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Controllability of a System with Nonlinear Damping Devices and Nonlinear Source Terms in Elasticity Problems: Existence, Time Blow-up, and Numerical Results

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Abstract. Swelling soil problems arise in various real-world applications, such as geomechanics, biomedical engineering, and hydrogel-based materials, where fluid interaction with elastic structures influences mechanical stability. In this study, we investigate a swelling soil system incorporating two nonlinear variable exponent damping and source terms, which provide a more adaptable framework for capturing heterogeneous material behaviors and evolving energy dissipation mechanisms. Using the Faedo-Galerkin method and the Banach Contraction Theorem, we establish the local existence and uniqueness of weak solutions under suitable conditions on the variable exponent functions. Furthermore, we demonstrate the global existence of solutions and identify conditions leading to finite-time blow-up, offering insights into stability and failure prediction in porous-elastic media. To validate our theoretical findings, we present numerical simulations illustrating the blow-up behavior, emphasizing the role of variable exponent damping in influencing system dynamics.

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1. Introduction

Swelling soils, also known as expansive soils, are characterized by an increase in volume when exposed to moisture. The clay minerals within these soils naturally attract and absorb water. As described succinctly by Handy [1], "when water is introduced to swelling soils, the water molecules are pulled into gaps between the soil plates. As more water is absorbed, the plates are forced further apart, leading to an increase in soil pore pressure." Consequently, swelling soils pose significant geotechnical and structural challenges, impacting both the environment and society, as illustrated in Figures (1)-(2).

Figure 1: Cracking in soil

Figure 2: Cracking in a building structure

Swelling soils are found worldwide. As reported by Nelson and Miller [2], the American Society of Civil Engineers estimates that one in four homes suffer damage caused by expansive soils. Typically, such soils result in greater financial losses for property owners than earthquakes, floods, hurricanes, and tornadoes combined. Therefore, it is crucial to explore effective methods to mitigate or eliminate the damage caused by swelling soils. Further theoretical background and details can be found in [3–5].

Although various chemical admixtures and other costly methods have been employed to mitigate swelling soil damage, these solutions are often ineffective in the long term. This research aims to stabilize swelling soils by utilizing damping mechanisms that are both effective and environmentally sustainable. Mathematically, swelling soil models consist of two coupled partial differential equations that describe the displacement of both the fluid and the elastic solid material.

To the best of our knowledge, the swelling soil system was first proposed by Ieş [6] and later simplified by Quintanilla [7]. The fundamental field equations for the linear theory of swelling soils are given by

$$\begin{cases} \rho_z z_{tt} = P_{1x} - G_1 + F_1, \\ \rho_u u_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1}$$

where the variables z and u represent the displacement of the fluid and the elastic solid material, respectively. The positive constants ρ_z and ρ_u denote the densities of each constituent. The terms (P_1, G_1, F_1) correspond to the partial tension, internal body forces, and external forces acting on the displacement, respectively. Similarly, (P_2, G_2, F_2) have analogous definitions for the elastic solid.

The constitutive equations of partial tensions are expressed as

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{\text{A}} \begin{bmatrix} z_x \\ u_x \end{bmatrix}, \tag{2}$$

where a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number. The matrix A is positive definite in the sense that $a_1 a_3 > a_2^2$. Quintanilla [7] investigated the system:

$$\begin{cases} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \xi(z_t - u_t) + a_3 z_{xxt}, \\ \rho_u u_{tt} = a_2 z_{xx} + a_3 u_{xx} + \xi(z_t - u_t), \end{cases} \tag{3}$$

where ξ is a positive coefficient. Under initial and homogeneous Dirichlet boundary conditions, he established an exponential stability result. Similarly, Wang and Guo [8] considered:

$$\begin{cases} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \rho_z \gamma(x) z_t, \\ \rho_u u_{tt} = a_2 z_{xx} + a_3 u_{xx}, \end{cases} \tag{4}$$

where $\gamma(x)$ represents an internal viscous damping function with a positive mean. Using spectral analysis, they established an exponential stability result.

Several recent studies have introduced new stabilization mechanisms for swelling soil models (1) that are effective, economical, and environmentally sustainable [9–12, 12]. In recent years, there has been growing interest in treating equations with variable exponent nonlinearities due to their applications in the mathematical modeling of non-Newtonian fluids. One prominent example is electro-rheological fluids, which can undergo drastic changes under the influence of external electromagnetic fields. In these cases, the variable exponent nonlinearity depends on physical parameters such as density, temperature, saturation, and electric field. For more results in this direction, we refer to [13–21].

Recently, Al-Mahdi et al. [22] established exponential and polynomial decay results for swelling soils governed by the system:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \alpha |z_t|^{m(\cdot)-2} z_t = \beta |z|^{m(\cdot)-2} z, & \text{in } (0,1) \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} = 0, & \text{in } (0,1) \times (0,\infty), \end{cases} \tag{5}$$

under suitable conditions on the variable exponents. However, an important question arises:

Does system (5) with $\gamma, \beta > 0$ admit global existence, stability, or blow-up results under certain conditions on the variable exponents?

In this paper, we address this question by considering the following swelling soil system of the form:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma |z_t|^{p(\cdot)-2} z_t = c|z|^{m(\cdot)-2} z, & \text{in } \Omega \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta |u_t|^{q(\cdot)-2} u_t = d|u|^{\ell(\cdot)-2} u, & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad z(x,0) = z_0(x), z_t(x,0) = z_1(x) & x \in \Omega, \\ z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0 & t \geq 0, \end{cases} \tag{6}$$

where Ω denotes the interval $(0,1)$. The positive constant coefficients ρ_u and ρ_z are the densities of each constituent. The coefficients a_1, a_2 and a_3 are positive constants satisfying specific conditions. The coefficients $\gamma, \beta, c, d > 0$, z_0, z_1, u_0, u_1 are given data and $p(.), q(.), m(.), \ell(.)$ are function satisfying some conditions to be specified in the next section.

System (6) exhibits several key differences from previously studied swelling porous-elastic models. Unlike classical models, where damping and source terms are typically governed by constant power-law exponents, our system introduces variable exponent functions $p(x)$, $q(x)$, $m(x)$, and $\ell(x)$. This formulation provides a more flexible and generalized framework for capturing energy dissipation and nonlinear interactions, accommodating spatial heterogeneity in material properties. As a result, our model is more adaptable to real-world applications, including soil swelling, biomechanics, and geomechanics.

The presence of variable exponent damping and source terms introduces new challenges in the global existence and the blow-up analysis. We establish conditions under which solutions exhibit finite-time blow-up, extending classical results from constant exponent cases to more general variable exponent settings.

These advancements mark a significant generalization of existing studies on swelling porous-elastic systems. By incorporating variable exponent nonlinearities, our work provides novel insights into the behavior of complex porous-elastic media under nonlinear stress conditions.

First, we establish the existence and uniqueness results of a weak solution and then we prove the global existence of the solutions under suitable assumptions on the variable exponents. Second, we show that the solutions with negative-initial energy blow-up in a finite time. Finally, we produce some numerical tests and examples to illustrate our blow-up results.

The interaction between the damping and the source terms was first considered by Levine [23, 24] in the linear damping case, $(m = 2)$, in the following wave equation

$$u_{tt} - \Delta u + a|u_t|^{m-2} u_t = b|u|^{p-2} u, \text{in } \Omega,\ t > 0, \tag{7}$$

where $a, b > 0$, $p > 2$, $m \geq 1$, Ω is a bounded domain in $\mathbb{R}^n$. He showed that solutions with negative initial energy blow up in finite time. Then, Georgiev and Todorova [25] extended Levine's result to the nonlinear damping case $(m > 2)$. In their work, the authors introduced a different method and determined suitable relations between m and p for which there is global existence or alternatively finite time blow up. More precisely: they showed that solutions with any initial data continue to exist globally (in time) if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [26] extended the blow up result of [25] to solutions with negative initial energy only. Messaoudi [27]considered the following nonlinear viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad x \in \Omega, \ t > 0, \tag{8}$$

where Ω is a bounded domain of $\mathbb{R}^n$. He proved that any weak solution with negative initial energy blows-up in finite time if $p > m$. Also the case of a stronger damping is considered and it is showed that solutions exist globally for any initial data, in the appropriate space, provided that $m \leq p$. For more results in blow-up, we refer the reader to see [28–32] and the references therein.

2. Preliminaries

In this section, we present some material needed in the proof of our results. Throughout this paper $\Omega = (0,1)$ and C is used to denote a generic positive constant.

- **(A1)**: $p, q, m, \ell : \overline{\Omega} \to [1,\infty)$ are measurable functions on Ω satisfying all the following conditions
  $2 \leq p_1 \leq p(x) \leq p_2 < m_1 \leq m(x) \leq m_2 < \infty$,
  $2 \leq q_1 \leq q(x) \leq q_2 < \ell_1 \leq \ell(x) \leq \ell_2 < \infty$, where

  $$p_1 := \text{essinf}_{x\in\Omega} p(x), \ p_2 := \text{esssup}_{x\in\Omega} p(x), \ m_1 := \text{essinf}_{x\in\Omega} m(x), \ m_2 := \text{esssup}_{x\in\Omega} m(x),$$

  $$q_1 := \text{essinf}_{x\in\Omega} q(x), \ q_2 := \text{esssup}_{x\in\Omega} q(x), \ \ell_1 := \text{essinf}_{x\in\Omega} \ell(x), \ \ell_2 := \text{esssup}_{x\in\Omega} \ell(x).$$

  and satisfy the log-Hölder continuity condition; that is for any δ with $0 < \delta < 1$, there exists a constant $A > 0$ such that,

  $$|f(x) - f(y)| \leq -\frac{A}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta. \tag{9}$$

- **(A2)**: The coefficients $a_i, \ i = 1, ..., 3$ satisfy $a_1 a_3 - a_2^2 > 0$.

