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On Common Fixed Point Theorems for Generalized Contractions Involving Rational Expressions and Auxiliary Functions

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Abstract. The purpose of this article is to prove common fixed point theorems for pair of maps (not necessary continuous), satisfying generalised contractions involving rational expressions and auxiliary functions (ψ, β) in the setting of both fuzzy b-metric spaces as well as partially ordered fuzzy b-metric spaces. To substantiate our finding, an example with graphical representation is given. The obtained results may generalize and improve some of the well known fixed-point results of the literature.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: common fixed point, auxiliary functions, rational expression, fuzzy b-metric, partially ordered

1. Introduction and Preliminaries

Fuzzy logic is a mathematical approach designed to address imprecise or uncertain information, providing a framework for representing vagueness and uncertainty in decision-making processes. Zadeh [1], in 1965, introduced the concept of fuzzy sets, laying the foundation for the development of fuzzy logic. Fuzzy logic is applied across a diverse array of fields, including image processing, natural control systems, medical diagnosis, language processing, and artificial intelligence.

Fixed point theorems are one of the most productive and successful tool in Mathematics which has large number of applications inside as well as the outer side of the Mathematics. The concept of fuzzy logic was also applied in the context of metric spaces. Kramosil and Michalek [2] was the first to introduced the notion of fuzzy metric space. Two decades

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later, it was further modified by George and Veeramani [3] with the aim of incorporating Hausdorff topology into fuzzy metric spaces. In 1988, Grabiec [4] made a significant advancement in the development of fixed-point theorems within the context of fuzzy metric space, by demonstrating analogue of Banach fixed-point theorem. Grabiec [4] first gives the concept of convergence in fuzzy metric spaces and subsequently utilized it to established a fixed-point result. Over the past three decades, numerous authors have extended and generalized various results from metric space theory within the context of fuzzy metric spaces. In 1994, Mishra et al. [5] gave the idea of compatible mappings in fuzzy metric space and proved a lemma and a common fixed point theorem. In 1995, Subrahmanyam[6] extended the result of Jungck [7] from complete metric spaces to fuzzy metric spaces. Chauhan and Joshi [8] utilized the notion of compatible mapping to prove some fixed point theorems in fuzzy M- metric Spaces. In 2010, Mihet [9], by utilizing the concept of the (E.A.) -property in fuzzy metric spaces, proved a common fixed point theorem.

The concept of a b-metric (an alternative formulation of the metric concept was derived by substituting the triangle inequality with a modified version) was first introduced through the works of Bakhtin [10], who laid the groundwork for its development. Later, Czerwik [11] formally defined the concept of b-metric space, offering a rigorous framework for its study. In contrast, Sedghi et al. [12] and Shobe et al. [13] introduced the concept of a fuzzy b-metric space, which is, in fact, broader than the concept of fuzzy metric spaces.

Definition 1. *[12] Let $\mathfrak{W}$ be a continuous conjunction, $b \geq 1$ is a real number, Y is an arbitrary (nonempty) set and Z is a fuzzy set on $Y^2 \times (0, \infty)$. Then a 3-tuple $(Y, Z, \mathfrak{W})$ is known as a fuzzy $b-$ metric space if for all $t, s > 0$ and for all $l, q, \varsigma \in Y$, following conditions hold:*

FB-1. $Z(l, q, t) > 0,$

FB-2. $Z(l, q, t) = 1$ *if and only if* $l = q,$

FB-3. $Z(l, q, t) = Z(q, l, t),$

FB-4. $\mathfrak{W}\left(Z\left(l, q, \frac{t}{b}\right), Z\left(q, \varsigma, \frac{s}{b}\right)\right) \leq Z(l, \varsigma, t+s),$

FB-5. $Z(l, q, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*

Following are few examples of fuzzy $b-$metric spaces.

Example 1. *[13] Suppose d is a b -metric on Y. Define $Z(l, q, t) = e^{-\frac{d(l,q)}{t}}$ and $t-$ norm as $\mathfrak{W}(l, q) = lq \quad \forall \;\; l, q \in [0, 1]$. Then Z is a fuzzy b-metric on Y.*

Definition 2. *[13] Suppose $(Y, Z, \mathfrak{W})$ is fuzzy b-metric space. Then we say that a sequence $\{l_i\} \in Y$:*

(i) converges to l if $Z(l_i, l, t) \to 1$ as $i \to \infty$ for each $t > 0$.

(ii) is called a Cauchy sequence, if for all $t > 0$ and $\varepsilon \in (0, 1)$, there exists $j_0 \in \mathbb{N}$ such that $1 - \varepsilon < Z(l_i, l_j, t)$ for all $i, j \geq j_0$.

Remark 1. *Triplet* $(Y, Z, \mathfrak{W})$ *is said to be complete fuzzy* b*-metric space, if every Cauchy sequence in* Y *is convergent.*

In past 10 years, this theory has been extended and generalized by many authors in various directions. Some of them are [14–20]. In 2020, Rakić et al. [21] proved a sufficient condition for a sequence to be Cauchy in fuzzy b-metric space and then gave number of extensions of fixed point theorems in such spaces. Recently, Mani et al. [22] proved some fixed point theorems to guarantee the existence and uniqueness of fixed point for a selfmap in complete fuzzy b-metric space using two different t -norms.

Definition 3. *Self mappings* $\mathcal{A}$ *and* $\mathcal{B}$ *of a fuzzy* b*-metric are said to satisfied property (E.A.) if there exist a sequence* $\{l_n\} \in Y$ *such that* $\lim_{n\to\infty} \mathcal{B}l_n = \lim_{n\to\infty} \mathcal{A}l_n = \varsigma \in Y$.

Definition 4. *Two mappings* $\mathcal{A}$ *and* $\mathcal{B}$ *of a fuzzy* b*-metric* $(Y, Z, *)$ *into itself are said to be compatible maps if* $\lim_{n\to\infty} Z(\mathcal{AB}l_n, \mathcal{BA}l_n, t) = 1$*, for all* $t > 0$*, where* $\{l_n\} \in Y$ *such that* $\lim_{n\to\infty} \mathcal{A}l_n = \lim_{n\to\infty} \mathcal{B}l_n = \varsigma \in Y$.

