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On Certified Perfect Domination in Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple graph. A perfect dominating set $J \subseteq V(G)$ is called a *certified perfect dominating set* of G if each vertex $a \in J$ has either no neighbors or at least two neighbors in $V(G) \setminus J$. The *certified perfect domination number* of G $\gamma_{cerp}(G)$ represents the smallest size of a certified perfect dominating set in G. A certified perfect dominating set of G that attains this minimum size, i.e., $|J| = \gamma_{cerp}(G)$ is referred to as a γ_{cerp}-set. In this paper, we first present some upper bounds for the certified perfect domination number of G, investigate the relationship between certified domination and certified perfect domination parameters, and determine graphs with small and large values of these parameters. Secondly, we characterize the graphs with $\gamma_{cerp}(G) = n$ and $\gamma_{cerp}(G) = \gamma_{cer}(G)$. Finally, we characterize the certified perfect dominating set under the lexicographic and Cartesian products of two graphs, determine its certified perfect domination number, and identify a non-γ_{cerp}-graph under these binary operations.

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Key Words and Phrases: Certified domination, perfect domination, certified perfect domination, Corona, Lexicographic product

1. Introduction

In 2020, Dettlaff et al. [1, 2] introduced the concept of certified domination in graphs. They determined the exact values of the certified domination number for certain classes of graphs and provided upper bounds for this parameter in arbitrary graphs. Additionally, they characterized a broad class of graphs in which the domination number and the certified domination number are equal and identified graphs with large certified domination numbers. They also analyzed how the certified domination number changes when a graph is modified by adding or deleting an edge or a vertex. Moreover, they established Nordhaus–Gaddum type inequalities for the certified domination number and explored

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the relationships among domination, upper domination, certified domination, and upper certified domination numbers.

In 2023, Hamja [3] introduced the concept of certified perfect domination in graphs. He characterized the certified perfect dominating set, determined the exact values of the certified perfect domination number for specific graphs, and examined this parameter in graphs formed by the join and corona operations. Furthermore, he investigated the relationship between the perfect dominating set and the certified perfect dominating set of a graph G

Suppose we have a situation modeled by a graph G, where a subset $S \subseteq V(G)$ represents a group of officials, and the remaining vertices $H = V(G) \setminus S$ represent civilians. Each civilian $x \in H$ must be assigned exactly one official $u \in S$ who is responsible for serving them. Additionally, when an official u serves a civilian x, there must exist another civilian $y \in H$ who only observes the service provided by the official u to the civilian x. This observer, or witness y, ensures that the service is performed ethically and without any misconduct from the official. This setup prompts the question: What is the smallest number of officials needed to ensure every civilian is served in such a monitored manner within a given social structure? This question motivated Hamja [3] to define a new concept in graph theory, known as the certified perfect dominating set of a graph G.

In this paper, we extend the work of Hamja [3] by presenting new results related to certified perfect domination. We establish some upper bounds for the certified perfect domination number of a graph G and explore the relationship between certified domination and certified perfect domination parameters. Additionally, we identify classes of graphs with both small and large values of these parameters. Furthermore, we provide a characterization of graphs for which $\gamma_{cerp}(G) = n$ and $\gamma_{cerp}(G) = \gamma_{cer}(G)$. We also examine the behavior of certified perfect dominating sets under graph operations such as the lexicographic and Cartesian products. In doing so, we determine the certified perfect domination number for these products and identify non-γ_{cerp}-graphs arising from these operations. This work contributes to the ongoing development of domination parameters in graphs and offers new insights into their structural properties.

2. Terminology and Notation

For standard graph theory terminology, we follow the book *Graph Theory* by Diestel [4]. Let $G = (V, E)$ be a simple, connected graph, where $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set* of G. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v. The *maximum degree* of G, denoted by $\Delta(G)$, is the highest degree among all vertices of G, defined as $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$. The *open neighborhood* of a vertex u, denoted by $N_G(u)$, is the set of all vertices adjacent to u, formally given by $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of a vertex u, denoted by $N_G[u]$, is the set containing u along with all vertices adjacent to it, expressed as $N_G[u] = N_G(u) \cup \{u\}$. Similarly, the *closed neighborhood* of a subset $S \subseteq V(G)$ is defined as the union of the closed neighborhoods of all vertices in S, i.e., $N_G[S] = \bigcup_{v\in S} N_G[v]$.

