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Hamilton-Jacobi Framework for Nonholonomic Dynamics

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Abstract. We derive generalized Hamilton–Jacobi equations for dynamical systems subject to nonholonomic constraints. The geometric formulation of Hamilton-Jacobi theory for nonholonomic constraints is developed, following the ideas of the authors in previous papers. It is shown that the equations of motion which follow from the principle of d'Alembert are identical to the equations which follow from the variational action principle. To illustrate the effectiveness of the proposed framework, we present and analyze two illustrative examples: the motion of a rolling disk on a horizontal plane and the motion of a knife edge on an inclined plane.

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1. Introduction

The theory of systems with nonholonomic constrains goes back to the 19th century [1–4] Nonholonomic constraints arise naturally in the context of mechanical systems with rigid bodies rolling without slipping over a surface. Another typical example is the Chaplygin sleigh. This is a rigid body where one of the contact points with the surface forms a knife edge. The nonholonomic constraint assumed that there is no motion perpendicular to the knife edge [5–10] or that the velocity of the contact point remains in the direction of x axis of the body. The equations of constraints in nonholonomic dynamics, written in terms of generalized coordinates q_i and generalized velocities $\dot{q}_i$.

Typical engineering problems that involve such constraints arise for example in robotics, where the wheels of a mobile robot are often required to roll without slipping, or if one interested in guiding the motion of a cutting tool.

Ghori [11]developed a generalization of the classical Hamilton-Jacobi method in order to cover mechanical systems subject to nonholonomic constraints. The method was based on a modification of the classical Hamilton-Jacobi equation by supplementing a modifying function. However, Nzaziev [12] proved that Ghori's theorem is incorrect. Recently, the authors [13, 14] introduced a new generalized Hamilton-Jacobi method for such systems.

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This method used a variational principle for the integral of action in connection with the introduction of unknown multipliers.

We will start from the actual solutions of the nonholonomic system, and apply a sort of Hamilton-Jacobi theory to arrive to the generating function, subsequently we will derive Hamilton's equations of motion.

This method was governed by a variational principle applied to a certain function. The resulting variational relation was then treated by introducing some unknown multipliers in connection with constraint relations. After the elimination of these multipliers the generalized momenta were found to be certain functions of the partial derivatives of the Hamilton Jacobi function with respect to the generalized coordinates and the time. Then the partial differential equation of the classical Hamilton-Jacobi method was modified by inserting these functions for the generalized momenta in the Hamiltonian of the system.

Unlike previous methods that changed the classical Hamilton-Jacobi equation using extra assumptions or added functions, this work develops the Hamilton-Jacobi approach in a clear and organized way, starting directly from a variational principle. It includes nonholonomic constraints by using Lagrange multipliers and shows that the resulting equations match those from the well-known Lagrange-d'Alembert principle. The main contribution of this work is applying this method to find exact solutions and Hamilton-Jacobi functions for real-world systems, making it easier and more understandable to solve problems involving constrained motion.

2. Formalism

Hamiltonian approaches to nonholonomic mechanical systems are developed by, for example [3,4,6,8]. The standard way to derive the equations of motion of a nonholonomic mechanical systems starts from Lagrange's equations and the addition of some unknown force that have to be exerted by the constraint in order to satisfy the nonholonomic constraint.

Consider a Lagrangian function

$$L = L(q_i, \dot{q}_i, t) \tag{1}$$

along with a set of nonholonomic constraints that are linear functions of the velocities:

$$f_\alpha(q_i, \dot{q}_i) = 0, \qquad \alpha = n+1, n+2, \ldots, n+m \tag{2}$$

The generalized momenta and the Hamiltonian of the system are defined by the relations:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3}$$

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{4}$$

When the system is nonholonomic, Hamilton's variational principle has to be expressed in the form [14]

$$\delta \int L \, dt = 0. \tag{5}$$

The Lagrangian formulation of these theories requires the configuration space formed by n generalized coordinates q_i and n generalized velocities $\dot{q}_i$.