In this article, to ensure the existence and uniqueness of fixed point for a pair of self-maps utilizing auxiliary functions (ψ, β) and a rational expression, two common fixed point theorems are established within the framework of fuzzy b -metric spaces. To illustrate the applicability of the main results, example accompanied by graphical representation is also given.

2. Common Fixed Point Theorem for a Pair of Weakly Increasing Mappings

We begin this section by presenting a fixed point result for a pair of weakly increasing mappings (not necessary continuous) in partially ordered fuzzy $b-$ metric space (not necessary complete) involving auxiliary functions (ψ, β).

Theorem 1. *Let* $(Z, Y, \mathfrak{W}, \preceq)$ *be a partially ordered fuzzy* b *- metric space with continuous* t *-norms* $\mathfrak{W}(l_1, l_2) = l_1 l_2$. *Let* $\mathcal{A}, \mathcal{B} : Y \to Y$ *be weakly increasing mappings of* Y. *Further assume that:*

(i) For every comparable pair $(l, q) \in Y$ *and for all* $\lambda \in (0, 1)$, *maps* $\mathcal{A}, \mathcal{B}$ *satisfies*

$$\psi(Z(\mathcal{A}l, \mathcal{B}q, \frac{t}{\lambda})) \geq \beta(N(l, q, \frac{t}{\lambda})), \tag{1}$$

where

$$N(l, q, \frac{t}{\lambda}) \in \left\{ \frac{Z(q, \mathcal{B}q, \frac{t}{\lambda}) Z(l, \mathcal{A}l, \frac{t}{\lambda})}{Z(l, q, \frac{t}{\lambda})}, \frac{Z(q, \mathcal{B}q, \frac{t}{\lambda})(1 + Z(l, \mathcal{A}l, \frac{t}{\lambda}))}{1 + Z(l, q, \frac{t}{\lambda})} \right\}$$

and $\psi, \beta : (0, 1] \to (0, 1]$ *are continuous functions such that* $\psi(1) = \beta(1) = 1$ *with*

$$\beta(r) > \psi(r), \quad \forall \quad r \in (0, 1) \tag{2}$$

(ii) For all comparable pair $(l, q) \in Y$, if

$$Z(l, q, \frac{t}{\lambda})Z(l, q, \frac{t}{\lambda}) \leq 1. \tag{3}$$

Then the maps $\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point in Z.

Proof. Let $l_0 \in Y$ be any arbitrary point. Define $\mathcal{A}l_0 = l_1$ and $\mathcal{B}l_1 = l_2$. Continuing in this manner, in general, we can construct sequences $l_{2n+1}, l_{2n+2} \in Y$ such that

$$l_{2n+1} = \mathcal{A}l_{2n} \quad \text{and} \quad l_{2n+2} = \mathcal{B}l_{2n+1}.$$

Lets assume that $l_{2n+1} \neq l_{2n+2}$.
For all $n \geq 0$, we can construct a sequence $\{q_n\} \in Y$ such that

$$\begin{cases} q_{2n} &= l_{2n+1} = \mathcal{A}l_{2n} \text{ and} \\ q_{2n+1} &= l_{2n+2} = \mathcal{B}l_{2n+1}. \end{cases} \tag{4}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are both weakly increasing mapping, therefore

$$q_0 = l_1 = \mathcal{A}l_0 \preceq \mathcal{B}l_1 = q_1 = l_2 \cdots.$$

By repeating this process, we obtain

$$q_0 \preceq q_1 \preceq q_2 \cdots \preceq q_{2n} \preceq q_{2n+1} \preceq \cdots.$$

Let us assume that,

$$q_{2n} = q_{2n+1} \text{ for no } n \in \mathbb{N}. \tag{5}$$

Since l_{2n} and l_{2n+1} are comparable, therefore on substituting $l = l_{2n}$ and $q = l_{2n+1}$ in Eq. (1), we obtain

$$\psi(Z(q_{2n}, q_{2n+1}, \frac{t}{\lambda})) = \psi(Z(\mathcal{A}l_{2n}, \mathcal{B}l_{2n+1}, \frac{t}{\lambda})) \geq \beta(N(l_{2n+1}, l_{2n+2}, \frac{t}{\lambda}), \tag{6}$$

where

$$\begin{aligned} N(l_{2n+1}, l_{2n+2}, \frac{t}{\lambda}) &\in \left\{ \begin{array}{c} \dfrac{Z(l_{2n+1}, \mathcal{B}l_{2n+1}, \frac{t}{\lambda})Z(l_{2n}, \mathcal{A}l_{2n}, \frac{t}{\lambda})}{Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda})}, \\ \dfrac{Z(l_{2n+1}, \mathcal{B}l_{2n+1}, \frac{t}{\lambda})(1 + Z(l_{2n}, \mathcal{A}l_{2n}, \frac{t}{\lambda}))}{1 + Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda})} \end{array} \right\} \\ &\in \left\{ \begin{array}{c} \dfrac{Z(l_{2n+1}, l_{2n+2}, \frac{t}{\lambda})Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda})}{Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda})}, \\ \dfrac{Z(l_{2n+1}, l_{2n+2}, \frac{t}{\lambda})(1 + Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda}))}{1 + Z(l_{2n}, l_{2n+1}, \frac{t}{\lambda})} \end{array} \right\} \end{aligned}$$

$$= Z(l_{2n+1}, l_{2n+2}, \frac{t}{\lambda}) = Z(q_{2n}, q_{2n+1}, \frac{t}{\lambda}).$$

Thus, from Eq.(6)

$$\psi(Z(q_{2n}, q_{2n+1}, \frac{t}{\lambda})) \geq \beta(Z(q_{2n}, q_{2n+1}, \frac{t}{\lambda})).$$