A vertex of degree $|V(G)|-1$ is called a *universal vertex* of G. A vertex with degree one is referred to as a *leaf*, and its unique adjacent vertex is called its *support vertex* (or simply, its *support*). A support vertex that is adjacent to at least two leaves is known as a *strong support*, whereas one that is adjacent to only a single leaf is called a *weak support*. The set of all leaves in G is denoted by $L(G)$. For a leaf $v \in L(G)$, its support vertex is represented as $\mathcal{S}_L$, while for a weak support v, the unique leaf adjacent to v is denoted by $\mathcal{W}_L$. The set of all strong support vertices in G is denoted by $\mathcal{S}_S$, and the set of all weak support vertices is denoted by $\mathcal{W}_S$, as defined by Dettlaff et al. (see [1]).

A subset J of $V(G)$ is called a *dominating set* if every vertex in G is either in J or adjacent to at least one vertex in J, i.e., $N_G[J] = V(G)$. A dominating set J is said to be *minimal* if no proper subset $J' \subset J$ remains a dominating set. The *domination number* $\gamma(G)$ represents the smallest size of a dominating set in G. A dominating set J that attains this minimum size, i.e., $|J| = \gamma(G)$, is referred to as a γ-*set*, as defined by Haynes et al. (see [5]). A subset J of $V(G)$ is called a *certified dominating set* if each vertex $a \in J$ has either no neighbors or at least two neighbors in $V(G) \setminus J$. The *certified domination number* $\gamma_{cer}(G)$ represents the smallest size of a certified dominating set in G. A certified dominating set of G that attains this minimum size, i.e., $|J| = \gamma_{cerp}(G)$ is referred to as a γ_{cer}-set, as defined by Dettlaff et al. (see [1]). A subset J of $V(G)$ is called an *independent set* if for each pair of distinct vertices $a, b \in J$, $ab \notin E(G)$. The *independence number* $\alpha(G)$ represents the maximum size of an independent set of G. Additionally, the *independent domination number* $\gamma_i(G)$ represents the minimum size of an independent set of G, as defined by Paraic et al. (see[6]). A subset J of $V(G)$ is called a *perfect dominating set* if each vertex in $V(G) \setminus J$ is adjacent to exactly one vertex in J. The *perfect domination number* $\gamma_p(G)$ represents the smallest size of a perfect dominating set in G. A perfect dominating set of G that attains this minimum size , i.e, $|J| = \gamma_{cerp}(G)$ is referred to as a γ_p-set, as defined by Livingston et al. (see [7]).

A perfect dominating set $J \subseteq V(G)$ is called an *independent perfect dominating set* if no two vertices in J are adjacent. The *independent perfect domination number* $\gamma_{ip}(G)$ represents the smallest size of an independent perfect dominating set of G. An independent perfect dominating set of G that attains this minimum size, i.e., $|J| = \gamma_{ip}(G)$ is referred to as an γ_{ip}-set, as defined by Armada et al (see [8]). An independent perfect dominating set $J \subseteq V(G)$ is called a *certified independent perfect dominating set* if every $a \in J$ has either zero or at least two neighbors in $V(G) \setminus J$. The *certified independence perfect number* the maximum size of a certified independent perfect set of G. Additionally, the *certified independent perfect domination number* $\gamma_{cerip}(G)$ represents the minimum size of a certified independent perfect dominating set of G. A certified independent perfect dominating set of G that attains this minimum size, i.e., $|J| = \gamma_{cerip}(G)$ is referred to as a γ_{cerip}-set.

A perfect dominating set $J \subseteq V(G)$ is called a *certified perfect dominating set* of G

if each vertex $a \in J$ has either no neighbors or at least two neighbors in $V(G) \setminus J$. The *certified perfect domination number* of G $\gamma_{cerp}(G)$ represents the smallest size of a certified perfect dominating set in G. A certified perfect dominating set of G that attains this minimum size, i.e., $|J| = \gamma_{cep}(G)$ is referred to as a γ_{cerp}-set, as defined by Hamja (see [3]). It is worth noting that if a graph G has no certified perfect dominating set, then we say that G is a non-γ_{cerp}-graph.

3. Preliminary Results

This section presents the bounds, properties, and exact values for various graphs. We begin by providing some upper bounds for the certified perfect domination in graphs.

Proposition 1. [3] *For a path P_n of order $n \geq 1$,*

$$\gamma_{cerp}(P_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ n, & \text{otherwise.} \end{cases}$$

Proposition 2. [3] *For a cycle C_n of order $n \geq 3$,*

$$\gamma_{cerp}(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3, \\ n, & \text{otherwise.} \end{cases}$$

Proposition 3. [3] *For a complete graph K_n of order n,*

$$\gamma_{cerp}(K_n) = \begin{cases} 1, & \text{if } n = 1 \text{ or } n \geq 3, \\ 2, & \text{if } n = 2. \end{cases}$$