According to the usual approach of the calculus of variations, it is found that the generalized Lagrange equations for such nonholonomic systems are given by [15, 16] by using the Hamilton's principle:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q} i}\right) - \frac{\partial L}{\partial q_i} = \lambda \alpha \frac{\partial f_\alpha}{\partial \dot{q}_i}\,, \tag{6}$$

these dynamic nonholonomic equations of motion are known as Lagrange-d'Alembert principle and are the correct equations to describe the time evolution a mechanical system under velocity dependent constraints. Where λ is Lagrange multiplier, to be determined that representing the force of constraint.

This equation represents a system of $n + m$ differential equations that can be solved for the $n + m$ unknown functions $q_i(t), \lambda_i(t)$.

We will consider a Lagrangian system with nonholonomic constraints whose motion obeys the Lagrange-d'Alembert principle in the case of completely integrable constraints.

The standard formulation of the Hamilton–Jacobi problem for a Hamiltonian system is based on the generating function $S(q, t)$ which is called the Hamilton-Jacobi function:

The set of the Hamilton-Jacobi partial differential equation can be written in compact form as [17–24]:

$$\frac{\partial S}{\partial t} + H(p_i, q_i, t) = 0\,. \tag{7}$$

We find the solution of constraint equations by integration whose solution has the following form:

$$q_i = q_i(\alpha_1, \alpha_2, \ldots, \alpha_{n-m}, \beta_1, \beta_2, \ldots, \beta_n, t)\,. \tag{8}$$

Where n is dimensional configuration space, subject to m nonholonomic constraints. $m < n - 1$

We may rewrite the above equations as:

$$X_i(\alpha_1, \alpha_2, \ldots, \alpha_{n-m}, q_i, t) = \beta_i\,. \tag{9}$$

Where α_i and β_i are initial constants of motion respectively related to the initial velocity and position of the particle.

The central issue in the procedure is to assume the existence of Hamilton's principal function from which the relations are obtained [13-14]

We can set:

$$\frac{\partial S}{\partial \alpha_i} = \beta_i\,. \tag{10}$$

From the above equations we can find the Hamilton-Jacobi function, we will see that the separation of variables is done in the following simple manner:

$$S(q_1, q_2, \ldots, q_n; \alpha_1, \alpha_2, \ldots, \alpha_n, t) = S_1(q_1, \alpha_1) + S_2(q_2, \alpha_2) + \cdots + S_n(q_n, \alpha_n). \tag{11}$$

The generalized momenta are determined by:

$$p_i = \frac{\partial S}{\partial q_i}\,. \tag{12}$$

We further have:

$$\alpha = \alpha(q_i, p_i, t). \tag{13}$$

Where we substitute α by their expression in terms of q_i and p_i.

The Hamiltonian is obtained as:

$$H = -\frac{\partial S}{\partial t}. \tag{14}$$

Following [7], the Hamilton's equations of motion are given by [14-16]:

$$\dot{q}i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \lambda\alpha \frac{\partial f_\alpha}{\partial \dot{q}_i} \tag{15}$$

This represents the equations of motion for nonholonmic constraints. It is worth to mention that the advantage of this method is that, in spite of the difficulties to solve a partial differential equation instead of an ordinary differential one, in many cases it works, being an extremely useful tool, indeed, in these cases the method provides an immediate way to integrate the equations of motion.

To demonstrate how the proposed scheme works, we will consider two examples.

3. Examples

In order to illustrate the general discussion we will examine two examples subject to nonholonmic constrains.

Example 1

As a first example of nonholonomic constraints consider the motion of a flat uniform disk rolls upright without slipping on horizontal plane; we assume that the mass, the moments of inertia and the radius of the disk are all equal to unity.

The Lagrangian describing the system is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\varphi}^2\right) \tag{16}$$

Where θ is the coordinate associated to rolling, φ to pivoting, x and y to point of contact of the disk with the horizontal plane.