This is a contradiction to our assumption (2). Thus our assumption in (5) is false and so $q_{2n} = q_{2n+1}$ for some $n \in \mathbb{N}$, say $n = k$. Consequently, with $q_{2k} = q_{2k+1}$ and $\varsigma = l_{2k+1}$, we get $\mathcal{B}\varsigma = \varsigma$, by (4). This proves that ς is a fixed point of $\mathcal{B}$.
Next, we prove that any fixed point of $\mathcal{B}$ is also a fixed point of $\mathcal{A}$.
Suppose not, *i.e* $\mathcal{A}\varsigma \neq \varsigma$.
If we take $l = q = \varsigma$ in (1), we get

$$\psi\left(Z(\mathcal{A}\varsigma, \varsigma, \frac{t}{\lambda})\right) = \psi\left(Z(\mathcal{A}\varsigma, \mathcal{B}\varsigma, \frac{t}{\lambda})\right) \geq \beta\left(N(\varsigma, \varsigma, \frac{t}{\lambda})\right), \tag{7}$$

where

$$\begin{aligned} N(\varsigma, \varsigma, \frac{t}{\lambda}) &\in \left\{ \frac{Z(\varsigma, \mathcal{B}\varsigma, \frac{t}{\lambda})Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})}{Z(\varsigma, \varsigma, \frac{t}{\lambda})}, \frac{Z(\varsigma, \mathcal{B}\varsigma, \frac{t}{\lambda})(1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varsigma, \frac{t}{\lambda})} \right\} \\ &\in \left\{ \frac{Z(\varsigma, \varsigma, \frac{t}{\lambda})Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})}{Z(\varsigma, \varsigma, \frac{t}{\lambda})}, \frac{Z(\varsigma, \varsigma, \frac{t}{\lambda})(1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varsigma, \frac{t}{\lambda})} \right\} \\ &\in \left\{ Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}), \frac{Z(\varsigma, \varsigma, \frac{t}{\lambda})(1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varsigma, \frac{t}{\lambda})} \right\} \end{aligned}$$

Here, two possible cases arise.

(i) If $N(\varsigma, \varsigma, \frac{t}{\lambda}) = Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})$ then from Eq. (7), we have

$$\psi\Big(Z(\mathcal{A}\varsigma, \varsigma, \frac{t}{\lambda})\Big) \geq \beta\Big(Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})\Big),$$

which contradicts Eq. (2). This means that $\mathcal{A}\varsigma = \varsigma$ for all $\varsigma \in Y$. Therefore, we get $\mathcal{A}\varsigma = \mathcal{B}\varsigma = \varsigma$, that is, ς is a common fixed point of $\mathcal{A}$ and $\mathcal{B}$.

(ii) If $N(\varsigma, \varsigma, \frac{t}{\lambda}) = \dfrac{Z(\varsigma, \varsigma, \frac{t}{\lambda})(1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varsigma, \frac{t}{\lambda})}$, then, again from Eq. (7), we get

$$\psi\left(Z(\mathcal{A}\varsigma, \varsigma, \frac{t}{\lambda})\right) \geq \beta\left(\frac{1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})}{2}\right) > \psi\left(\frac{1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})}{2}\right)$$

which is a contradiction. This means that $\mathcal{A}\varsigma = \varsigma$ for all $\varsigma \in Y$. Therefore, we get $\mathcal{A}\varsigma = \mathcal{B}\varsigma = \varsigma$, that is, ς is a common fixed point of $\mathcal{A}$ and $\mathcal{B}$.

In order to prove uniqueness of the fixed point, suppose on contrary that $\varsigma \neq \varrho$ are two fixed points of $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}\varsigma = \mathcal{B}\varsigma = \varsigma$ and $\mathcal{A}\varrho = \mathcal{B}\varrho = \varrho$.
That is, $Z(\varsigma, \varrho, \frac{t}{\lambda}) \neq 1$.
Let us proceed by considering two distinct possibilities:

(i) If ς is comparable to ϱ, then $\mathcal{A}\varsigma = \varsigma$ is comparable to $\varrho = \mathcal{B}\varrho$. Also from Eq. (1), and above discussion it is clear that if ϱ is a fixed point of $\mathcal{B}$, then $\mathcal{A}\varsigma = \varrho$ for all ς comparable with ϱ.
Again on substituting $l = \varsigma$ and $q = \varrho$ in Eq. (1), we have

$$\psi\Big(Z(\varsigma, \varrho, \frac{t}{\lambda})\Big) = \psi\Big(Z(\mathcal{A}\varsigma, \mathcal{B}\varrho, \frac{t}{\lambda})\Big) \geq \beta\Big(N(\varsigma, \varrho, \frac{t}{\lambda})\Big), \tag{8}$$

where

$$\begin{aligned} N(\varsigma, \varrho, \frac{t}{\lambda}) &\in \left\{ \frac{Z(\varrho, \mathcal{B}\varrho, \frac{t}{\lambda})Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})}{Z(\varsigma, \varrho, \frac{t}{\lambda})}, \frac{Z(\varrho, \mathcal{B}\varrho, \frac{t}{\lambda})(1 + Z(\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varrho, \frac{t}{\lambda})} \right\} \\ &\in \left\{ \frac{Z(\varrho, \varrho, \frac{t}{\lambda})Z(\varsigma, \varrho, \frac{t}{\lambda})}{Z(\varsigma, \varrho, \frac{t}{\lambda})}, \frac{Z(\varsigma, \varrho, \frac{t}{\lambda})(1 + Z(\varsigma, \varrho, \frac{t}{\lambda}))}{1 + Z(\varsigma, \varrho, \frac{t}{\lambda})} \right\} = 1. \end{aligned}$$

Eq (8) implies that

$$\psi\Big(Z(\varsigma, \varrho, \frac{t}{\lambda})\Big) \geq \beta(1) = 1.$$

This is possible only if $Z(\varsigma, \varrho, \frac{t}{\lambda}) = 1$. This gives, $\varsigma = \varrho$.