Proposition 4. [3] *For a complete bipartite graph $K_{m,n}$ of orders $m+n$,*

$$\gamma_{cerp}(K_{m,n}) = \begin{cases} 1, & \text{if } m = 1 \text{ or } n = 1, \\ 4, & \text{if } m = n = 2, \\ 2, & \text{otherwise.} \end{cases}$$

Proposition 5. *If G is a connected graph, then each support vertex of G is contained in every certified perfect dominating set of G.*

Proof. Let J be a certified perfect dominating set of G. Assume, for the sake of contradiction, that there exists a support vertex $a \in V(G)$ such that $a \notin J$. Since a is a support vertex, there must be at least one leaf $b \in V(G)$ with $N_G(b) = \{a\}$. As $a \notin J$, it follows that $b \in J$. However, this implies that b has exactly one neighbor in $V(G) \setminus J$, say a, contradicting the definition of a certified perfect dominating set. Thus, every support vertex of G is included in every certified perfect dominating set of G. □

Proposition 6. *Let k be a positive integer and G be a graph of order n. If the strong support vertices of G are collectively adjacent to k leaves, then*

$$\gamma_{cerp}(G) \leq n - k.$$

Moreover,

$$\gamma_{cerp}(G) \leq n - 2|\mathcal{S}_S|.$$

Proof. Let L_G be the set of all leaves adjacent to strong support vertices in G, where $|L_G| = k$. Consider the set $V(G) \setminus L_G$. Since each leaf in L_G is adjacent to exactly one strong support vertex, the set $V(G) \setminus L_G$ forms a certified perfect dominating set of G. Therefore, $\gamma_{cerp}(G) \leq |V(G) \setminus L_G| = n - k$.

Moreover, each strong support vertex in $\mathcal{S}_S$ is adjacent to at least two leaves, implying $k \geq 2|\mathcal{S}_S|$. Hence, it follows that $\gamma_{cerp}(G) \leq n - 2|\mathcal{S}_S|$. □

Theorem 1. *Let G be a graph. Then for every maximum certified independent perfect set J of G is a certified perfect dominating set of G. Moreover,*

$$\gamma_{cerp}(G) \leq \alpha_{cerp}(G).$$

Proof. Let G be a graph and J a maximum certified independent perfect set of G. Since J is an independent set, for any $a, b \in J$, we have $d_G(a, b) \neq 1$. Furthermore, because J is a certified independent perfect set, it follows that $d_G(a, b) \neq 2$ for all $a, b \in D$. Now, let $y \in V(G) \setminus J$. Since J is a maximum-size certified independent perfect set, y must be dominated by at least one vertex in J. Hence, there exists some $z \in J$ such that $y \in N_G(z)$. Therefore, $N_G[J] = V(G)$, which implies that J is a dominating set of G. Additionally, since J is an independent perfect set, each vertex in $V(G) \setminus J$ is dominated by exactly one vertex in J. Thus, J is an independent perfect dominating set of G. Given that J is also a certified independent perfect set, it follows that J is a certified perfect dominating set of G. Let J be a maximum certified perfect independent set of G. Therefore, $|J| = \alpha_{cerp}(G)$. Since J is also a certified perfect dominating set, we have

$$\gamma_{cerp}(G) \leq \alpha_{cerp}(G).$$

This completes the proof. □

The next result shows the relationship between certified domination and certified perfect domination parameters.

Remark 1. *Every certified perfect dominating set of G is also a certified dominating set of G. Hence, $\gamma_{cer}(G) \leq \gamma_{cerp}(G)$.*

Theorem 2. *Let a and b be positive integers with $3 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{cer}(G) = a$ and $\gamma_{cerp}(G) = b$.*

Proof. For $a = b$, consider $G = K_a$. In this case, $\gamma_{cer}(G) = a$ and $\gamma_{cerp}(G) = a$, so $\gamma_{cer}(G) = \gamma_{cerp}(G)$.

For $a < b$, we examine the following cases:

Case 1. a is odd.

Let $m = b - a$, and consider the graph G as illustrated in Figure 1. Define $J_1 = \{x_1, x_2, \dots, x_a\}$ and $J_2 = \{x_1, x_2, \dots, x_a, y_1, y_2, \dots, y_m\}$. Then, J_1 and J_2 are the γ_{cer}-set and γ_{cerp}-set of G, respectively. Therefore, we have $\gamma_{cer}(G) = |J_1| = a$ and $\gamma_{cerp}(G) = |J_2| = a + m = b$.