The system has two nonholonomic constraints:

$$f_1 = \dot{x} - \dot{\theta}\cos\varphi = 0 \tag{17}$$

$$f_2 = \dot{y} - \dot{\theta}\sin\varphi = 0 \tag{18}$$

The constraint equations (17) and (18) can be rewritten as

$$\dot{x} = \dot{\theta}\cos\varphi, \tag{19}$$
$$\dot{y} = \dot{\theta}\sin\varphi. \tag{20}$$

First we derive the equations of motion as a result of varying the coordinates of the extended phase space including the multipliers q_i and λ_α according to equation (6), we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda_1 \frac{\partial f_1}{\partial \dot{x}}, \tag{21}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \lambda_2 \frac{\partial f_2}{\partial \dot{y}}, \tag{22}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \dot{\theta}} + \lambda_2 \frac{\partial f_2}{\partial \dot{\theta}}, \tag{23}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0\,. \tag{24}$$

These give:

$$\ddot{x} = \lambda_1\,, \tag{25}$$
$$\ddot{y} = \lambda_2\,, \tag{26}$$
$$\ddot{\theta} = -\lambda_1\cos\varphi - \lambda_2\sin\varphi, \tag{27}$$
$$\ddot{\varphi} = 0. \tag{28}$$

According to equations (25) and (26), equation (27) can be expressed as:

$$\ddot{\theta} = -\ddot{x}\cos\varphi - \ddot{y}\sin\varphi. \tag{29}$$

Differentiating constraint functions in equations (19) and (20) with respect to time, we find

$$\ddot{x} = \ddot{\theta}\cos\varphi - \dot{\varphi}\dot{\theta}\sin\varphi, \tag{30}$$
$$\ddot{y} = \ddot{\theta}\sin\varphi + \dot{\varphi}\dot{\theta}\cos\varphi, \tag{31}$$

By inserting equations (30) and (31) in equation (29), we arrive at the following equation

$$\ddot{\theta} = 0. \tag{32}$$

So we obtain the constrained equations:

$$\ddot{\theta} = 0, \qquad \ddot{\varphi} = 0, \qquad \dot{x} = \cos\varphi\dot{\theta}, \qquad \dot{y} = \sin\varphi\dot{\theta}. \tag{33}$$

This is a mixed set of first and second-order differential equations. From these equations, we can solve $\theta(t), \varphi(t), x(t), y(t)$ as follows:

The first two equations can be easily integrated to give:

$$\theta(t) = \alpha_1 t + \beta_1, \tag{34}$$
$$\varphi(t) = \alpha_2 t + \beta_2. \tag{35}$$

Differentiating equations (34) and (35) with respect to time, we get

$$\dot{\theta} = \alpha_1, \tag{36}$$
$$\dot{\varphi} = \alpha_2. \tag{37}$$

Substituting equations (35) and (36) into equation (19) and integrating the result equation, we obtain

$$\int \dot{x} dt = \int \alpha_1 \cos(\alpha_2 t + \beta_2) dt, \tag{38}$$

and this gives:

$$x(t) = \alpha_3 \sin \varphi + \beta_3, \tag{39}$$

where

$$\alpha_3 = \frac{\alpha_1}{\alpha_2}.$$

Also substituting equations (35), (36) into equation (20) and integrating the result equation, we have:

$$\int \dot{y} dt = \int \alpha_1 \sin(\alpha_2 t + \beta_2) dt, \tag{40}$$

which leads to

$$y(t) = \alpha_4 \cos \varphi + \beta_4\,, \tag{41}$$

where

$$\alpha_4 = -\frac{\alpha_1}{\alpha_2}.$$

Here $\beta_1, \beta_2, \beta_3$ and β_4 are constants of integration related to the initial values of θ, φ, x, y respectively while $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the initial values of velocities.