3. Technical Lemmas

In this section, we present and establish some lemmas needed for the proof of our main results.

Lemma 1. *The energy of the problem* (6) *is defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega \left[\rho_z z_t^2 + \rho_u u_t^2 + a_3 u_x^2 + a_1 z_x^2 + 2a_2 z_x u_x\right] dx \\ & -c\int_\Omega \frac{|z|^{m(x)}}{m(x)}dx - d\int_\Omega \frac{|u|^{\ell(x)}}{\ell(x)}dx, \end{aligned} \tag{10}$$

and satisfies the following

$$E'(t) = -\gamma\int_\Omega |z_t|^{p(\cdot)}dx - \beta\int_\Omega |u_t|^{q(\cdot)}dx \leq 0. \tag{11}$$

Proof. Multiplying the equations in (6) by z_t and u_t respectively and then integrating over the interval $\Omega = (0,1)$ to get

$$\begin{aligned} &\rho_z\int_\Omega z_t z_{tt}dx - a_1\int_\Omega z_t z_{xx}dx - a_2\int_\Omega z_t u_{xx}dx + \gamma\int_\Omega z_t|z_t|^{p(\cdot)-2}z_t dx \\ &+ \rho_u\int_\Omega u_t u_{tt}dx - a_3\int_\Omega u_t u_{xx}dx - a_2\int_\Omega u_t z_{xx}dx + \beta\int_\Omega u_t|u_t|^{q(\cdot)-2}u_t dx \\ &= c\int_\Omega z_t|z|^{m(\cdot)-2}z dx + d\int_\Omega u_t|u|^{\ell(\cdot)-2}u dx. \end{aligned} \tag{12}$$

Using the integration by parts and the boundary conditions and summing up all the results, Eq. (12) becomes

$$\begin{aligned} &\rho_z\int_\Omega z_t z_{tt}dx + a_1\int_\Omega z_x z_{xt}dx + a_2\int_\Omega z_{xt}u_x dx + \gamma\int_\Omega z_t|z_t|^{p(\cdot)-2}z_t dx \\ &\rho_u\int_\Omega u_t u_{tt}dx + a_3\int_\Omega u_{xt}u_x dx + a_2\int_\Omega u_{xt}z_x dx + \beta\int_\Omega u_t|u_t|^{q(\cdot)-2}u_t dx \\ &= c\int_\Omega z_t|z|^{m(\cdot)-2}z dx + d\int_\Omega u_t|u|^{\ell(\cdot)-2}u dx. \end{aligned} \tag{13}$$

Using the following differential equations:

$$\begin{aligned} &\rho_z\int_\Omega z_t z_{tt}dx = \frac{\rho_z}{2}\frac{d}{dt}\int_\Omega z_t^2 dx, \quad \rho_u\int_\Omega u_t u_{tt}dx = \frac{\rho_u}{2}\frac{d}{dt}\int_\Omega u_t^2 dx, \\ &a_1\int_\Omega z_x z_{xt}dx = \frac{a_1}{2}\frac{d}{dt}\int_\Omega z_x^2 dx, \quad a_3\int_\Omega u_x u_{xt}dx = \frac{a_3}{2}\frac{d}{dt}\int_\Omega u_x^2 dx, \\ &a_2\int_\Omega (z_{xt}u_x + u_{xt}z_x)\, dx = a_2\frac{d}{dt}\int_\Omega u_x z_x dx, \\ &c\int_\Omega z_t|z|^{m(\cdot)-2}z dx = c\frac{d}{dt}\int_\Omega \frac{|z|^{m(x)}}{m(x)}dx, \text{ and } d\int_\Omega u_t|u|^{\ell(\cdot)-2}u dx = d\frac{d}{dt}\int_\Omega \frac{|u|^{\ell(x)}}{\ell(x)}dx. \end{aligned} \tag{14}$$

Combing (14) and (13), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\left[\rho_z z_t^2+\rho_u u_t^2+a_3u_x^2+a_1z_x^2+2a_2z_xu_x\right]dx-\frac{d}{dt}\left[c\int_{\Omega}\frac{|z|^{m(x)}}{m(x)}dx-d\int_{\Omega}\frac{|u|^{\ell(x)}}{\ell(x)}dx\right]$$
$$=-\gamma\int_{\Omega}|z_t|^{p(\cdot)}dx-\beta\int_{\Omega}|u_t|^{q(\cdot)}dx.$$

This gives

$$\frac{d}{dt}E(t)=E'(t)=-\gamma\int_{\Omega}|z_t|^{p(\cdot)}dx-\beta\int_{\Omega}|u_t|^{q(\cdot)}dx\leq 0,$$

where $E(t)$ is defined in (10) and this completes the proof of (11).

Lemma 2. *For any $z,u\in H_0^1(\Omega)$ and $p(\cdot),q(\cdot)$ satisfying **(A1)**, we have*

$$\begin{aligned}\int_{\Omega}|z|^{p(x)}dx&\leq C_e^{p_1}||z_x||_2^{p_1}+C_e^{p_2}||z_x||_2^{p_2}\\ \int_{\Omega}|u|^{q(x)}dx&\leq C_e^{q_1}||u_x||_2^{q_1}+C_e^{q_2}||u_x||_2^{q_2},\end{aligned}\tag{15}$$

where C_e is the embedding constant.

Proof. The proof of this lemma can be found in [33].

As in [33], we have the following:

$$\varrho(z):=\int_{\Omega}|z|^{m(x)}dx\geq C||z||_{m_1}^{m_1},\tag{16}$$

and

$$\varrho(u):=\int_{\Omega}|u|^{\ell(x)}dx\geq C||u||_{p_1}^{p_1}.\tag{17}$$

Lemma 3. *Assume that **(A1)** holds. Then, we have*

$$\begin{aligned}\int_{\Omega}|z|^{p(x)}dx&\leq C\left(\varrho(z)^{\frac{p_1}{m_1}}+\varrho(z)^{\frac{p_2}{m_1}}\right),\\ \int_{\Omega}|u|^{q(x)}dx&\leq C\left(\varrho(u)^{\frac{q_1}{\ell_1}}+\varrho(u)^{\frac{q_2}{\ell_1}}\right).\end{aligned}\tag{18}$$

Proof. The proof of this lemma can be found in [33].

4. Local Existence

In this section, we give a detailed proof of the local existence theorem by using the Faedo- Galerkin approximations and the Banach-Fixed-point theorem. We multiply the first equation in (6) by $\phi \in C_0^\infty(\Omega)$ and the second equation by $\psi \in C_0^\infty(\Omega)$, integrate each result over Ω, use Green's formula and the boundary conditions to obtain the following definition:

Definition 1. *Let $T > 0$. Any pair of functions*

$$z, u \in L^\infty([0,T), H_0^1(\Omega)), \qquad z_t \in L^\infty([0,T), L^2(\Omega)) \cap L^p(\Omega \times (0,T))$$

and $u_t \in L^\infty([0,T), L^2(\Omega)) \cap L^q(\Omega \times (0,T))$ is called a weak solution of system (6), if

$$\begin{cases} \frac{d}{dt}\int_\Omega \rho_z z_t \phi(x)dx + a_1 \int_\Omega z_x \phi_x(x)dx + a_2 \int_\Omega u_x \phi_x(x)dx \\ +\gamma \int_\Omega |z_t|^{p(.)-2} z_t \phi(x)dx = c\int_\Omega |z|^{m(.)-2} z\phi(x)dx \\ \frac{d}{dt}\int_\Omega \rho_u u_t \psi(x)dx + a_3 \int_\Omega u_x \psi_x(x)dx + a_2 \int_\Omega z_x \psi_x(x)dx \\ +\beta \int_\Omega |u_t|^{q(.)-2} u_t \psi(x)dx = d\int_\Omega |u|^{\ell(.)-2} u\psi(x)dx \\ z(0) = z_0,\ z_t(0) = z_1,\ u(0) = u_0,\ u_t(0) = u_1, \end{cases} \tag{19}$$

*for a.e. $t \in [0,T]$ and all test functions $\phi, \psi \in H_0^1(\Omega)$. Note that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. In addition, the spaces $H_0^1(\Omega) \subset L^{p(.)}(\Omega) \cap L^{q(.)}(\Omega)$, under the conditions **(A1)** and **(A2)**.*

Before establishing the existence theorem of a local weak solution of problem (6), we first consider, the following initial-boundary-value problem:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma |z_t|^{p(\cdot)-2} z_t = f(x,t), & \text{in } \Omega \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta |u_t|^{q(\cdot)-2} u_t = g(x,t), & \text{in } \Omega \times (0,\infty), \\ \\ z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0 & t \geq 0\ , \\ (z(0), u(0)) = (z_0, u_0), (z_t(0), u_t(0)) = (z_1, u_1), & \text{in } \Omega, \end{cases} \tag{Q}$$

where $f, g \in L^2(\Omega \times (0,T))$ and $(u_0,u_1), (v_0,v_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Theorem 1. *Assume that **(A1)** and **(A2)** hold and let $(u_0,u_1), (v_0,v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then problem (Q) has a unique local weak solution (u,v) on $[0,T)$.*

Proof. **UNIQUENESS:** Suppose that (Q) has two solutions (z_1,u_1) and (z_2,u_2). Then, $(z,u) = (z_1 - z_2, u_1 - u_2)$ satisfies, in the sense of distribution, the following problem:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma |z_{1t}|^{p(x)-2} z_{1t} - \gamma |z_{2t}|^{p(x)-2} z_{2t} = 0, & \text{in } \Omega \times (0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta |u_{1t}|^{q(x)-2} u_{1t} - \beta |u_{2t}|^{q(x)-2} u_{2t} = 0, & \text{in } \Omega \times (0,\infty), \\ \\ z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0 & t \geq 0\ , \\ (z(0), u(0)) = (z_0, u_0), (z_t(0), u_t(0)) = (z_1, u_1), & \text{in } \Omega. \end{cases}$$