(ii) If ς is not comparable to ϱ, then there exists $\vartheta \in Y$ comparable to ς and ϱ such that $\mathcal{B}\vartheta = \vartheta$ is comparable to $\varsigma = \mathcal{B}\varsigma$ and $\mathcal{A}\varrho = \varrho$. We claim that $\vartheta = \varsigma$ and $\vartheta = \varrho$. Indeed, uniqueness of limit gives that $\varsigma = \varrho$.
Suppose $\vartheta \neq \varrho$, then $Z(\varrho, \vartheta, \frac{t}{\lambda}) \neq 1$.
Once again from Eq. (1), on substituting $l = \varrho$ and $q = \vartheta$, we have

$$\psi\Big(Z(\varrho, \vartheta, \frac{t}{\lambda})\Big) = \psi\Big(Z(\mathcal{A}\varrho, \mathcal{B}\vartheta, \frac{t}{\lambda})\Big) \geq \beta\Big(N(\varrho, \vartheta, \frac{t}{\lambda})\Big), \tag{9}$$

where

$$\begin{aligned} N(\varrho, \vartheta, \frac{t}{\lambda}) &\in \left\{ \frac{Z(\vartheta, \mathcal{B}\vartheta, \frac{t}{\lambda})Z(\varrho, \mathcal{A}\varrho, \frac{t}{\lambda})}{Z(\varrho, \vartheta, \frac{t}{\lambda})}, \frac{Z(\vartheta, \mathcal{B}\vartheta, \frac{t}{\lambda})(1 + Z(\varrho, \mathcal{A}\varrho, \frac{t}{\lambda}))}{1 + Z(\varrho, \vartheta, \frac{t}{\lambda})} \right\} \\ &\in \left\{ \frac{Z(\vartheta, \vartheta, \frac{t}{\lambda})Z(\varrho, \varrho, \frac{t}{\lambda})}{Z(\varrho, \vartheta, \frac{t}{\lambda})}, \frac{Z(\vartheta, \vartheta, \frac{t}{\lambda})(1 + Z(\varrho, \varrho, \frac{t}{\lambda}))}{1 + Z(\varrho, \vartheta, \frac{t}{\lambda})} \right\} \\ &\in \left\{ \frac{1}{Z(\varrho, \vartheta, \frac{t}{\lambda})}, \frac{2}{1 + Z(\varrho, \vartheta, \frac{t}{\lambda})} \right\}. \end{aligned}$$

This implies that

$$\text{either } N(\varrho, \vartheta, \frac{t}{\lambda}) = \frac{1}{Z(\varrho, \vartheta, \frac{t}{\lambda})} \quad \text{or} \quad N(\varrho, \vartheta, \frac{t}{\lambda}) = \frac{2}{1 + Z(\varrho, \vartheta, \frac{t}{\lambda})}$$

First assume that $N(\varrho,\vartheta,\frac{t}{\lambda}) = \frac{1}{Z(\varrho,\vartheta,\frac{t}{\lambda})}$

Then from Eq. (9), we get

$$\psi\Big(Z(\varrho,\vartheta,\frac{t}{\lambda})\Big) \geq \beta\left(\frac{1}{Z(\varrho,\vartheta,\frac{t}{\lambda})}\right)$$

Continuity of ψ, and Eq. (2), gives that

$$Z(\varrho,\vartheta,\frac{t}{\lambda})Z(\varrho,\vartheta,\frac{t}{\lambda}) > 1$$

which is contradiction to our assumption (Eq. 3).

Thus $Z(\varrho,\vartheta,\frac{t}{\lambda}) = 1$, *i.e.* $\vartheta = \varrho$.

Secondly, assume that $N(\varrho,\vartheta,\frac{t}{\lambda}) = \frac{2}{1+Z(\varrho,\vartheta,\frac{t}{\lambda})}$.

Again from Eq. (9), we get

$$\psi\left(Z(\varrho,\vartheta,\frac{t}{\lambda})\right) \geq \beta\left(\frac{2}{1+Z(\varrho,\vartheta,\frac{t}{\lambda})}\right) > \psi\left(\frac{2}{1+Z(\varrho,\vartheta,\frac{t}{\lambda})}\right)$$

On using the fact ψ is continuous, and on simplification we arrive at contradiction that $Z(\varrho,\vartheta,\frac{t}{\lambda}) > 1$. Thus our assumption is wrong. Hence $\vartheta = \varrho$. Following the same argument as above, we can prove that $\vartheta = \varsigma$.

This completes the proof of the Theorem 1.

3. Common Fixed Point Theorem for a Pair of Mappings Satisfying (E.A.) and Weakly Compatible Property

In our next result, we utilized the concept of (E.A.) property and weakly compatible for pair of three maps to guarantee the existence and uniqueness of fixed point involving auxiliary functions (ψ,β).

Theorem 2. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are three self maps defined on a fuzzy b-metric space $(Y, Z, \mathfrak{W})$ such that for each $l, q \in Y$, $t > 0$ and $\lambda \in (0,1)$, following holds:*

(i) $\mathcal{A}(Y) \subset \mathcal{C}(Y)$ *and* $\mathcal{B}(Y) \subset \mathcal{C}(Y)$.

(ii) $(\mathcal{A},\mathcal{C})$ *or* $(\mathcal{B},\mathcal{C})$ *satisfies E.A property with*

$$\psi\left(Z(\mathcal{A}l,\mathcal{B}q,\frac{t}{\lambda})\right) \geq \beta\left(M(l,q,\frac{t}{\lambda})\right), \tag{10}$$

where

$$M(l,q,\frac{t}{\lambda}) = \min\left\{\begin{array}{c} Z(\mathcal{C}l,\mathcal{C}q,\frac{t}{\lambda}), Z(\mathcal{C}l,\mathcal{B}q,\frac{t}{\lambda}), Z(\mathcal{C}q,\mathcal{B}q,\frac{t}{\lambda}), \\ \dfrac{Z(\mathcal{C}l,\mathcal{C}q,\frac{t}{\lambda}).Z(\mathcal{C}q,\mathcal{B}q,\frac{t}{\lambda})}{Z(\mathcal{C}l,\mathcal{B}q,\frac{t}{\lambda})} \end{array}\right\}$$

and $\psi, \beta : [0,1] \to [0,1]$ *are continuous functions such that* $\psi(1) = \beta(1) = 1$, $\psi(0) = \beta(0) = 0$ *with*

$$\beta(r) > \psi(r), \quad \forall \quad r \in (0,1). \tag{11}$$

(iii) $(\mathcal{A}, \mathcal{C})$ *and* $(\mathcal{B}, \mathcal{C})$ *are weakly compatible.*

Further, if any of the ranges of $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *is a complete subspace of* Y, *then there exist a unique* $\vartheta \in Y$ *such that* $\mathcal{A}\vartheta = \mathcal{B}\vartheta = \mathcal{C}\vartheta = \vartheta$.