The essential point of our procedure for obtaining the Hamiltonian formalism for nonholonomic systems is to recognize that the above equations (34), (35), (39), (41) can be obtained in the Hamilton-Jacobi formalism. We may rewrite these equations respectively as:

$$\beta_1 = \theta - \alpha_1 t = \frac{\partial S_1}{\partial \alpha_1}, \tag{42}$$

$$\beta_2 = \varphi - \alpha_2 t = \frac{\partial S_2}{\partial \alpha_2}, \tag{43}$$

$$\beta_3 = x - \alpha_3 \sin \varphi = \frac{\partial S_3}{\partial \alpha_3}\,, \tag{44}$$

$$\beta_4 = y - \alpha_4 \cos \varphi = \frac{\partial S_4}{\partial \alpha_4}\,. \tag{45}$$

Solving equations (42-45) simultaneously, we obtain:

$$S_1(\theta,\alpha_1) = \theta\alpha_1 - \frac{1}{2}\alpha_1^2 t\,, \tag{46}$$

$$S_2(\varphi,\alpha_2) = \varphi\alpha_2 - \frac{1}{2}\alpha_2^2 t\,, \tag{47}$$

$$S_3(x,\alpha_3) = x\alpha_3 - \frac{1}{2}\alpha_3^2 \sin\varphi\,, \tag{48}$$

$$S_4(y,\alpha_4) = y\alpha_4 - \frac{1}{2}\alpha_4^2 \cos\varphi. \tag{49}$$

We saw that it is possible to separate the variables in the Hamilton-Jacobi equations.

With these results, the Hamilton-Jacobi function S becomes

$$S = \theta\alpha_1 + \varphi\alpha_2 + x\alpha_3 + y\alpha_4 - \frac{1}{2}t(\alpha_1^2 + \alpha_2^2) - \frac{1}{2}\alpha_3^2 \sin\varphi - \frac{1}{2}\alpha_4^2 \cos\varphi. \tag{50}$$

We interpret $S(\theta, \varphi, x, y, \alpha_1, \alpha_2, \alpha_3, \alpha_4, t)$ as the solution of Hamilton-Jacobi equation. The generalized momenta are derived as

$$p_\theta = \frac{\partial S}{\partial \theta} = \alpha_1\,, \tag{51}$$

$$p_\varphi = \frac{\partial S}{\partial \varphi} = \alpha_2 - \frac{1}{2}\alpha_3^2 \cos\varphi + \frac{1}{2}\alpha_4^2 \sin\varphi, \tag{52}$$

$$p_x = \frac{\partial S}{\partial x} = \alpha_3\,, \tag{53}$$

$$p_y = \frac{\partial S}{\partial y} = \alpha_4\,. \tag{54}$$

From the above equations we can obtain $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as functions of p_i and q_i:

$$\alpha_1 = p_\theta\,, \tag{55}$$

$$\alpha_2 = p_\varphi + \frac{1}{2}\alpha_3^2 \cos\varphi - \frac{1}{2}\alpha_4^2 \sin\varphi\,, \tag{56}$$

$$\alpha_3 = p_x\,, \tag{57}$$

$$\alpha_4 = p_y\,. \tag{58}$$

Following the equation (14), the Hamiltonian is defined as:

$$H = -\frac{\partial S}{\partial t} = \frac{1}{2}(\alpha_1^2 + \alpha_2^2)\,. \tag{59}$$

Substituting equation (55) and (56) into equation (59), this leads the following expression for the Hamiltonian:

$$H = \frac{1}{2}\left[P_\theta^2 + \left(p_\varphi + \frac{1}{2}P_x^2 \cos\varphi - \frac{1}{2}p_y^2 \sin\varphi\right)^2\right]. \tag{60}$$

The Hamilton's equations of motion can be found from equation (15). So the generalized velocities corresponding to this Hamiltonian are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = P_\theta \,, \tag{61}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \left(p_\varphi + \frac{1}{2}P_x^2 \cos\varphi - \frac{1}{2}p_y^2 \sin\varphi\right) \,, \tag{62}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \left(p_\varphi + \frac{1}{2}P_x^2 \cos\varphi - \frac{1}{2}p_y^2 \sin\varphi\right) (P_x \cos\varphi) \,, \tag{63}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \left(p_\varphi + \frac{1}{2}P_x^2 \cos\varphi - \frac{1}{2}p_y^2 \sin\varphi\right) (-p_y \sin\varphi). \tag{64}$$