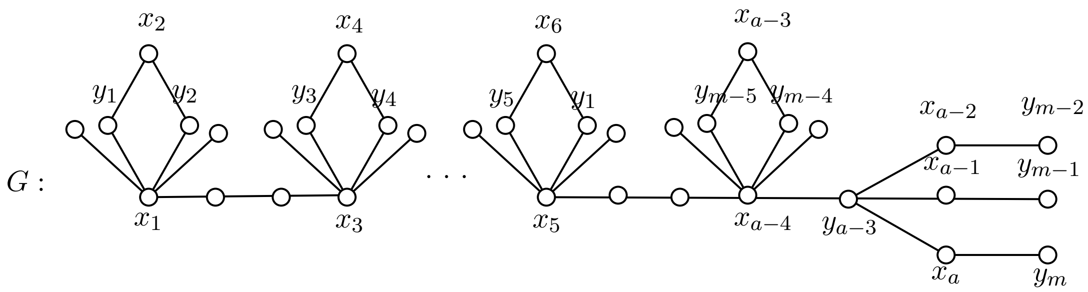

Figure 1: A graph G with $\gamma_{cer}(G) < \gamma_{cerp}(G)$.

Case 2. a is even.

Let $m = b - a$, and consider the graph G' as shown in Figure 2. Define $J'_1 = \{x_1, x_2, \dots, x_a\}$ and $J'_2 = \{x_1, x_2, \dots, x_a, y_1, y_2, \dots, y_m\}$. Then, J'_1 and J'_2 are the γ_{cer}-set and γ_{cerp}-set of G, respectively. Consequently, $\gamma_{cer}(G) = |J'_1| = a$ and $\gamma_{cerp}(G) = |J'_2| = a + m = b$.

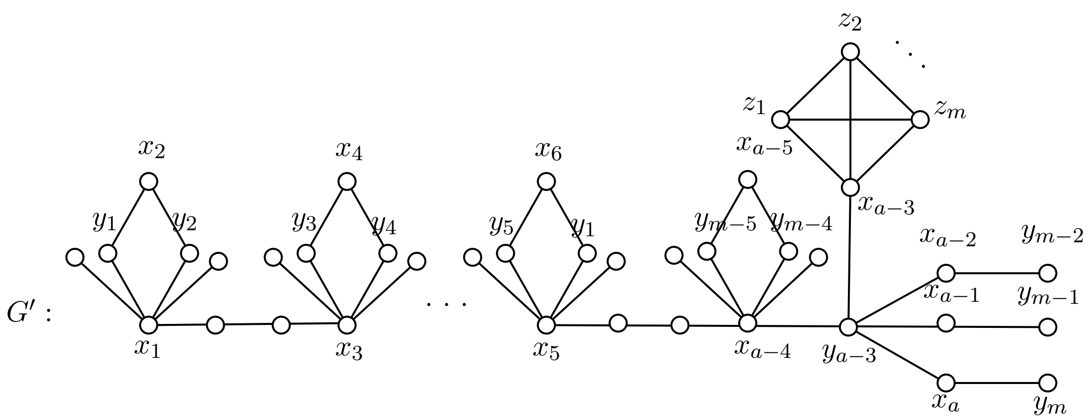

Figure 2: A graph G' with $\gamma_{cer}(G') < \gamma_{cerp}(G')$.

This completes the proof. ∎

The following result is a direct consequence of Theorem 2.

Corollary 1. *For any non-negative integer c, there exists a connected graph G such that* $\gamma_{cerp}(G) - \gamma_{cer}(G) = c$*. In other words, the difference between* $\gamma_{cerp}(G)$ *and* $\gamma_{cer}(G)$ *can be made arbitrarily large.*

We will now determine the small and large values of graph G.

Theorem 3. *Let G be a graph with order $n \geq 1$. Then the following hold:*

(i) $1 \leq \gamma_{cerp}(G) \leq n$;

(ii) $\gamma_{cerp}(G) = 1$ *if and only if* $\gamma_{cer}(G) = 1$;

(iii) *If* $\gamma_{cerp}(G) = 2$*, then* $\gamma_{cer}(G) = 2$*. However, the reverse is not necessarily true.*

(iv) *If* $\gamma_{cer}(G) = n$*, then* $\gamma_{cerp}(G) = n$*. However, the reverse is not necessarily true.*

Proof. (i) Let G be a graph with $|V(G)| = n$. Since the empty set is not a certified dominating set of G, we conclude that $\gamma_{cer}(G) \geq 1$. Additionally, note that any certified perfect dominating set J is a subset of the vertex set of G, i.e., $J \subseteq V(G)$. Thus, we have $\gamma_{cerp}(G) \leq |V(G)| = n$. Therefore, $1 \leq \gamma_{cerp}(G) \leq n$.

(ii) Assume that $\gamma_{cerp}(G) = 1$. Then $J = \{x\}$ is the minimum certified perfect dominating set of G. This implies that J is also a minimum certified dominating set of G, so $\gamma_{cer}(G) = |J| = 1$. Conversely, suppose $\gamma_{cer}(G) = 1$. Then $J' = \{x'\}$. For every $y' \in V(G) \setminus J'$, the only vertex dominating y' is $x' \in J'$. Hence, J' is a perfect dominating set of G, which implies $\gamma_{cerp}(G) = |J'| = 1$.