Proof. Since $\mathcal{B}(Y) \subset \mathcal{C}(Y)$, then for some sequence $\{q_n\}$ in Y,

$$\mathcal{B}l_n = \mathcal{C}q_n = \vartheta.$$

Also, the pair $(\mathcal{B}, \mathcal{C})$ satisfies (E.A.) property, therefore there exist sequences $\{l_n\} \in Y$ such that, for some $\vartheta \in Y$

$$\begin{cases} \lim_{n\to\infty} \mathcal{B}l_n = \lim_{n\to\infty} \mathcal{C}l_n = \vartheta. \\ \lim_{n\to\infty} \mathcal{C}q_n = \lim_{n\to\infty} \mathcal{C}l_n = \vartheta. \end{cases} \tag{12}$$

Next, we prove that $\lim_{n\to\infty} \mathcal{A}q_n = \vartheta$.
Substitute $l = q_n$ and $q = l_n$ in Eq. (10), we obtain

$$\psi\left(Z(\mathcal{A}q_n, \vartheta, \frac{t}{\lambda})\right) = \psi\left(Z(\mathcal{A}q_n, \mathcal{B}l_n, \frac{t}{\lambda})\right) \geq \beta\left(M(q_n, l_n, \frac{t}{\lambda})\right), \tag{13}$$

where

$$(M(q_n, l_n, \frac{t}{\lambda}) = \min \left\{ \begin{array}{c} Z(\mathcal{C}q_n, \mathcal{C}l_n, \frac{t}{\lambda}), Z(\mathcal{C}q_n, \mathcal{B}l_n, \frac{t}{\lambda}), Z(\mathcal{C}l_n, \mathcal{B}l_n, \frac{t}{\lambda}), \\ \frac{Z(\mathcal{C}q_n, \mathcal{C}l_n, \frac{t}{\lambda}).Z(\mathcal{C}l_n, \mathcal{B}l_n, \frac{t}{\lambda})}{Z(\mathcal{C}q_n, \mathcal{B}l_n, \frac{t}{\lambda})} \end{array} \right\}.$$

Taking $\lim_{n\to\infty}$ in above equality, and make use of Eq. (11), we get

$$\lim_{n\to\infty} M(l, q, \frac{t}{\lambda}) = \min \left\{ \begin{array}{c} Z(\vartheta, \vartheta, \frac{t}{\lambda}), Z(\vartheta, \vartheta, \frac{t}{\lambda}), Z(\vartheta, \vartheta, \frac{t}{\lambda}), \\ \frac{Z(\vartheta, \vartheta, \frac{t}{\lambda}).Z(\vartheta, \vartheta, \frac{t}{\lambda})}{Z(\vartheta, \vartheta, \frac{t}{\lambda})}. \end{array} \right\} = 1.$$

Thus from Eq. (12), we get

$$\lim_{n\to\infty} \psi\left(Z(\mathcal{A}q_n, \vartheta, \frac{t}{\lambda})\right) \geq \beta(1) = 1.$$

This is possible only if, $\lim_{n\to\infty} \mathcal{A}q_n = \vartheta = \lim_{n\to\infty} \mathcal{B}l_n$.
Suppose that $\mathcal{C}(Y)$ is a complete subspace of Y, then $\mathcal{C}\varsigma = \vartheta$, for at least one $\varsigma \in Y$. Thus we have

$$\lim_{n\to\infty} \mathcal{A}q_n = \vartheta = \lim_{n\to\infty} \mathcal{B}l_n = \lim_{n\to\infty} \mathcal{C}q_n = \lim_{n\to\infty} \mathcal{C}l_n = \mathcal{C}\varsigma. \tag{14}$$

Next we claim that $\mathcal{A}\varsigma = \mathcal{C}\varsigma$.
Consider,

$$\begin{aligned}\psi\left(Z(\mathcal{A}\varsigma, \mathcal{C}\varsigma, \frac{t}{\lambda})\right) &= \lim_{n\to\infty} \psi\left(Z(\mathcal{A}\varsigma, \mathcal{B}l_n, \frac{t}{\lambda})\right)\\ &\geq \lim_{n\to\infty} \beta\left(M(\varsigma, l_n, \frac{t}{\lambda})\right)\\ &\geq \beta\left(\lim_{n\to\infty} M(\varsigma, l_n, \frac{t}{\lambda})\right),\end{aligned}$$

where

$$\lim_{n\to\infty} M(\varsigma, l_n, \frac{t}{\lambda}) = \lim_{n\to\infty} \min\left\{\begin{array}{c} Z(\mathcal{C}\varsigma, \mathcal{C}l_n, \frac{t}{\lambda}), Z(\mathcal{C}\varsigma, \mathcal{B}l_n, \frac{t}{\lambda}), Z(\mathcal{C}l_n, \mathcal{B}l_n, \frac{t}{\lambda}),\\ \dfrac{Z(\mathcal{C}\varsigma, \mathcal{C}l_n, \frac{t}{\lambda})Z(\mathcal{C}l_n, \mathcal{B}l_n, \frac{t}{\lambda})}{Z(\mathcal{C}\varsigma, \mathcal{B}l_n, \frac{t}{\lambda})}\end{array}\right\} = 1.$$

This is possible only if $Z(\mathcal{A}\varsigma, \mathcal{C}\varsigma, \frac{t}{\lambda}) = 1$, implies that $\mathcal{A}\varsigma = \mathcal{C}\varsigma$.
Since the pair $(\mathcal{A}, \mathcal{C})$ is weakly compatible, therefore $\mathcal{A}\mathcal{C}\varsigma = \mathcal{C}\mathcal{A}\varsigma$.
This implies that

$$\mathcal{A}\mathcal{A}\varsigma = \mathcal{A}\mathcal{C}\varsigma = \mathcal{C}\mathcal{A}\varsigma = \mathcal{C}\mathcal{C}\varsigma.$$

