obtained for $S_{-1}(1+i,1)$ using the recurrence relation in Theorem 3.2 as shown below:

$$
\begin{aligned}
S_{-1}\big(1+i,1\big) &= S_{-1}\big((1+i)-1\;,\;(1-1)\big) \;+\; \big(1-(-1)\big)S_{-1}\big((1+i)-1\;,\;1\big)\\
&= S_{-1}\big(i\;,0\big) \;+\; (2)\;S_{-1}\big(i\;,1\big)\\
&= 1 \;+\; 2\big[-1\;+\cos{(\ln 2)}+i\sin{(\ln 2)}\big]\\
&= -1+2\cos{(\ln 2)}+2i\sin{(\ln 2)}.
\end{aligned}
$$

Theorem 3.3 can be generalized further by substituting k with a complex number y, as presented in the subsequent theorem.

Theorem 3.4. *For complex numbers x, y and a with $Re(a) < 0$, we have*

$$S_a(x,y) = \frac{1}{y!}\sum_{j=0}^{\infty}(-1)^{y-j}\binom{y}{j}(j-a)^{x}.$$

Proof. Since y is a complex number, we use the generalized binomial theorem due to Newton to obtain

$$(e^u-1)^y = \sum_{j=0}^{\infty}\binom{y}{j}(-1)^{y-j}(e^u)^j.$$

In effect, using Definition 3.1, we have

$$\begin{aligned}
S_a(x,y) &= \frac{1}{2\pi i}\frac{x!}{y!}\int_{\mathcal{H}} e^{-au}\left(\sum_{j=0}^{\infty}(-1)^{y-j}e^{uj}\binom{y}{j}\right)\frac{du}{u^{x+1}} \\
&= \frac{1}{2\pi i}\frac{x!}{y!}\sum_{k=0}^{\infty}\int_{\mathcal{H}} e^{-au}(-1)^{y-j}e^{uj}\binom{y}{j}\frac{du}{u^{x+1}} \\
&= \frac{1}{y!}\sum_{k=0}^{\infty}\left[(-1)^{y-j}e^{uj}\binom{y}{j}\cdot\frac{x!}{2\pi i}\int_{\mathcal{H}} e^{(j-a)u}\frac{du}{u^{x+1}}\right].
\end{aligned}$$

Consequently, by Lemma 4,

$$S_a(x,y) = \frac{1}{y!}\sum_{j=0}^{\infty}(-1)^{y-j}\binom{y}{j}(j-a)^x.$$

4. Conclusion

In this work, we extended the non-central Stirling numbers of the second kind to complex arguments using an integral representation with a Hankel contour. This extension broadens the use of these numbers and offers new insights into their properties in the complex domain. We also explored how important properties, such as recurrence relations, are preserved and adapted, showing the usefulness of this approach for studying combinatorial structures.

Looking ahead, it would be beneficial to extend non-central Bell numbers (as defined in [10]) to complex arguments, building on the work in [11]. These Bell numbers are important for counting partitions and other combinatorial structures, and extending them could uncover new identities, recurrence relations, and lead to deeper insights in areas like number theory, asymptotics, and complex analysis. Additionally, expanding the parameters of $(r_1, \cdots, r_p)$-Stirling numbers of the second kind from [12] and the generalized Apostol-type Frobenius-Euler polynomials in [13] to complex arguments is also an interesting direction for future research.

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