Multiplying the first differential equation by z_t and the second by u_t and then integrating the result over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho_z}{2}||z_t||_2^2+\frac{\rho_u}{2}||u_t||_2^2+\frac{a_1}{2}||z_x||_2^2+\frac{a_3}{2}||u_x||_2^2+a_2\int_\Omega u_x z_x dx\right]\\&+\gamma\int_\Omega\left(|z_{1t}|^{p(x)-2}z_{1t}-|z_{2t}|^{p(x)-2}z_{2t}\right)(z_{1t}-z_{2t})dx\\&+\beta\int_\Omega\left(|u_{1t}|^{q(x)-2}u_{1t}-|u_{2t}|^{q(x)-2}u_{2t}\right)(u_{1t}-u_{2t})dx=0.\end{aligned}\tag{20}$$

Integrating (20) over $(0,t)$, to get

$$\begin{aligned}&\frac{\rho_z}{2}||z_t||_2^2+\frac{\rho_u}{2}||u_t||_2^2+\frac{a_1}{2}||z_x||_2^2+\frac{a_3}{2}||u_x||_2^2+a_2\int_\Omega u_x z_x dx\\&+\gamma\int_0^t\int_\Omega\left(|z_{1t}|^{p(x)-2}z_{1t}-|z_{2t}|^{p(x)-2}z_{2t}\right)(z_{1t}-z_{2t})dxds\\&+\beta\int_0^t\int_\Omega\left(|u_{1t}|^{q(x)-2}u_{1t}-|u_{2t}|^{q(x)-2}u_{2t}\right)(u_{1t}-u_{2t})dxds=0.\end{aligned}\tag{21}$$

By using the following inequality

$$\left[|Y|^{b(x)-2}Y-|Z|^{b(x)-2}Z\right](Y-Z)\geq 0,\ b(x)\geq 2,\tag{22}$$

for all $x\in\Omega$ and $Y,Z\in\mathbb{R}$, we have

$$\frac{\rho_z}{2}||z_t||_2^2+\frac{\rho_u}{2}||u_t||_2^2+\frac{a_1}{2}||z_x||_2^2+\frac{a_3}{2}||u_x||_2^2+a_2\int_\Omega u_x z_x dx\leq 0.\tag{23}$$

Applying the following Cauchy-Schwarz' inequality:

$$|\langle v_1,v_2\rangle|\leq\|v_1\|\|v_2\|,\quad\forall v_1,v_2\in L^2(0,1),\tag{24}$$

and the following Young inequality:

$$|ab|\leq\frac{1}{2}\left(\epsilon a^2+\frac{1}{\epsilon}b^2\right),\quad\forall a,b\in\mathbb{R},\ \forall\epsilon>0,\tag{25}$$

we see that, for any $\epsilon>0$,

$$a_3\|u_x\|^2+a_1\|z_x\|^2+2a_2\int_\Omega u_x z_x dx\geq\left(a_3-\frac{|a_2|}{\epsilon}\right)\|u_x\|^2+(a_1-|a_2|\epsilon)\|z_x\|^2,$$

by choosing $\epsilon=\frac{1}{2|a_0|}\left(a_2-a_1+\sqrt{(a_1-a_3)^2+4a_2^2}\right)$ (ϵ is well defined and positive, since $a_2\neq 0$), we obtain

$$a_3\|u_x\|^2+a_1\|z_x\|^2+2a_2\int_\Omega u_x z_x dx\geq\tilde{C}\left(\|u_x\|^2+\|z_x\|^2\right),\tag{26}$$

where $\tilde{C} := \frac{1}{2}\left(a_1 + a_3 - \sqrt{(a_1 - a_3)^2 + 4a_2^2}\right)$.
Combining (23) and (26), we find

$$\rho_z \left\|z_t\right\|_2^2 + a_1 \left\|z_x\right\|_2^2 = 0.$$

Similarly, we obtain

$$\rho_u \left\|u_t\right\|_2^2 + a_3 \left\|u_x\right\|_2^2 = 0.$$

Therefore, $z_t(x,.) = u_t(x,.) = 0$ on Ω and $u_x(.,t) = z_x(.,t) = 0,$ for a.e $t \in (0,T)$. This implies $u = z = 0$ on $\Omega \times (0,T)$, since $u = z = 0$ on $\partial\Omega \times (0,T)$. This proves the uniqueness.
EXISTENCE: The proof of the existence of a weak solution of (Q) consists of four steps:
Step 1. Approximate problem: In this step, we consider $\{w_j\}_{j=1}^{\infty}$ an orthogonal basis of $H_0^1(\Omega)$ and define, for all $k \geq 1$, (z^k, u^k) a sequence in the finite - dimensional subspace $(V_k \times V_k)$, where $V_k = span\{w_1, w_2, ..., w_k\}$, as follows:

$$z^k(x,t) = \sum_{j=1}^{k} a_j(t) w_j, \qquad u^k(x,t) = \sum_{j=1}^{k} b_j(t) w_j,$$

for all $x \in \Omega$ and $t \in (0,T)$ satisfying the following approximate problem:

$$\begin{cases} \rho_z \langle z_{tt}^k, w_j \rangle_{L^2(\Omega)} + a_1 \langle {z_x}^k, {w_j}_x \rangle_{L^2(\Omega)} + a_2 \langle u_x^k, {w_j}_x \rangle_{L^2(\Omega)} \\ + \gamma \langle |z_t^k|^{p(x)-2} z_t^k, w_j \rangle_{L^2(\Omega)} = \langle f(x,t), w_j \rangle_{L^2(\Omega)}, \quad j = 1,2,...,k, \\ \rho_u \langle u_{tt}^k, w_j \rangle_{L^2(\Omega)} + a_3 \langle {u_x}^k, {w_j}_x \rangle_{L^2(\Omega)} + a_2 \langle z_x^k, {w_j}_x \rangle_{L^2(\Omega)} \\ + \beta \langle |u_t^k|^{q(x)-2} u_t^k, w_j \rangle_{L^2(\Omega)} = \langle g(x,t), w_j \rangle_{L^2(\Omega)}, \quad j = 1,2,...,k, \\ z^k(0) = z_0^k, \; z_t^k(0) = z_1^k, \; u^k(0) = u_0^k, \; u_t^k(0) = u_1^k, \end{cases} \tag{27}$$

where $\langle \ , \ \rangle$ is the inner product in $L^2(\Omega)$ and

$$z_0^k = \sum_{i=1}^{k} \langle z_0, w_i \rangle w_i, u_0^k = \sum_{i=1}^{k} \langle u_0, w_i \rangle w_i, z_1^k = \sum_{i=1}^{k} \langle z_1, w_i \rangle w_i, u_1^k = \sum_{i=1}^{k} \langle u_1, w_i \rangle w_i.$$

By the Projection Theorem in Hilbert spaces, the approximated initial data $z_0^k, u_0^k, z_1^k, u_1^k$ are obtained via orthogonal projection onto the finite-dimensional subspace V_k, providing the best approximation in the corresponding norm. Since V_k is dense in $H_0^1(\Omega)$ and $L^2(\Omega)$, and due to the finite-dimensional approximation property, the projections converge strongly to the original initial data. Furthermore, the Banach–Alaoglu theorem guarantees weak compactness of bounded sequences, but in this case, weak convergence implies the following strong convergence,

$$\begin{cases} z_0^k \to z_0 \text{ and } u_0^k \to u_0 \text{ in } H_0^1(\Omega) \\ \text{and} \\ z_1^k \to z_1 \text{ and } u_1^k \to u_1 \text{in } L^2(\Omega). \end{cases} \tag{28}$$

Based on standard existence theory for ordinal differential equations, the system (27) admits a unique local solution (z^k, u^k) on a maximal time interval $[0, T_k)$, $0 < T_k < T$, for each $k \in \mathbb{N}$.

Step 2. A priori Estimates: In this step, we show, by a priory estimates, that $T_k = T$, for each $k \in \mathbb{N}$. We multiply the first equation by $a_j'(t)$ and the second equation by $b_j'(t)$ in (27), sum over $j = 1, 2, ...k$ and add the two equations to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\rho_z\|z_t^k\|_2^2 + \rho_u\|u_t^k\|_2^2 + a_1\|z_x^k\|_2^2 + a_3\|u_x^k\|_2^2 + 2a_2\int_\Omega u_x^k z_x^k dx\right]\\ &= -\gamma\int_\Omega |z_t^k(x,t)|^{p(.)}dx - \beta\int_\Omega |u_t^k(x,t)|^{q(.)}dx + \int_\Omega \left(z_t^k f(x,t) + u_t^k g(x,t)\right)dx.\end{aligned} \tag{29}$$

Integration of (29) over $(0, t)$ leads to

$$\begin{aligned}&\frac{1}{2}\left(\rho_z||z_t^k||_2^2 + \rho_u||u_t^k||_2^2 + a_1||z_x^k||_2^2 + a_3||u_x^k||_2^2 + 2a_2\int_\Omega u_x^k z_x^k dx\right)\\ &\quad + \gamma\int_0^t\int_\Omega |z_t^k(s)|^{p(.)}dxds + \beta\int_0^t\int_\Omega |u_t^k(s)|^{q(.)}dxds\\ &= \frac{1}{2}\left(\rho_z||z_1^k||_2^2 + \rho_u||u_1^k||_2^2 + \rho_z||z_{0x}^k||_2^2 + \rho_u||u_{0x}^k||_2^2 + 2a_2\int_\Omega z_{0x}^k u_{0x}^k dx\right)\\ &\quad + \int_0^t\int_\Omega \left(z_t^k f(x,t) + u_t^k g(x,t)\right)dxds, \text{ for all } t \le T_k.\end{aligned} \tag{30}$$