But, as $\mathcal{A}(Y) \subset \mathcal{C}(Y)$, therefore $\exists$ a $\varrho \in Y$ such that $\mathcal{A}\varsigma = \mathcal{C}\varrho$.
By the similar arguments as above, we can prove that $\mathcal{C}\varrho = \mathcal{B}\varrho$.
Thus,

$$\mathcal{A}\varsigma = \mathcal{C}\varsigma = \mathcal{C}\varrho = \mathcal{B}\varrho.$$

Further, on using the fact that the pair $(\mathcal{B}, \mathcal{C})$ is weakly compatible, we can deduce that

$$\mathcal{B}\mathcal{C}\varrho = \mathcal{C}\mathcal{B}\varrho = \mathcal{C}\mathcal{C}\varrho = \mathcal{B}\mathcal{B}\varrho.$$

Next we prove that $\mathcal{A}\mathcal{A}\varsigma = \mathcal{A}\varsigma$.
Suppose not, that is, $\mathcal{A}\mathcal{A}\varsigma \neq \mathcal{A}\varsigma$,implies Z$(\mathcal{A}\mathcal{A}\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}) \neq 1$.
From Eq. (10), we have

$$\psi\left(Z(\mathcal{A}\varsigma, \mathcal{A}\mathcal{A}\varsigma, \frac{t}{\lambda})\right) = \psi\left(Z(\mathcal{A}\mathcal{A}\varsigma, \mathcal{B}\varrho, \frac{t}{\lambda})\right) \geq \beta\left(M(\mathcal{A}\varsigma, \varrho, \frac{t}{\lambda})\right), \tag{15}$$

where

$$\begin{aligned}M(\mathcal{A}\varsigma, \varrho, \frac{t}{\lambda}) &= \min\left\{\begin{array}{c} Z(\mathcal{C}\mathcal{A}\varsigma, \mathcal{C}\varrho, \frac{t}{\lambda}), Z(\mathcal{C}\mathcal{A}\varsigma, \mathcal{B}\varrho, \frac{t}{\lambda}), Z(\mathcal{C}\varrho, \mathcal{B}\varrho, \frac{t}{\lambda}),\\ \dfrac{Z(\mathcal{C}\mathcal{A}\varsigma, \mathcal{C}\varrho, \frac{t}{\lambda})Z(\mathcal{C}\varrho, \mathcal{B}\varrho, \frac{t}{\lambda})}{Z(\mathcal{C}\mathcal{A}\varsigma, \mathcal{B}\varrho, \frac{t}{\lambda})}\end{array}\right\}\\ &= \min\left\{Z(\mathcal{A}\mathcal{A}\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda}), 1\right\}.\end{aligned}$$

This implies that, either $M(\mathcal{A}\varsigma, v, \frac{t}{\lambda}) = 1$ or $M(\mathcal{A}\varsigma, \varrho, \frac{t}{\lambda}) = Z(\mathcal{A}\mathcal{A}\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})$.
If $M(\mathcal{A}\varsigma, v, \frac{t}{\lambda}) = 1$, then from Eq. (15)

$$\psi\left(Z(\mathcal{A}\varsigma, \mathcal{A}\mathcal{A}\varsigma, \frac{t}{\lambda})\right) \geq \beta(1) = 1. \tag{16}$$

This is true only if Z$(\mathcal{A}\varsigma, \mathcal{A}\mathcal{A}\varsigma, \frac{t}{\lambda}) = 1$.
Further, if $M(\mathcal{A}\varsigma, \varrho, \frac{t}{\lambda}) = Z(\mathcal{A}\mathcal{A}\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})$, then again from Eq.(15), we have

$$\psi\left(Z(\mathcal{A}\varsigma, \mathcal{A}\mathcal{A}\varsigma, \frac{t}{\lambda})\right) \geq \beta\left(Z(\mathcal{A}\mathcal{A}\varsigma, \mathcal{A}\varsigma, \frac{t}{\lambda})\right),$$

which contradict the assumption Eq(11). Thus for both possibilities, we have our claim that $\mathcal{A}\mathcal{A}\varsigma = \mathcal{A}\varsigma$.
Similarly, we can prove that $\mathcal{B}\mathcal{A}\varsigma = \mathcal{A}\varsigma$ and $\mathcal{C}\mathcal{A}\varsigma = \mathcal{A}\varsigma$. This proves our claim.
For uniqueness suppose there exist ϑ and ϱ as two fixed point of maps $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Therefore

$$\mathcal{A}\vartheta = \mathcal{B}\vartheta = \mathcal{C}\vartheta = \vartheta \text{ and } \mathcal{A}\varrho = \mathcal{B}\varrho = \mathcal{C}\varrho = \varrho.$$

Consider,

$$\psi\left(Z(\vartheta, \varrho, \frac{t}{\lambda})\right) = \psi\left(Z(\mathcal{A}\vartheta, \mathcal{B}\varrho, \frac{t}{\lambda})\right) \geq \beta\left(M(\vartheta, \varrho, \frac{t}{\lambda})\right), \tag{17}$$

where

$$\begin{aligned} M(\vartheta, \varrho, \frac{t}{\lambda}) &= \min\left\{ \begin{array}{c} Z(\mathcal{C}\vartheta, \mathcal{C}\varrho, \frac{t}{\lambda}), Z(\mathcal{C}\vartheta, \mathcal{B}\varrho, \frac{t}{\lambda}), Z(\mathcal{C}\varrho, \mathcal{B}\varrho, \frac{t}{\lambda}), \\ \dfrac{Z(\mathcal{C}\vartheta, \mathcal{C}\varrho, \frac{t}{\lambda}) Z(\mathcal{C}\varrho, \mathcal{B}\varrho, \frac{t}{\lambda})}{Z(\mathcal{C}\vartheta, \mathcal{B}\varrho, \frac{t}{\lambda})} \end{array} \right\} \\ &= \min\left\{ \begin{array}{c} Z(\vartheta, \varrho, \frac{t}{\lambda}), Z(\vartheta, \varrho, \frac{t}{\lambda}), Z(\varrho, \varrho, \frac{t}{\lambda}), \\ \dfrac{Z(\vartheta, \varrho, \frac{t}{\lambda}) Z(\varrho, \varrho, \frac{t}{\lambda})}{Z(\vartheta, \varrho, \frac{t}{\lambda})} \end{array} \right\} \\ &= \min\left\{1, Z(\vartheta, \varrho, \frac{t}{\lambda})\right\}. \end{aligned}$$