(iii) Assume that $\gamma_{cerp}(G) = 2$. Then $J = \{x, y\}$ is the minimum certified perfect dominating set of G. Since every certified perfect dominating set is also a certified dominating set, we conclude that $\gamma_{cer}(G) = 2$. However, the reverse does not always hold. For example, consider $G = C_5$. In this case, $\gamma_{cer}(C_5) = 2$, but $\gamma_{cerp}(C_5) = 5$.

(iv) Suppose $\gamma_{cer}(G) = n$. Since $\gamma_{cerp}(G) \geq \gamma_{cer}(G)$ and $\gamma_{cerp}(G) \leq n$, it follows that $\gamma_{cerp}(G) = n$. To show that the reverse is not necessarily true, consider $G = P_5$. In this case, $\gamma_{cerp}(P_5) = 5$, but $\gamma_{cer}(P_5) = 2$.

In what follows the following result will be used.

Theorem 4. [3] *Let G be a graph consisting of components $G_1, G_2, \ldots, G_k$, where $k \geq 2$. Then*

$$\gamma_{cerp}(G) = \sum_{i=1}^{k} \gamma_{cerp}(G_i).$$

Proposition 7. *Let $G = \overline{K}_n$ be the complement of a complete graph of order n. Then*

$$\gamma_{cerp}(\overline{K}_n) = n.$$

Proof. Let $G = \overline{K_n}$ be the complement of a complete graph of order n. Then G consists of n isolated vertices. Since every vertex is an isolated, each component is a trivial graph K_1. From [3], we know that $\gamma_{\text{cerp}}(K_1) = 1$ for each component. Since G has exactly n such components, Theorem 4 implies that

$$\gamma_{cerp}(G) = \sum_{i=1}^{n} \gamma_{cerp}(K_1) = n.$$

Thus, we conclude that $\gamma_{cerp}(G) = \gamma_{cerp}(\overline{K}_n) = n$. □

As previously noted, for any graph G with order n, we have $\gamma_{cerp}(G) \leq n$ and $\gamma_{cerp}(G) \neq n-1$. Furthermore, there exist graphs such as a path P_n, a cycle C_n for $n \not\equiv 0 \pmod 3$ (see [3]), the complement of a complete graph on n vertices, where $\gamma_{cerp}(G) = n$. This motivates the investigation into the characterization of all graphs for which $\gamma_{cerp}(G) = n$, which is addressed in this section.

Recall that F. Harary (see [9]) defined the *corona* of two graphs G and H, denoted by $G \circ H$, is constructed by taking one copy of G and $|V(G)|$ copies of H, then connecting each vertex i of G to every vertex in the corresponding ith copy of H. For each $v \in V(G)$, let H^v represent the copy of H whose vertices are individually linked to v. The subgraph of the corona $G \circ H$ associated with the join $\langle\{v\}\rangle + H^v$ is denoted by $v + H^v$, where $v \in V(G)$ [9]. In particular, when H is a single-vertex graph, $H = K_1$, the corona $G \circ K_1$ is referred to as the corona of G.

Theorem 5. [3] *If G is a connected graph with $|V(G)| = m$, and H is a trivial graph, then $\gamma_{cerp}(G \circ H) = 2m$.*

Corollary 2. *If G is the corona of a graph with $|V(G)| = n$, then $\gamma_{cerp}(G) = n$.*

Proof. Let D be a γ_{cerp}-set of G. Let G be the corona of some graph. Then $G = H \circ k_1$ for some graph H. By Theorem 5,

$$|D| = \gamma_{cerp}(G) = \gamma_{cerp}(H \circ K_1) = 2m = |V(G)| = n.$$

Therefore, $\gamma_{cerp}(G) = n$.

Remark 2. *Let G be a graph with $|V(G)| = n$. If G is either the corona of a graph, the complement of a complete graph, or the union of these two types, then $\gamma_{cerp}(G) = n$.*

Remark 3. *It is important to observe that the result above indicates the sharpness of the upper bound in the inequality $\gamma_{cerp}(G) \leq 2\gamma(G)$, because for the corona G of any graph that does not contain an isolated vertex, we have $\gamma_{cerp}(G) = 2\gamma(G)$.*

4. Graphs with $\gamma_{cerp}(G) = \gamma_{cer}(G)$

We proceed with our investigation of the certified perfect domination number by examining the class of graphs where $\gamma_{cerp}(G) = \gamma_{cer}(G)$.