Using the identity (26) and Young's inequality on the last two terms, Eq. (30) becomes for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{1}{2}\left(\rho_z||z_t^k||_2^2 + \rho_u||u_t^k||_2^2 + \tilde{C}\left(\|u_x\|^2 + \|z_x\|^2\right)\right)\\ &\quad + \gamma\int_0^{T_k}\int_\Omega |z_t^k(s)|^{p(.)}dxds + \beta\int_0^{T_k}\int_\Omega |u_t^k(s)|^{q(.)}dxds\\ &\le \frac{1}{2}\left(\rho_z||z_1^k||_2^2 + \rho_u||u_1^k||_2^2 + \rho_z||z_{0x}^k||_2^2 + \rho_u||u_{0x}^k||_2^2\right) + c\left(||z_{0x}^k||_2^2 + ||u_{0x}^k||_2^2\right)\\ &\quad + \varepsilon\int_0^{T_k}\left(\rho_u\left\|u_t^k\right\|_2^2 + \rho_z\left\|z_t^k\right\|_2^2\right)ds + C_\varepsilon\int_0^{T_k}\int_\Omega \left(|f(x,t)|^2 + |g(x,t)|^2\right)dxds.\end{aligned} \tag{31}$$

Using (22) and recalling that $f, g \in L^2(\Omega \times (0, T))$, we have

$$z_0^k \longrightarrow z_0 \text{ and } u_0^k \longrightarrow u_0 \ \ in\ H_0^1(\Omega),$$

$$z_1^k \longrightarrow z_1 \text{ and } u_1^k \longrightarrow u_1 \ \ in\ L^2(\Omega)$$

and appling Gronwall's lemma, estimate (31) becomes, for some $C > 0$,

$$C\sup_{(0,T_k)}\left[||z_t^k||_2^2 + ||u_t^k||_2^2 + ||z_x^k||_2^2 + ||u_x^k||_2^2\right]$$

$$+\int_0^{T_k}\int_\Omega \left(\gamma \left|z_t^k(x,t)\right|^{p(x)} + \beta \left|u_t^k(x,t)\right|^{q(x)}\right) dxds$$
$$\leq C_\varepsilon + \varepsilon T \sup_{(0,T_k)} \left(\rho_u \left\|u_t^k\right\|_2^2 + \rho_z \left\|z_t^k\right\|_2^2\right) \ \forall T_k \leq T, k \geq 1.$$

Choosing $\varepsilon = \frac{1}{4T}$, we find

$$\sup_{(0,T_k)} \left[||z_t^k||_2^2 + ||u_t^k||_2^2 + ||z_x^k||_2^2 + ||u_x^k||_2^2\right] \leq C.$$

Therefore, the local solution (z^k, u^k) of system (27) can be extended to $(0,T)$, for all $k \geq 1$. Furthermore, we have

$$(z^k), (u^k) \text{ are bounded in } L^\infty((0,T), H_0^1(\Omega)),$$
$$(z_t^k) \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L^{p(.)}(\Omega \times (0,T)),$$
$$(u_t^k) \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L^{q(.)}(\Omega \times (0,T)).$$

Consequently, we have, up to two subsequences,

$$z^k \to z \text{ and } u^k \to u \text{ weakly * in } L^\infty((0,T), H_0^1(\Omega)),$$
$$z_t^k \to z_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{p(.)}(\Omega \times (0,T)),$$
$$u_t^k \to u_t \text{ weakly * in } L^\infty((0,T), L^2(\Omega)) \text{ and weakly in } L^{q(.)}(\Omega \times (0,T)).$$

Step 3. The Nonlinear terms: In this step, we show that

$$| z_t^k |^{p(.)-2} z_t^k \to | z_t |^{p(.)-2} z_t \text{ weakly in } L^{\frac{p(.)}{p(.)-1}}(\Omega \times (0,T)),$$
$$| u_t^k |^{q(.)-2} u_t^k \to | u_t |^{q(.)-2} u_t \text{ weakly in } L^{\frac{q(.)}{q(.)-1}}(\Omega \times (0,T))$$

and that (z,u) satisfies the partial differential equations of (Q) on $\Omega \times (0,T)$.
Since (z_t^k) is bounded in $L^{p(.)}(\Omega \times (0,T))$, then $(|z_t^k|^{p(.)-2} z_t^k)$ is bounded in $L^{\frac{p(.)}{p(.)-1}}(\Omega \times (0,T))$. Hence, up to a subsequence,

$$|z_t^k|^{p(.)-2} z_t^k \rightharpoonup \chi_1 \ \text{ in } L^{\frac{p(.)}{p(.)-1}}(\Omega \times (0,T)). \tag{32}$$

Similarly, we have

$$|u_t^k|^{q(.)-2} u_t^k \rightharpoonup \chi_2 \ \text{ in } L^{\frac{q(.)}{q(.)-1}}(\Omega \times (0,T)). \tag{33}$$

We can show that $\chi_1 = |z_t|^{p(.)-2}z_t$ and $\chi_2 = |u_t|^{q(.)-2}u_t$ by following the same steps as in [34]. Now, integrate (27) on $(0,t)$ to obtain $\forall j < k$,

$$\begin{aligned}
&\int_\Omega z_t^k w_j(x)dx - \int_\Omega z_1^k w_j(x)dx + a_1\int_0^t\int_\Omega z_x^k w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega u_x^k w_{j_x}(x)dxds \\
&\quad + \gamma\int_0^t\int_\Omega |z_t^k|^{p(.)-2}z_t^k w_j(x)dxds = \int_0^t\int_\Omega w_j f(x,t)dxds, \\
&\int_\Omega u_t^k w_j(x)dx - \int_\Omega u_1^k w_j(x)dx + a_3\int_0^t\int_\Omega u_x^k w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega z_x^k w_{j_x}(x)dx \\
&+ \beta\int_0^t\int_\Omega |u_t^k|^{q(.)-2}u_t^k w_j(x)dxds = \int_0^t\int_\Omega w_j g(x,t)dxds.
\end{aligned}$$

As k goes to $+\infty$, we easily check that $\forall j < k$,

$$\begin{aligned}
&\int_\Omega z_t w_j(x)dx - \int_\Omega z_1 w_j(x)dx + a_1\int_0^t\int_\Omega z_x w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega u_x w_{j_x}(x)dxds \\
&+ \gamma\int_0^t\int_\Omega |z_t|^{p(.)-2}z_t w_j(x)dxds = \int_0^t\int_\Omega w_j f(x,t)dxds, \\
&\int_\Omega u_t w_j(x)dx - \int_\Omega u_1 w_j(x)dx + a_3\int_0^t\int_\Omega u_x w_{j_x}(x)dxds + a_2\int_0^t\int_\Omega z_x w_{j_x}(x)dx \\
&+ \beta\int_0^t\int_\Omega |u_t|^{q(.)-2}u_t w_j(x)dxds = \int_0^t\int_\Omega w_j g(x,t)dxds.
\end{aligned}$$

Consequently, we have $\forall w \in H_0^1(\Omega)$

$$\begin{aligned}
&\int_\Omega z_t w(x)dx - \int_\Omega z_1 w(x)dx + a_1\int_0^t\int_\Omega z_x w_x(x)dxds + a_2\int_0^t\int_\Omega u_x w_x(x)dxds \\
&\quad + \gamma\int_0^t\int_\Omega |z_t|^{p(.)-2}z_t w(x)dxds = \int_0^t\int_\Omega w f(x,t)dxds, \\
&\int_\Omega u_t w(x)dx - \int_\Omega u_1 w(x)dx + a_3\int_0^t\int_\Omega u_x w_x(x)dxds + a_2\int_0^t\int_\Omega z_x w_x(x)dxds \\
&+ \beta\int_0^t\int_\Omega |u_t|^{q(.)-2}u_t w(x)dxds = \int_0^t\int_\Omega w g(x,t)dxds.
\end{aligned}$$

All terms define absolute continuous functions, so we get, for a.e. $t \in [0,T]$ and $\forall w \in$

$H_0^1(\Omega)$,

$$\int_\Omega z_{tt}w(x)dx + a_1\int_\Omega z_x w_x(x)dx + a_2\int_\Omega u_x w_x(x)dx + \gamma\int_\Omega |z_t|^{p(.)-2}z_t w(x)dx$$
$$= \int_\Omega wf(x,t)dx,$$
$$\int_\Omega u_{tt}w(x)dx + a_3\int_\Omega u_x w_x(x)dx + a_2\int_\Omega z_x w_x(x)dx + \beta\int_\Omega |u_t|^{q(.)-2}u_t w(x)dx$$
$$= \int_\Omega wg(x,t)dx.$$

This implies that

$$\rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma|z_t|^{p(\cdot)-2}z_t = f, \text{ in } D'(\Omega\times(0,T))$$
$$\rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta|u_t|^{q(\cdot)-2}u_t = g, \text{ in } D'(\Omega\times(0,T)).$$

This implies that (z,u) satisfies the two differential equations in (Q), on $\Omega\times(0,T)$.
Step 4. The Initial Conditions: We can handle the initial conditions like the one in [34]. Hence, we deduce that (z,u) is the unique local solution of (Q). This completes the proof of Theorem 1.

Now, we proceed to establish the local existence result for problem (P), we first recall the following elementary inequalities:

$$\left||a|^k - |b|^k\right| \le C\,|a-b|\left(|a|^{k-1} + |b|^{k-1}\right), \tag{34}$$

for some constant $C>0$, all $k\ge 1$ and all $a,b\in\mathbb{R}$. Also

$$\left||a|^{k_0}a - |b|^{k_0}b\right| \le C\,|a-b|\left(|a|^{k_0} + |b|^{k_0}\right), \tag{35}$$

for some constant $C>0$, all $k_0\ge 0$ and all $a,b\in\mathbb{R}$.