This implies that either $M(\vartheta, \varrho, \frac{t}{\lambda}) = 1$ or $M(\vartheta, \varrho, \frac{t}{\lambda}) = Z(\vartheta, \varrho, \frac{t}{\lambda})$.
If $M(\vartheta, \varrho, \frac{t}{\lambda}) = 1$, then from Eq. (17), we get

$$\psi\left(Z(\vartheta, \varrho, \frac{t}{\lambda})\right) \geq \beta(1) = 1$$

This is true only if $Z(\vartheta, \varrho, \frac{t}{\lambda}) = 1$, which proves that $\vartheta = \varrho$. Further, if $M(\vartheta, \varrho, \frac{t}{\lambda}) = Z(\vartheta, \varrho, \frac{t}{\lambda})$, then again from Eq. (17), we get

$$\psi\left(Z(\vartheta, \varrho, \frac{t}{\lambda})\right) = \psi\left(Z(\mathcal{A}\vartheta, \mathcal{B}\varrho, \frac{t}{\lambda})\right) \geq \beta\left(Z(\vartheta, \varrho, \frac{t}{\lambda})\right),$$

which is contradiction for Eq(11).
This proves the uniqueness and completes the proof of result. □

4. Numerical Illustrations

Example 2. *Let* $Y = [1,\infty)$. *We define partial order* $\preceq$ *on* Y *as* $l \preceq q$ *if and only if* $q \leq l$ *for all* $l, q \in Y$. *Define fuzzy metric as*

$$Z(l,q,\frac{t}{\lambda}) = \exp\frac{-(l-q)^2}{\frac{t}{\lambda}}$$

Then clearly, $(Y, Z, \mathfrak{W}, \preceq)$ *is a partially ordered fuzzy b-metric spaces.*

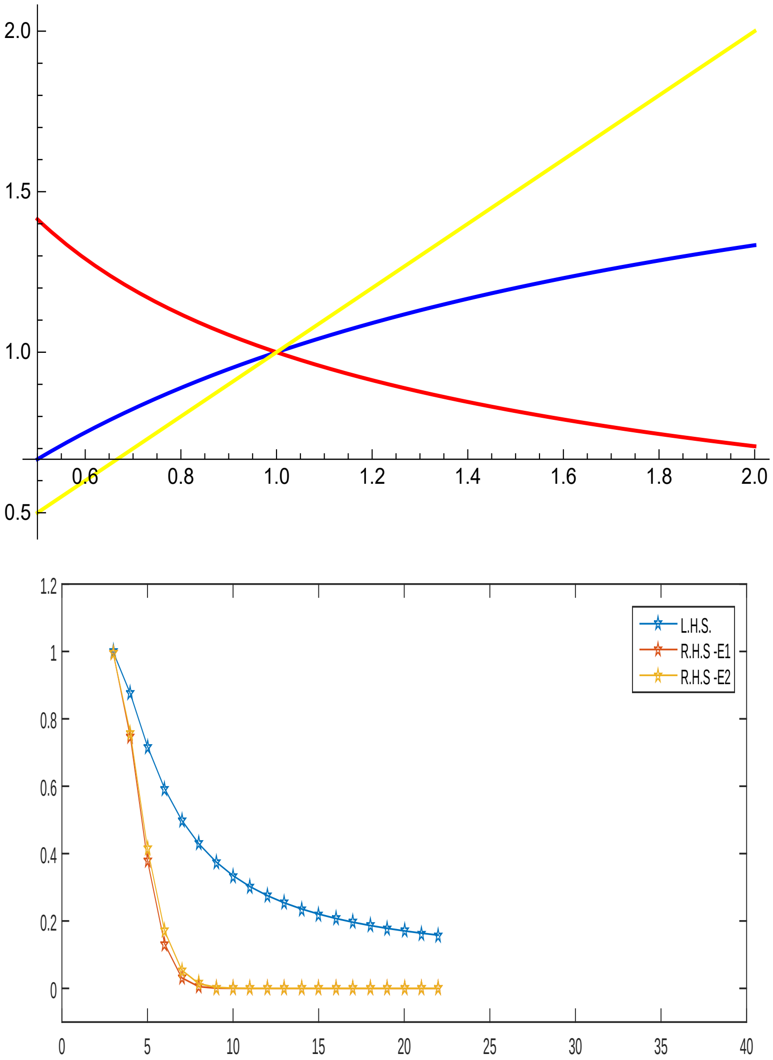

Figure 1: Graph of the functions defined in Example 2 and the inequality (1)

Consider two maps $\mathcal{A}, \mathcal{B} : Y \to Y$*, defined as*

$$\mathcal{A}l = \frac{2l}{l+1}; \qquad \mathcal{B}l = \frac{1}{\sqrt{l}}, \ \forall \ l \in Y.$$

Let $\psi, \beta : [0,1] \to [0,1]$ *be defined as:*

$$\psi(r) = r^2; \qquad \beta(r) = r \quad \psi(1) = \beta(1) = 1, \psi(0) = \beta(0) = 0.$$

Then for all $r \in (0,1)$*,* $\beta(r) > \psi(r)$*.*
Without loss of generality, if we assume that $l > q$*, then all the condition of Theorem 1 are satisfied. Also, form Figure 1 it can be observed that* $\mathcal{A}(1) = 1 = \mathcal{B}(1)$*. Thus* 1 *is the only one common fixed point of maps* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Example 3. *Let* $Y = [1, \infty)$*. Define fuzzy metric as*

$$Z(l, q, \frac{t}{\lambda}) = \exp \frac{-(l-q)^2}{\frac{t}{\lambda}}$$

Then clearly, $(Y, Z, \mathfrak{W})$ *is a fuzzy b-metric spaces.*
Consider three self maps $\mathcal{A}, \mathcal{B}, \mathcal{C} : Y \to Y$*, defined as*

$$\mathcal{A}l = \frac{2l}{l+1}; \qquad \mathcal{B}l = \frac{1}{\sqrt{l}}; \quad \mathcal{C}l = l, \ \forall \ l \in Y.$$