Theorem 6. *Let G be a connected graph with $|V(G)| \geq 3$. Then $\gamma_{cer}(G) = \gamma_{cerp}(G)$ if and only if there exists a γ_{cer}-set J of G such that each vertex $u \in V(G) \setminus J$ is dominated by exactly one vertex $v \in J$.*

Proof. Assume that $\gamma_{cer}(G) = \gamma_{cerp}(G)$. Let J be a γ_{cerp}-set of G. Since J is a γ_{cerp}-set and $\gamma(G) = \gamma_{cer}(G)$, it follows that J is also a γ_{cer}-set of G. Now, for the sake of contradiction, assume that there exists a vertex $u \in V(G) \setminus J$ that is dominated by at least two distinct vertices $v_1, v_2 \in J$. In this case, J would not be a certified perfect dominating set, which contradicts our assumption that J is a γ_{cerp}-set. Therefore, every vertex $u \in V(G) \setminus J$ is dominated by exactly one vertex in J.

Conversely, suppose that J is a γ_{cer}-set of G such that each vertex $u \in V(G) \setminus J$ is dominated by exactly one vertex in J. For contradiction, assume that there exists a vertex $v \in J$ such that $|N_G(v) \cap (V(G) \setminus J)| = 1$. This implies that the unique neighbor of v in $V(G) \setminus J$ contradicts our assumption that every vertex in $V(G) \setminus J$ is dominated by exactly one vertex in J. Hence, we conclude that $|N_G(v) \cap (V(G) \setminus J)| \geq 2$ for all $v \in J$, meaning that J is a certified perfect dominating set.

Therefore, $\gamma_{cerp}(G) \leq |J| = \gamma_{cer}(G)$. By Remark 1, we also have $\gamma_{cer}(G) \leq \gamma_{cerp}(G)$. Thus, $\gamma_{cer}(G) = \gamma_{cerp}(G)$. □

Corollary 3. *Let G be a connected graph with $|V(G)| \geq 3$. If G has a γ_{ip}-set that does not include any leaf of G, then $\gamma_{cer}(G) = \gamma_{cerp}(G)$.*

Proof. Let J be a γ_{ip}-set of G that does not contain any leaf of G. Since J is independent, every vertex $v \in J$ has all of its neighbors in $V(G) \setminus J$. This implies that $N_G(v) \subseteq V(G) \setminus J$. Furthermore, since v is neither a leaf nor an isolated vertex, we have $|N_G(v)| \geq 2$. Thus, $|N_G(v) \cap (V(G) \setminus J)| = |N_G(v)| \geq 2$. Therefore, J is a certified perfect dominating set. By Theorem 6, we conclude that $\gamma_{cer}(G) = \gamma_{cerp}(G)$. □

The following result immediately follows from Theorem 6 and Corollary 3.

Corollary 4. *Let G be a graph where $\delta(G) \geq 2$. If G contains a γ_p-set J such that each vertex in J has at least two neighbors in $V(G) \setminus J$, then $\gamma_{cerp}(G) = \gamma_{cer}(G)$.*

5. Lexicographic Product of Two Graphs

In this section, we characterize the lexicographic product of two graphs and determine its certified perfect domination number.

Recall that F. Harary (see [9]) defined the *Lexicographic product* or *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following condition:

$$(x, u)(y, v) \in E(G[H]) \text{ if and only if } xy \in E(G) \text{ or } x = y \text{ and } uv \in E(H).$$

Any subset C of $V(G[H])$ can be expressed as $Q = \bigcup_{x \in S} \{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 7. *Let G and H be connected nontrivial graphs. A subset $Q = \bigcup_{u\in J}\left[\{u\} \times T_u\right]$ of $V(G[H])$, where $J \subseteq V(G)$ and $T_u \subseteq V(H)$ for each $u \in J$, is a certified perfect dominating set of $G[H]$ if and only if J is a certified independent perfect dominating set of G and T_u is a dominating set of H satisfying $|T_x| = 1$ for all $u \in J$.*