Remark 1. *For a.e. $x\in\Omega$ and $m(x)$ and $\ell(x)$ satisfying **(A1)**, the functions $h_1(s) = c|s|^{m(x)-2}$ and $h_2(s) = d|s|^{\ell(x)-2}$ are differentiable and $|h_1'(s)| = |c||m(x)-1||s|^{m(x)-2}$, $|h_2'(s)| = |d||\ell(x)-1||s|^{\ell(x)-2}$.*

Theorem 2. *Let $(u_0,u_1),(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$ be given. Assume that **(A1)**-**(A2)** hold. Then, problem (P) has a unique weak local solution (z,u) on $[0,T)$, in the sense of Definition 1, for some $T>0$.*

Proof. **EXISTENCE:** Let $v_1,v_2\in L^\infty\left([0,T),H_0^1(\Omega)\right)$. Using Lemma 15 and the embedding property, we have

$$\begin{aligned} ||h_1(v_1)||_2^2 &= \int_\Omega |v_1|^{2(m(x)-1)}dx \le |c|\left(\int_\Omega |v_1|^{2(m_2-1)}dx + \int_\Omega |v_1|^{2(m_1-1)}dx\right) < +\infty \\ ||h_2(v_2)||_2^2 &= \int_\Omega |v_2|^{2(\ell(x)-1)}dx \le |d|\left(\int_\Omega |v_2|^{2(\ell_2-1)}dx + \int_\Omega |v_2|^{2(\ell_1-1)}dx\right) < +\infty. \end{aligned} \tag{36}$$

Hence,

$$h_1(v_1), h_2(v_2) \in L^\infty([0,T), L^2(\Omega)) \subset L^2(\Omega \times (0,T)).$$

Therefore, for each $v_1, v_2 \in L^\infty([0,T), H_0^1(\Omega))$, there exists a unique solution

$$(z,u) \in L^\infty([0,T), H_0^1(\Omega)), \quad z_t \in L^\infty([0,T), L^2(\Omega)) \cap L^p(\Omega \times (0,T))$$

and $u_t \in L^\infty([0,T), L^2(\Omega)) \cap L^q(\Omega \times (0,T))$ satisfying the following nonlinear problem

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma \left|z_t\right|^{p(x)-2} z_t = h_1(v_1) & \text{in } \Omega \times (0,T)\,, \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta \left|u_t\right|^{q(x)-2} u_t = h_2(v_2) & \text{in } \Omega \times (0,T)\,, \\ u = z = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u\left(0\right) = u_0 \text{ and } u_t\left(0\right) = u_1 & \text{in } \Omega, \\ z\left(0\right) = z_0 \text{ and } z_t\left(0\right) = z_1 & \text{in } \Omega. \end{cases} \qquad (R)$$

Now, let

$$W_T = \left\{ w \in L^\infty((0,T), H_0^1(\Omega)) / w_t \in L^\infty((0,T), L^2(\Omega)) \right\},$$

and define the map $K : W_T \times W_T \longrightarrow W_T \times W_T$ by $K(v_1, v_2) = (z, u)$. We note that W_T is a Banach space with respect to the following norm

$$||w||_{W_T}^2 = \sup_{(0,T)} \int_\Omega |w_x|^2 dx + \sup_{(0,T)} \int_\Omega |w_t|^2 dx,$$

and K is well defined by virtue of Theorem 1. In what follows, we prove that K is a contraction mapping from a closed bounded ball $B(0,M)$ into itself, where

$$B(0,M) = \left\{ (v_1, v_2) \in W_T \times W_T / \left\|(v_1, v_2)\right\|_{W_T \times W_T} \leq M \right\},$$

for $M > 1$ and $T_0 > 0$ to be fixed later. Multiplying the first equation in (R) by z_t, the second one by u_t and integrating the two results over $\Omega \times (0,t)$ we get, for all $t \leq T$,

$$\begin{aligned} &\frac{\rho_u}{2}\left\|u_t\right\|_2^2 + \frac{\rho_z}{2}\left\|z_t\right\|_2^2 + \frac{a_3}{2}\left\|u_x\right\|_2^2 + \frac{a_1}{2}\left\|z_x\right\|_2^2 + a_2 \int_\Omega u_x z_x dx - \frac{\rho_u}{2}\left\|u_1\right\|_2^2 - \frac{\rho_z}{2}\left\|z_1\right\|_2^2 \\ &- \frac{\rho_u}{2}\left\|u_{0x}\right\|_2^2 - \frac{\rho_z}{2}\left\|z_{0x}\right\|_2^2 - a_2 \int_\Omega u_{0x} z_{0x} dx + \gamma \int_0^t \int_\Omega |z_t|^{p(x)} dxds + \beta \int_0^t \int_\Omega |u_t|^{q(x)} dxds \\ &= \int_0^t \int_\Omega z_t h_1(v_1) dxds + \int_0^t \int_\Omega u_t h_2(v_2) dxds. \end{aligned} \qquad (37)$$

Using the definitions of $h_i, i = 1,2$, Lemma 15 and Young's and Poincaré's inequalities, we have for $\varepsilon > 0$,

$$\begin{aligned} \int_\Omega |v_1|^{m(x)-2} v_1 z_t dx &\leq \frac{\varepsilon \rho_z}{4} \int_\Omega z_t^2 dx + \frac{C}{\varepsilon} \int_\Omega |v_1|^{2(m(x)-1)} dx \\ &\leq \frac{\varepsilon \rho_Z}{4} \int_\Omega z_t^2 dx + \frac{C_e}{\varepsilon} \left[||v_{1x}||_2^{2m_2-2} + ||v_{1x}||_2^{2m_1-2} \right]. \end{aligned} \qquad (38)$$

Similarly, we have

$$\begin{aligned}\int_\Omega |v_2|^{\ell(x)-2} v_2 u_t dx &\leq \frac{\varepsilon\rho_u}{4}\int_\Omega u_t^2 dx + \frac{C}{\varepsilon}\int_\Omega |v_2|^{2(\ell(x)-1)} dx \\ &\leq \frac{\varepsilon\rho_u}{4}\int_\Omega u_t^2 dx + \frac{C_e}{\varepsilon}\left[||v_{2x}||_2^{2\ell_2-2} + ||v_{2x}||_2^{2\ell_1-2}\right].\end{aligned} \tag{39}$$

Thus, (37) becomes

$$\begin{aligned}&\frac{\rho_u}{2}\|u_t\|_2^2 + \frac{\rho_z}{2}\|z_t\|_2^2 + \frac{a_3}{2}\|u_x\|_2^2 + \frac{a_1}{2}\|z_x\|_2^2 + a_2\int_\Omega u_x z_x dx \\ &\leq \lambda_0 + \frac{\varepsilon T\rho_z}{4}\sup_{(0,T)}\int_\Omega z_t^2 dx + \frac{\varepsilon T\rho_u}{4}\sup_{(0,T)}\int_\Omega u_t^2 dx \\ &+ \frac{C_e}{\varepsilon}\left[||v_{1x}||_2^{2m_1-2} + ||v_{1x}||_2^{2m_2-2} + ||v_{2x}||_2^{2\ell_2-2} + ||v_{2x}||_2^{2\ell_1-2}\right],\end{aligned}$$

where, by using (26),

$$\lambda_0 = \frac{\rho_u}{2}\|u_1\|_2^2 + \frac{\rho_z}{2}\|z_1\|_2^2 + \frac{\rho_u}{2}\|u_{0x}\|_2^2 + \frac{\rho_z}{2}\|z_{0x}\|_2^2 + a_2\int_\Omega u_{0x}z_{0x}dx \geq 0,$$

and C_e is the embedding constant. Choosing ε such that $\varepsilon T = 1$, we get

$$||z||_{W_T}^2 + ||u||_{W_T}^2 \leq \lambda_0 + TC\left(||v_1||_{W_T}^{2(m_1-1)} + ||v_1||_{W_T}^{2(m_2-1)} + ||v_2||_{W_T}^{2(\ell_1-1)} + ||v_2||_{W_T}^{2(\ell_2-1)}\right).$$

Suppose that $\max\{||v_1||_{W_T}, ||v_2||_{W_T}\} \leq M$, for some M large. Then, we have for large $M > 0$,

$$||u||_{W_T}^2 + ||z||_{W_T}^2 \leq \lambda_0 + TC\widetilde{M} \leq M^2,$$

where $\widetilde{M} = \max\{M^{2(m_2-1)}, M^{2(\ell_2-1)}\}$ and $M^2 > \lambda_0$ and $T \leq T_0 < \frac{M^2-\lambda_0}{C\widetilde{M}}$. Hence, we conclude that that K maps $B(0,M)$ into $B(0,M)$.