Clearly, $\mathcal{A}(Y) \subset \mathcal{C}(Y)$ *and* $\mathcal{B}(Y) \subset \mathcal{C}(Y)$*.*
Define a sequence

$$l_n = 1 + \frac{1}{n} \in Y$$

such that

$$\lim_{n \to \infty} l_n = 1 \in Y.$$

Further,

$$\lim_{n \to \infty} \mathcal{A}l_n = \lim_{n \to \infty} \mathcal{B}l_n = \lim_{n \to \infty} \mathcal{C}l_n = 1.$$

Thus the pair $(\mathcal{A}, \mathcal{C})$ *and* $(\mathcal{B}, \mathcal{C})$ *satisfies (E.A.) property.*
Moreover,

$$\mathcal{A}\mathcal{C}l_n = \frac{2(1+\frac{1}{n})}{(1+\frac{1}{n})+1} = \frac{2+\frac{2}{n}}{2+\frac{1}{n}}$$

and

$$\mathcal{C}\mathcal{A}l_n = \frac{2(1+\frac{1}{n})}{(1+\frac{1}{n})+1} = \frac{2+\frac{2}{n}}{2+\frac{1}{n}}$$

Therefore,

$$\lim_{n \to \infty} Z(\mathcal{A}\mathcal{C}l_n, \mathcal{C}\mathcal{A}l_n, t) = 1.$$

Similarly, we can have

$$\lim_{n \to \infty} Z(\mathcal{B}\mathcal{C}l_n, \mathcal{C}\mathcal{B}l_n, t) = 1.$$

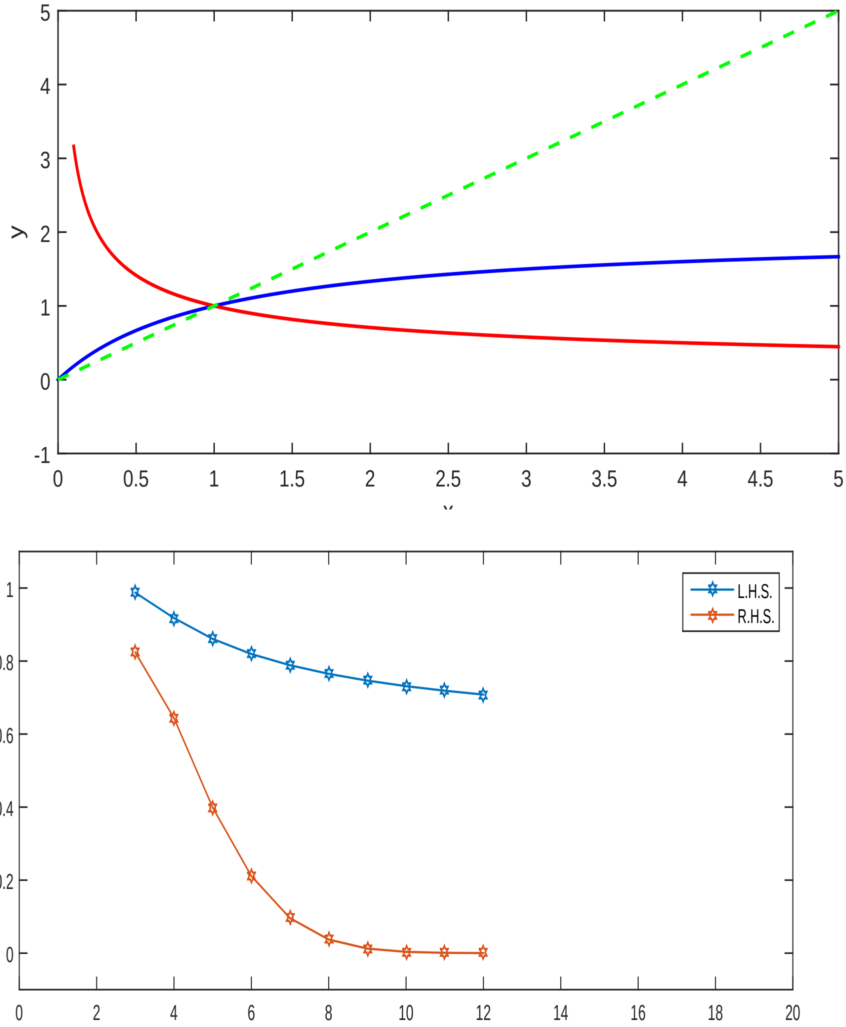

Figure 2: Graph of the functions defined in Example 3 and the inequality (10)

Thus the pairs $(\mathcal{A}, \mathcal{C})$ *and* $(\mathcal{B}, \mathcal{C})$ *are weakly compatible.*
Further, define maps $\psi, \beta : [0, 1] \to [0, 1]$ *as:*

$$\psi(r) = r^2; \qquad \beta(r) = r$$

such that $\psi(1) = \beta(1) = 1, \psi(0) = \beta(0) = 0.$. *Clearly, for all* $r \in (0, 1)$, $\beta(r) > \psi(r)$.
On following the graph of the inequality (11) in Figure 2, we can say that all the condition of Theorem 2 are satisfied. Also, $\mathcal{A}(1) = 1 = \mathcal{B}(1) = \mathcal{C}(1)$. *that is,* 1 *is the only one common fixed point of maps* $\mathcal{A}, \mathcal{B}$ *and* $\mathcal{C}$.

5. Conclusion

Utilizing the concept of auxiliary functions, we have established the existence and uniqueness of fixed points for a pair of self-mappings in both partially ordered fuzzy b - metric spaces and in fuzzy b - metric spaces. Furthermore, two supporting examples with graphical representations have been provided to illustrate the main findings.

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