Proof. Suppose that Q is a certified perfect dominating set of $G[H]$. Let $a \in V(G) \setminus J$ and select any $b \in V(H)$. Since Q is a perfect dominating set, there exists a vertex $(y, z) \in Q$ such that $N_{G[H]}(a, b) \cap Q = \{(y, z)\}$. This implies that $N_G(a) \cap J = \{y\}$, meaning that each vertex $a \in V(G) \setminus J$ is dominated by exactly one vertex in J. Therefore, J is a perfect dominating set in G. Next, assume that there exist vertices $r, s \in J$ such that $rs \in E(G)$. Since H is connected, there exist vertices $k, l \in V(H)$ such that $kl \in E(H)$, and consequently, $(r, k)(s, k) \in E(G[H])$ with both (r, k) and (s, k) belonging to Q. This leads to a contradiction, as both (r, l) and (s, l) are dominated by (r, k) and (s, k) in Q. Thus, $rs \notin E(G)$ for any $r, s \in J$, meaning that J is an independent set. Therefore, J is an independent perfect dominating set in G, and hence a certified independent perfect dominating set of G since $N_G(J) \geq 2$. Now, let $u \in J$, and suppose that $|T_u| \geq 2$. Choose distinct $c, d \in T_u$, so that both (u, f) and (u, g) belong to Q. Consider an adjacent vertex $h \in V(G)$ such that $uh \in E(G)$. Since J is independent, we have $h \notin J$. Clearly, $(h, f) \notin Q$, and it must be dominated by both (u, f) and (u, g) in Q, which contradicts the assumption that Q is a perfect dominating set. Therefore, $|T_u| = 1$ for each $u \in J$. Furthermore, since J is an independent perfect dominating set of G and Q is a dominating set in $G[H]$, it follows that each T_u is a dominating set in H, with $|T_u| = 1$ for each $u \in J$.

Conversely, suppose that J is a certified independent perfect dominating set in G and that T_u is a dominating set of H with $|T_u| = 1$ for all $u \in J$. Let $T_u = \{w\}$ for each $u \in J$, where w is the vertex in H that dominates all other vertices of H.

Consider an arbitrary vertex $(r, s) \in V(G[H]) \setminus Q$, and consider the following cases:

Case 1: $r \notin J$. Since J is a perfect dominating set, there exists $u \in J$ such that $N_G(r) \cap J = \{u\}$. Hence, $N_{G[H]}(r, s) \cap Q = \{(u, w)\}$. Therefore, (r, s) is dominated by exactly one vertex in Q, and thus Q is a perfect dominating set.

Case 2: $r \in J$. Since $T_u = \{w\}$ for each $u \in J$ and J is a perfect dominating set, we have $N_{G[H]}(r, s) \cap Q = \{(r, w)\}$, meaning that (r, s) is dominated by exactly one vertex (r, w) in Q. Therefore, Q is a perfect dominating set.

Finally, since J is a certified independent perfect dominating set in G and T_u is a dominating set of H with $|T_u| = 1$ for all $u \in J$, it follows that Q is a certified perfect dominating set of $G[H]$. This completes the proof. □

The following results immediately follow from Theorem 7.

Corollary 5. *If G and H are connected graphs. Then*

$$\gamma_{cerp}(G[H]) = \begin{cases} \gamma_{cerp}(G), & \text{if } H = K_1, \\ \gamma_{cerp}(H), & \text{if } G = K_1, \\ \gamma_{cerp}(G), & \text{if } G \text{ contains } \gamma_{cerip} - \text{set and } \gamma(H) = 1. \end{cases}$$

Corollary 6. *Let G and H be connected graphs. Then $G[H]$ is a non-γ_{cerp}-graph if and only if one of the following conditions holds:*

(i) $\gamma(H) \geq 2$

(ii) $\gamma(H) = 1$ *and G does not possess a certified independent perfect dominating set.*

The following result is a direct consequence of Corollary 6 (ii).

Corollary 7. *Let G be a connected graph that does not have a γ_{cerip}-set, and let H be any graph. Then $G[H]$ is a non-γ_{cerp}-graph.*

6. Cartesian Product of Two Graphs

Recall that F. Harary (see [9]) defined the *Cartesian product* $G\Box H$ of two graphs G and H is a graph with the vertex set $V(G\Box H) = V(G) \times V(H)$. Two vertices (u_i, v_j) and (u_k, v_l) are adjacent in $G\Box H$ if and only if one of the following conditions holds:

(i) $u_i = u_k$ and $v_j v_l \in E(H)$,

(ii) $v_j = v_l$ and $u_i u_k \in E(G)$.

Proposition 8. *Let G be a graph with $\gamma_{cerp}(G) = 1$. Then $\gamma_{cerp}(G\Box K_1) = 1$.*

Proof. Let G be a graph with $\gamma_{cerp}(G) = 1$. Since $G\Box K_1 = G$, it follows that $\gamma_{cerp}(G\Box K_1) = 1$. □

Theorem 8. *Let $p \geq 2$ be an integer. Then $K_p\Box K_p$ is a non-γ_{cerp}-graph.*

Proof. Consider the complete graph K_p with p vertices. The graph $K_p\Box K_p$ has p^2 vertices. For the sake of contradiction, assume that $K_p\Box K_p$ is a γ_{cerp}-graph. Let (u, v) be any vertex in $V(K_p\Box K_p)$, where u is a vertex in the first K_p and v is a vertex in the second K_p. The vertex (u, v) is adjacent to $p - 1$ vertices of the form (u, v_i), where v_i is one of the $p - 1$ vertices of the second K_p. Therefore, (u, v) dominates $2(p - 1)$ vertices in $K_p\Box K_p$, and so (u, v) must be included in the dominating set J.