Next, we prove, for T_0(even smaller), K is a contraction. For this purpose, let $(z_1,u_1) = K(v_1,\tilde{v_1})$ and $(z_2,u_2) = K(v_2,\tilde{v_2})$ and set $(Z,U) = (v_1 - v_2, \tilde{v_1} - \tilde{v_2})$ then (Z,U) satisfies the following

$$\begin{cases}\rho_z Z_{tt} - a_1 Z_{xx} - a_2 U_{xx} + \gamma\left(|v_{1t}|^{p(\cdot)-2}v_{1t} - |v_{2t}|^{p(\cdot)-2}v_{2t}\right) = c\left(|v_1|^{m(\cdot)-2}v_1 - |v_2|^{m(\cdot)-2}v_2\right), \\ \rho_u U_{tt} - a_3 U_{xx} - a_2 Z_{xx} + \beta\left(|\tilde{v_{1t}}|^{q(\cdot)-2}\tilde{v_{1t}} - |\tilde{v_{2t}}|^{q(\cdot)-2}\tilde{v_{2t}}\right) = d\left(|\tilde{v_1}|^{\ell(\cdot)-2}\tilde{v_1} - |\tilde{v_2}|^{\ell(\cdot)-2}\tilde{v_2}\right), \\ \\ U(x,0) = U_0(x), U_t(x,0) = U_1(x), \quad Z(x,0) = Z_0(x), Z_t(x,0) = Z_1(x), \\ Z(0,t) = Z(1,t) = U(0,t) = U(1,t) = 0.\end{cases} \tag{40}$$

Multiplication the first equation by Z_t and the second by U_t, integration over $\Omega \times (0,t)$ and addition of the two equations yield

$$\begin{aligned}&\frac{\rho_u}{2}\|U_t\|_2^2+\frac{\rho_z}{2}\|Z_t\|_2^2+\frac{a_3}{2}\|U_x\|_2^2+\frac{a_1}{2}\|Z_x\|_2^2+a_2\int_\Omega U_xZ_xdx\\&+\gamma\int_0^t\int_\Omega\left(|v_{1t}|^{p(\cdot)-2}v_{1t}-|v_{2t}|^{p(\cdot)-2}v_{2t}\right)Z_tdxds+\beta\int_0^t\int_\Omega\left(|\tilde{v_{1t}}|^{q(\cdot)-2}\tilde{v_{1t}}-|\tilde{v_{2t}}|^{q(\cdot)-2}\tilde{v_{2t}}\right)U_tdxds\\&=\int_0^t\int_\Omega\left(h_1(v_1)-h_1(v_2)\right)Z_tdxds+\int_0^t\int_\Omega\left(h_2(\tilde{v_1})-h_2(\tilde{v_2})\right)U_tdxds.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{\rho_u}{2}\|U_t\|_2^2+\frac{\rho_z}{2}\|Z_t\|_2^2+\frac{a_3}{2}\|U_x\|_2^2+\frac{a_1}{2}\|Z_x\|_2^2+a_2\int_\Omega U_xZ_xdx\\&\leq\int_0^t\int_\Omega\left(h_1(v_2)-h_1(v_2)\right)Z_tdxds+\int_0^t\int_\Omega\left(h_2(\tilde{v_1})-h_2(\tilde{v_2})\right)U_tdxds.\end{aligned}\tag{41}$$

Now, we evaluate $I_1=\int_\Omega|h_1(v_1)-h_1(v_2)||Z_t|$ and $I_2=\int_\Omega|h_2(\tilde{v_1})-h_2(\tilde{v_2})||U_t|$.
Therefore, $I_1=\int_\Omega|h_1(v_1)-h_1(v_2)||Z_t|=\int_\Omega|h_1'(\xi)||v||Z_t|$,where $v=v_1-v_2$ and $\xi=\alpha v_1-(1-\alpha)v_2$, $0\leq\alpha\leq1$.
Applying Young's inequality, we get for any $\delta>0$ and some positive constant C,

$$\begin{aligned}I_1&\leq\frac{\delta}{2}\int_\Omega Z_t^2dx+\frac{2}{\delta}\int_\Omega|h_1'(\xi)|^2|v|^2dx\\&\leq\frac{\delta}{2}\int_\Omega Z_t^2dx+\frac{C}{\delta}\int_\Omega|\alpha v_1-(1-\alpha)v_2|^{2(m(x)-2)}|v|^2dx\\&\leq\frac{\delta}{2}\int_\Omega Z_t^2dx\\&+C_\delta\left(\int_\Omega|v|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\times\left[\left(\int_\Omega|\alpha v_1+(1-\alpha)v_2|^{n(m_2-2)}\right)^{\frac{2}{n}}+\left(\int_\Omega|\alpha v_1+(1-\alpha)v_2|^{n(m_1-2)}\right)^{\frac{2}{n}}\right].\end{aligned}$$

By recalling Lemma (15), we arrive at

$$\begin{aligned}I_1&\leq\frac{\delta}{2}\int_\Omega Z_t^2dx+C_\delta C_e\|v_x\|_2^2\left(\|v_{1x}\|_2^{2(m_2-2)}+\|v_{1x}\|_2^{2(m_1-2)}+\|v_{2x}\|_2^{2(m_2-2)}+\|v_{2x}\|_2^{2(m_1-2)}\right)\\&\leq\frac{\delta}{2}\int_\Omega Z_t^2dx+C_\delta C_e\widetilde{M}\|v_x\|_2^2.\end{aligned}$$

Similarly, we can show that

$$I_2\leq\frac{\delta}{2}\int_\Omega U_t^2dx+C_\delta C_e\widetilde{M}\|\tilde{v}_x\|_2^2,$$

where $\tilde{v}=\tilde{v_1}-\tilde{v_2}$. Therefore, (41) takes the form

$$\|(Z,U)\|_{W_T\times W_T}^2\leq\frac{\delta}{2}T_0C\|(Z,U)\|_{W_T\times W_T}^2+4C_\delta\widetilde{M}T_0C\|(v,\tilde{v})\|_{W_T\times W_T}^2.$$

Choosing δ small enough, we arrive at

$$\|(Z,U)\|^2_{W_T\times W_T} \leq CT_0\|(v,\tilde{v})\|^2_{W_T\times W_T}.$$

Taking T_0 small enough, we get, for some $0<k<1$,

$$\|(Z,U)\|_{W_T\times W_T} \leq k\|(v,\tilde{v})\|^2_{W_T\times W_T}.$$

Thus K is a contraction. The Banach fixed theorem implies the existence of a unique $(z,u)\in D(0,M)$, such that $K(z,u)=(z,u)$. Hence,, (z,u) is a weak solution of system (6). The uniqueness of this solution can be obtained by applying the energy method.

5. Global existence

In this section, we prove that system (6) has a global solution if $p\leq m$ and $q\leq \ell$.

Proposition 1. *Assume that $p\leq m$ and $q\leq\ell$. Then, system 6 admits a unique global solution.*

Proof. Similar to [25], we define the following

$$\begin{aligned}\mathcal{E}(t) &:= E(t)+2c\int_\Omega \frac{|z|^{m(x)}}{m(x)}dx+2d\int_\Omega\frac{|u|^{\ell(x)}}{\ell(x)}dx\\ &= \frac{1}{2}\int_\Omega\left[\rho_z z_t^2+\rho_u u_t^2+a_3u_x^2+a_1z_x^2+2a_2z_xu_x\right]dx\\ &\quad +c\int_\Omega\frac{|z|^{m(x)}}{m(x)}dx+d\int_\Omega\frac{|u|^{\ell(x)}}{\ell(x)}dx. \end{aligned} \tag{42}$$

Therefore,

$$\mathcal{E}'(t) = -\gamma\int_\Omega|z_t|^{p(\cdot)}dx-\beta\int_\Omega|u_t|^{q(\cdot)}dx+2c\int_\Omega|z|^{m(\cdot)-2}z_tdx+2d\int_\Omega|u|^{\ell(\cdot)-2}u_tdx.$$

By using Young's inequality, we obtain for any $\varepsilon,\delta>0$,

$$\begin{aligned}\mathcal{E}'(t) &\leq -\gamma\int_\Omega|z_t|^{p(\cdot)}dx-\beta\int_\Omega|u_t|^{q(\cdot)}dx && (43)\\ &\quad+\varepsilon\int_\Omega|z_t|^{m(\cdot)}dx+\delta\int_\Omega|u_t|^{\ell(\cdot)}dx && (44)\\ &\quad+\int_\Omega C_\varepsilon(x)|z|^{m(\cdot)}dx+\int_\Omega C_\delta(x)|u|^{\ell(\cdot)}dx. && (45)\end{aligned}$$

By noting that $p\leq m$ and $q\leq\ell$, we have

$$\mathcal{E}'(t) \leq -\gamma\int_\Omega|z_t|^{p(\cdot)}dx-\beta\int_\Omega|u_t|^{q(\cdot)}dx+C\varepsilon\int_\Omega|z_t|^{p(\cdot)}dx+C\delta\int_\Omega|u_t|^{q(\cdot)}dx \tag{46}$$

$$+\int_\Omega C_\varepsilon(x)|z|^{m(\cdot)}dx + \int_\Omega C_\delta(x)|u|^{\ell(\cdot)}dx. \tag{47}$$

Choosing ε and δ such that $\gamma - C\varepsilon > 0$ and $\beta - C\delta > 0$, we obtain

$$\mathcal{E}'(t) \leq C\left(m_2\int_\Omega \frac{|z|^{m(x)}}{m(x)}dx + \ell_2\int_\Omega \frac{|u|^{\ell(\cdot)}}{\ell(x)}dx\right) \leq C\mathcal{E}(t). \tag{48}$$

A simple integration gives

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{Ct}. \tag{49}$$

The last estimate together with the continuation principle completes our proof.

6. Blow-up

In this section, we show that the solution of system (6) blows up in a finite time. Our blow-up result reads as follows:

Theorem 3. *Assume that **(A1)** and **(A2)** hold and $E(0) < 0$. Then the solution of system 6 blows-up in a finite time.*

Proof. We set $H(t) := -E(t)$ then $H'(t) = -E'(t) \geq 0$ and then for every $t \in [0, T)$, we have

$$0 < H(0) \leq H(t) \leq \frac{c}{m_1}\int_\Omega |z|^{m(x)}dx + \frac{d}{\ell_1}\int_\Omega |u|^{\ell(x)}dx. \tag{50}$$

We then define

$$F(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega (\rho_z z z_t + \rho_u u u_t)dx, \tag{51}$$

for $0 < \alpha < 1$ and a positive number ε to be chosen later. By taking the derivative of F and using Eq. (6), we obtain

$$\begin{aligned}
F'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega \rho_z z_t^2 dx + \varepsilon\int_\Omega \rho_u u_t^2 dx - \varepsilon a_3\int_\Omega u_x^2 dx - \varepsilon a_1\int_\Omega z_x^2 dx \\
&\quad -2\varepsilon a_2\int_\Omega u_x z_x dx - \varepsilon\gamma\int_\Omega z|z_t|^{p(\cdot)-2}z_t dx - \varepsilon\beta\int_\Omega u|u_t|^{q(\cdot)-2}u_t dx \\
&\quad +\varepsilon c\int_\Omega |z|^{m(\cdot)}dx + \varepsilon c\int_\Omega |u|^{\ell(\cdot)}dx.
\end{aligned} \tag{52}$$

Adding and subtracting $\varepsilon(1-\theta)m_1\ell_1 H(t)$, for $0 < \theta < 1$, to the right-hand side of (52), we arrive at

$$F'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon(1-\theta)m_1\ell_1 H(t)$$

$$
\begin{aligned}
&+\varepsilon\rho_z\left(1+\frac{(1-\theta)m_1\ell_1}{2}\right)\int_\Omega z_t^2dx+\varepsilon\rho_u\left(1+\frac{(1-\theta)m_1\ell_1}{2}\right)\int_\Omega u_t^2dx\\
&-\varepsilon a_3\left(\frac{(1-\theta)m_1\ell_1}{2}-1\right)\int_\Omega u_x^2dx-\varepsilon a_1\left(\frac{(1-\theta)m_1\ell_1}{2}-1\right)\int_\Omega z_x^2dx\\
&-2\varepsilon a_2\left(\frac{(1-\theta)m_1\ell_1}{2}-1\right)\int_\Omega u_xz_xdx+\varepsilon c\ell_1\int_\Omega |z|^{m(\cdot)}dx+\varepsilon cm_1\int_\Omega |u|^{\ell(\cdot)}dx\\
&-\varepsilon\gamma\int_\Omega z|z_t|^{p(\cdot)-2}z_tdx-\varepsilon\beta\int_\Omega u|u_t|^{q(\cdot)-2}u_tdx. \qquad (53)
\end{aligned}
$$

For θ small enough, we have for some positive constant η

$$
\begin{aligned}
F'(t) \;\geq\;& (1-\alpha)H^{-\alpha}(t)H'(t)\\
&+\varepsilon\eta\left[H(t)+||u_t||_2^2+||z_t||_2^2+||u_x||_2^2+||z_x||_2^2+\int_\Omega |z|^{m(\cdot)}dx+\int_\Omega 1|u|^{\ell(\cdot)}dx\right]\\
&-\varepsilon\gamma\int_\Omega z|z_t|^{p(\cdot)-2}z_tdx-\varepsilon\beta\int_\Omega u|u_t|^{q(\cdot)-2}u_tdx, \qquad (54)
\end{aligned}
$$

Now, by using Young's inequality, we estimate the last term in (54) as follows:
For any $\sigma_1, \sigma_2 > 0$, we have

$$
\begin{aligned}
\int_\Omega |z||z_t|^{p(\cdot)-1}dx &\leq \frac{1}{p_1}\int_\Omega \sigma_1^{p(x)}|z|^{p(x)}dx+\frac{(p_2-1)}{p_2}\int_\Omega \sigma_1^{\frac{-p(x)}{p(x)-1}}|z_t|^{p(x)}dx,\\
\int_\Omega |u||u_t|^{q(\cdot)-1}dx &\leq \frac{1}{q_1}\int_\Omega \sigma_2^{q(x)}|u|^{q(x)}dx+\frac{(q_2-1)}{q_2}\int_\Omega \sigma_2^{\frac{-q(x)}{q(x)-1}}|u_t|^{q(x)}dx. \qquad (55)
\end{aligned}
$$

Therefore, by choosing σ_1 and σ_2 such that

$$
\sigma_1^{\frac{-p(x)}{p(x)-1}}=\xi_1H^{-\alpha}(t), \quad \sigma_2^{\frac{-p(x)}{p(x)-1}}=\xi_2H^{-\alpha}(t) \qquad (56)
$$

for sufficiently large constants ξ_i, $i=1,2$, to be specified later, and substituting these into (55), we obtain:

$$
\begin{aligned}
&\int_\Omega |z||z_t|^{p(\cdot)-1}dx+\int_\Omega |u||u_t|^{q(\cdot)-1}dx\\
&\leq \frac{1}{p_1}\int_\Omega \xi_1^{1-p(x)}|z|^{p(x)}H^{\alpha(p(x)-1)}dx+\frac{(p_2-1)}{cp_2}\xi_1H^{-\alpha}(t)H'(t)\\
&+\frac{1}{q_1}\int_\Omega \xi_2^{1-q(x)}|u|^{q(x)}H^{\alpha(q(x)-1)}dx+\frac{(q_2-1)}{dq_2}\xi_2H^{-\alpha}(t)H'(t). \qquad (57)
\end{aligned}
$$

Combining (54) and (57), we obtain

$$
F'(t) \;\geq\; \left[(1-\alpha)-\varepsilon\left(\frac{(p_2-1)}{p_2}\xi_1\right)-\varepsilon\left(\frac{(q_2-1)}{q_2}\xi_2\right)\right]H^{-\alpha}(t)H'(t)
$$

$$+\varepsilon\eta\left[H(t)+||u_t||_2^2+||z_t||_2^2+||u_x||_2^2+||z_x||_2^2+\int_\Omega |z|^{m(\cdot)}dx+\int_\Omega |u|^{\ell(\cdot)}dx\right]$$
$$-c\varepsilon\frac{\xi_1^{1-p_1}}{p_1}H^{\alpha(p_2(x)-1)}(t)\int_\Omega |z|^{p(x)}dx-d\varepsilon\frac{\xi_2^{1-q_1}}{q_1}H^{\alpha(q_2(x)-1)}(t)\int_\Omega |u|^{q(x)}dx. \quad (58)$$

Recalling (16) and (17), then Eq. (18) becomes

$$\begin{aligned}
&H^{\alpha(p_2(x)-1)}(t)\int_\Omega |z|^{p(x)}dx+H^{\alpha(q_2(x)-1)}(t)\int_\Omega |u|^{q(x)}dx\\
&\leq H^{\alpha(p_*(x)-1)}(t)\left[\int_\Omega |z|^{p(x)}dx+\int_\Omega |u|^{q(x)}dx\right]\\
&\leq C\left(\varrho(z)^{\frac{p_1}{m_1}+\alpha(p_*(x)-1)}+\varrho(z)^{\frac{p_2}{m_1}+\alpha(p_*(x)-1)}+\varrho(u)^{\frac{q_1}{\ell_1}+\alpha(p_*(x)-1)}+\varrho(u)^{\frac{q_2}{\ell_1}+\alpha(p_*(x)-1)}\right)\\
&\leq C\left(||z_x||_2^2+\varrho(z)+||u_x||_2^2+\varrho(u)\right),
\end{aligned}$$

where $p_*(x)=\max\{p_2(x),q_2(x)\}$.
Using (58), we arrive at

$$\begin{aligned}
F'(t) \;\geq\; &\left[(1-\alpha)-\varepsilon\left(\frac{(p_2-1)}{p_2}\xi_1\right)-\varepsilon\left(\frac{(q_2-1)}{q_2}\xi_2\right)\right]H^{-\alpha}(t)H'(t)\\
&+\varepsilon(\eta-\lambda)\left[H(t)+||u_t||_2^2+||z_t||_2^2+||u_x||_2^2+||z_x||_2^2+\varrho(z)+\varrho(u)\right], \quad (59)
\end{aligned}$$

$\lambda := c\frac{\xi^{1-p_1}}{p_1}+d\frac{\xi^{1-q_1}}{q_1}>0$ and $\xi=\max\{\xi_1,\xi_2\}$.
Now, we choose ξ large enough such that

$$\mu:=\eta-\lambda>0.$$

Then, we pick ε small enough so that

$$(1-\alpha)-\varepsilon\left[\left(\frac{(p_2-1)}{p_2}\xi_1\right)+\left(\frac{(q_2-1)}{q_2}\xi_2\right)\right]\geq 0,$$

and

$$F(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega(\rho_z z_0 z_1+\rho_u u_0 u_1)dx>0.$$

Using the last results, recalling (17) and (16), Eq. (60) becomes

$$\begin{aligned}
F'(t)&\geq \mu\varepsilon\left[H(t)+||u_t||_2^2+||z_t||_2^2+||u_x||_2^2+||z_x||_2^2+\varrho(z)+\varrho(u)\right]\\
&\geq \mu\varepsilon\left[H(t)+||u_t||_2^2+||z_t||_2^2+||u_x||_2^2+||z_x||_2^2+||z||_{m_1}^{m_1}+||u||_{\ell_1}^{\ell_1}\right]>0. \quad (60)
\end{aligned}$$

Hence, F is non-decreasing; that is, we have

$$F(t)\geq F(0),\ \forall\, t\geq 0. \quad (61)$$

Using (51) and (60), we have

$$F'(t) \geq \nu F^{\frac{1}{1-\alpha}}, \ \forall \ t \geq 0. \tag{62}$$

A simple integration of (62), we obtain

$$F^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{F^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\nu t \alpha}{1-\alpha}}, \tag{63}$$

where $0 < \alpha < 1$ and $\nu > 0$. Therefore, (63) shows that F blows-up in the finite time

$$T* \leq \frac{1-\alpha}{\nu\alpha \left[F(0)\right]^{\frac{\alpha}{1-\alpha}}}. \tag{64}$$

This completes the proof.