Now, we need to select additional vertices for J from the remaining $p^2 - 2(p - 1)$ vertices. Without loss of generality, assume that (u, v) is among these remaining vertices. Since (u, v) is adjacent to at least one vertex that is already dominated by another vertex in J, this contradicts the assumption that J is a perfect dominating set. Thus, no certified perfect dominating set exists in $K_p\Box K_p$, which implies that $K_p\Box K_p$ is a non-γ_{cerp}-graph for all $p \geq 2$. □

Theorem 9. *Let $m \geq 3$, $n \geq 2$ be integers. Then $K_m\Box P_n$ is a non-γ_{cerp}-graph.*

Proof. Let $V(K_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots, y_n\}$, so that $K_m \Box P_n$ consists of mn vertices. Assume, for the sake of contradiction, that $K_m \Box P_n$ is a γ_{cerp}-graph. Without loss of generality, suppose $(k_1, p_1) \in J$. Then (k_1, p_1) dominates both (k_1, p_2) and (k_j, p_1) for some $j = 2, 3, \ldots, m$. Note that $(k_i, p_2) \notin J$, as (k_i, p_1) is adjacent to (k_i, p_2) for some $i = 1, 2, \ldots, m$. In this scenario, (k_i, p_3) could dominate (k_i, p_2), but (k_1, p_2) is already dominated by (k_1, p_1), violating the condition for perfect domination.

This contradiction implies that no certified perfect dominating set exists in $K_m \Box P_n$, concluding that $K_m \Box P_n$ is not a γ_{cerp}-graph. □

The next result is a direct consequence of Theorems 8 and 9.

Corollary 8. *For every even integer n, $K_2 \Box P_n$ is a non-γ_{cerp}-graph.*

Proposition 9. *Let n be a positive odd integer. Then*

$$\gamma_{cerp}(K_2 \Box P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Open Questions and Problems:

Problem 1. Characterize the trees T, certain classes of graphs, and other binary operations that have not yet been studied, and determine their certified perfect domination numbers.

Problem 2. Establish Nordhaus-Gaddum type results for $\gamma_{cerp}(G)$.

Problem 3. Investigate how the certified perfect domination number is affected when a graph is modified by adding or removing an edge or vertex.

Conclusion: In this paper, we extended the study of certified perfect domination in graphs by presenting new upper bounds for the certified domination number $\gamma_{cerp}(G)$. We explored the relationship between certified domination and certified perfect domination parameters, identifying graphs with both small and large values of these parameters. Our results provide a deeper understanding of the structural properties of graphs in the context of certified perfect domination. We also characterize graphs for which $\gamma_{cerp}(G) = n$ and $\gamma_{cerp}(G) = \gamma_{cer}(G)$, offering insight into special classes of graphs where the certified perfect domination number attains these specific values. Furthermore, our investigation into the lexicographic and Cartesian products of graphs revealed the behavior of certified perfect dominating sets under these operations. We determined the certified perfect domination number for such products and identified non-γ_{cerp}-graphs arising from these binary operations.

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References

[1] M. Dettlaff, M. Lemańska, J. Topp, R. Ziemann, and P. Żyliński. Certified domination. *AKCE International Journal of Graphs and Combinatorics*, pages 1–12, 2020.

[2] M. Dettlaff, M. Lemańska, M. Miotk, J. Topp, R. Ziemann, and P. Żyliński. Graphs with equal domination and certified domination numbers. *arXiv*, pages 1–15, 2017. Retrieved from http://arxiv.org/abs/1710.02059v2.

[3] J. J. Hamja. Certified perfect domination in graphs. *European Journal of Pure and Applied Mathematics*, 16:2763–2774, 2023.

[4] R. Diestel. *Graph Theory*. Springer, Heidelberg, 2012.

[5] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. *Topics in Domination in Graphs*, volume 64 of *Developments in Mathematics*. Springer, 2020.

[6] S. Paraico and S. Canoy Jr. Super dominating sets in some products of graphs. *Far East Journal of Mathematical Sciences*, 102(10):2393–2402, 2017.

[7] M. Livingston and Q. F. Stout. Perfect dominating sets. *Congressus Numerantium*, pages 187–203, 1990.

[8] C. L. Armada and J. J. Hamja. Perfect isolate domination in graphs. *European Journal of Pure and Applied Mathematics*, 16:1326–1341, 2023.

[9] F. Harary. *Graph Theory*. Addison-Wesley Publishing Company, Inc., Massachusetts, 1969.