Remark 2. *In the context of swelling porous-elastic systems, blow-up represents the onset of mechanical failure, which can manifest physically as excessive deformation, material rupture, or cracking, depending on the specific application. When a solution blows up in finite time, it signifies that certain physical quantities, such as stress, strain, or displacement, become unbounded, indicating an irreversible breakdown of the material structure. Thus, blow-up in our model provides a mathematical framework for predicting critical thresholds beyond which the material loses stability, aiding in failure analysis and design optimization of porous-elastic structures.*

7. Numerical Tests

In the following section, we illustrate the blow up results proved in Theorems 6.1. We perform four numerical tests for the blow up behavior of a one-dimensional real-valued function. We discretize the system 1.6 using a second order finite difference method explicit in time and in space. For more stability, we combine the finite difference method with the conservative scheme of Lax-Wendroff. For more details, we refer to our previous works [12, 35]. We examine the following four tests. For these test, we define the parameters $a_1 = a_3 = 1$ and $a_2 = 0.95$ satisfying the conditions (**A2**). The used spatial-temporal domain is $[0,1]^2 \times [0,1]$:

- **TEST 1**: In the first test, we examine the blow up of the energy function using the nonlinear exponent function

$$p(x) = q(x) = 1 + \frac{1}{(1+x^2)} < m(x) = \ell(x) = 2 + \frac{2}{(1+x^2)},$$

which satisfies the condition (**A1**).

- **TEST 2**: In the second numerical test, we modify the inequality in the first test

$$p(x) = q(x) = m(x) = 1 + \frac{1}{(1+x^2)} < \ell(x) = 2 + \frac{2}{(1+x^2)},$$

which satisfies the condition (**A1**).

- **TEST 3**: Similarly, in the third numerical test, we use

$$p(x) = q(x) = \ell(x) = 1 + \frac{1}{(1+x^2)} < m(x) = 2 + \frac{2}{(1+x^2)},$$

which satisfies the condition (**A1**).

- **TEST 4**: In the fourth numerical test, we set the following equality of the non-linear exponent functions

$$p(x) = q(x) = m(x) = \ell(x) = 1 + \frac{1}{(1+x^2)},$$

which satisfies the condition (**A1**).

We run our code using the following initial solution:

$$u(x,0) = \sin(\pi x), \quad (65)$$

$$z(x,0) = 2x(x-1) \quad (66)$$

As mentioned in the system (1.6), we set $u_1(x,0) = u(x,,0)$.

We also have to mention that to visualize the blow up behavior of the initial solution (65), we use a very small and constant temporal step $\Delta t = 10^{-5}$ and the equidistant spatial step $\Delta x = \Delta y = 10^{-2}$.

In the left two column of the Figures 3-6, we plot the cross section of the evolution of the solution at different time steps $t = 0$, $t = 0.25$ and $t = 0.75$. In the right column of the Figures 3-6, we plot the blow up of the energies for the four tests.

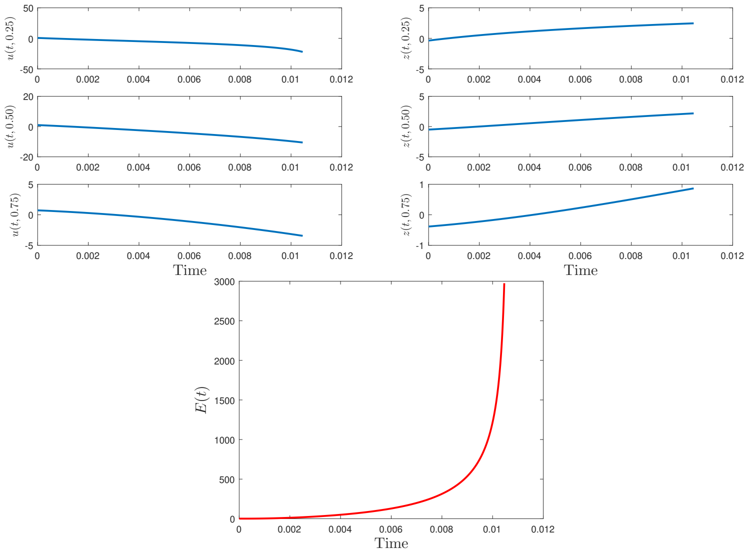

Figure 3: Test 1: Blow up under $p(x) < m(x)$ and $\ell(x) < m(x)$.

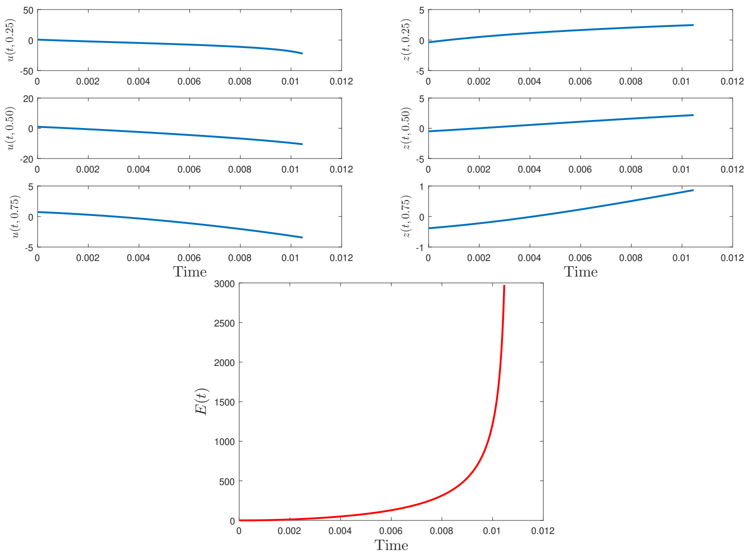

Figure 4: Test 2: Blow up under $p(x) = m(x)$ and $\ell(x) < m(x)$.

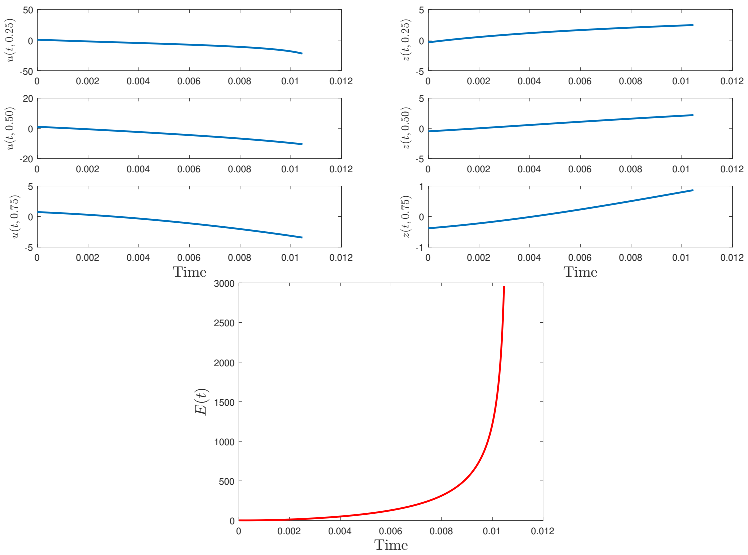

Figure 5: Test 3: Blow up under $p(x) < m(x)$ and $\ell(x) = m(x)$.

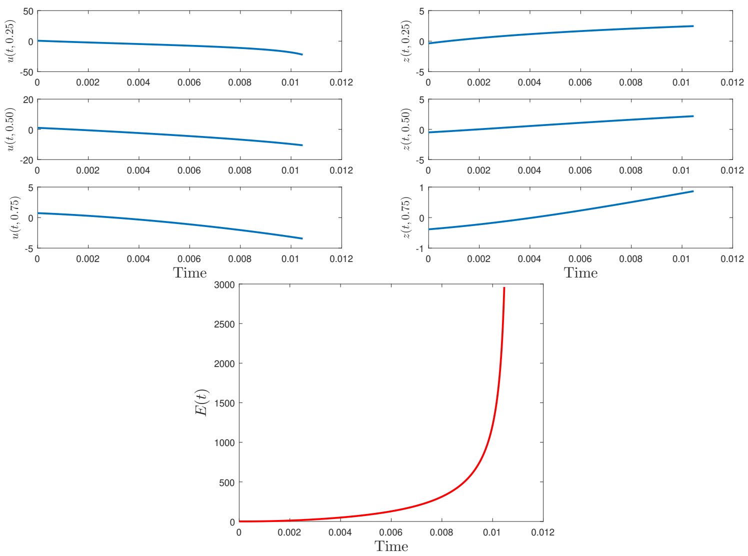

Figure 6: Test 4: Blow up under $p(x) = m(x)$ and $\ell(x) = m(x)$.

Finally, we remarked that even for a fine spatial and temporal discretization, the blow up occurs after a finite number of steps for all the Tests.

Conclusion

In this work, we investigated a swelling soil system incorporating two nonlinear damping and source terms of variable exponent-type. By employing the Faedo-Galerkin method and the Banach Contraction Theorem, we established the local existence and uniqueness of weak solutions under suitable conditions on the variable exponent functions. Furthermore, we demonstrated the global existence of solutions and identified conditions leading to finite-time blow-up. A key contribution of this study is the consideration of damping terms with variable exponents, which significantly generalizes classical models with constant exponent damping. This formulation allows for a more flexible and realistic representation of energy dissipation, capturing heterogeneous material properties and dynamic changes in the system. The presence of variable exponent damping plays a crucial role in influencing the stability and long-term behavior of solutions. Finally, we provided numerical simulations to illustrate the blow-up behavior, further validating our theoretical findings. While this study establishes significant results on the existence, uniqueness, and blow-up of solutions for swelling porous-elastic systems with variable exponent damping and source terms, several questions remain open for further investigation:

I. **Extension to Higher Dimensions and General Domains**
The current analysis is restricted to one-dimensional settings. Extending the results to higher-dimensional porous-elastic systems with variable exponent nonlinearity is a significant challenge that could lead to new theoretical developments.

II. **Impact of Additional Nonlinear Effects**
The current model does not include nonlocal effects, memory terms, or fractional diffusion. Introducing these effects could provide more accurate representations of real-world swelling porous media and lead to new mathematical difficulties in existence and blow-up analysis.

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Data Availability

No data were used to support this study.

Conflict of interest

The authors declare that there is no conflict of interest.

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