Substituting equations (52–54) into equation (62), we find

$$\dot{\varphi} = \left(\alpha_2 - \frac{1}{2}\alpha_3^2 \cos\varphi + \frac{1}{2}\alpha_4^2 \sin\varphi + \frac{1}{2}\alpha_3^2 \cos\varphi - \frac{1}{2}\alpha_4^2 \sin\varphi\right) \,,$$

or,

$$\dot{\varphi} = \alpha_2 \,. \tag{65}$$

Substituting equations (52–54) into equation (63), and using α_2 for the bracket term (from eq. 65) and $P_x = \alpha_3$ (from eq. 57, assuming $P_x = p_x$):

$$\dot{x} = \alpha_2 \alpha_3 \cos\varphi.$$

Using the definition $\alpha_3 = \alpha_1/\alpha_2$ (from the context of eq. 39), this simplifies to:

$$\dot{x} = \alpha_1 \cos\varphi \,. \tag{66}$$

According to equation (36), equation (66) can be finally written as follows:

$$\dot{x} = \dot{\theta} \cos\varphi. \tag{67}$$

Substituting equations (52–54) into equation (64), we get:

$$\dot{y} = \alpha_2(-\alpha_4 \sin\varphi) = -\alpha_2 \alpha_4 \sin\varphi,$$

or, using $\alpha_4 = -\alpha_1/\alpha_2$ (from the context of eq. 41):

$$\dot{y} = \alpha_1 \sin\varphi \,. \tag{68}$$

According to equation (36), equation (68) can be finally written as follows:

$$\dot{y} = \dot{\theta} \sin\varphi. \tag{69}$$

Using equation (15), this yields

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} + \lambda_1 \frac{\partial f_1}{\partial \dot{\theta}} + \lambda_2 \frac{\partial f_2}{\partial \dot{\theta}} \,, \tag{70}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} + \lambda_1 \frac{\partial f_1}{\partial \dot{\varphi}} + \lambda_2 \frac{\partial f_2}{\partial \dot{\varphi}}\,, \tag{71}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} + \lambda_1 \frac{\partial f_1}{\partial \dot{x}}\,, \tag{72}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial \dot{y}}\,. \tag{73}$$

These give

$$\dot{p}_\theta = -\lambda_1 \cos\varphi - \lambda_2 \sin\varphi\,, \tag{74}$$

$$\dot{p}_\varphi = 0\,, \tag{75}$$

$$\dot{p}_x = \lambda_1\,, \tag{76}$$

$$\dot{p}_y = \lambda_2\,. \tag{77}$$

These equations are similar to equations (27), (28), (25) and (26) respectively, so that:

$$\dot{p}_\theta = \ddot{\theta}\,, \tag{78}$$

$$\dot{p}_\varphi = \ddot{\varphi}\,, \tag{79}$$

$$\dot{p}_x = \ddot{x}\,, \tag{80}$$

$$\dot{p}_y = \ddot{y}\,. \tag{81}$$

We see that the equations of motion that obtained by the Hamilton-Jacobi methods are equivalent to those which obtained by the Lagrange-d'Alembert principle.

Example 2

As a second example for nonholonomic constraints consider the knife edge on an inclined plane. The plane corresponds physically to a blade moving in the xy-plane at an angle φ to the x-axis.

The Lagrangian of the system is:

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J \dot{\varphi}^2, \tag{82}$$

Where J is the moment of inertia of the blade about a vertical axis through the point of contact. We assume that the mass and the moments of inertia J are equal to unity. Thus the Lagrangian of the system is given by:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2)\,. \tag{83}$$

The system has the following nonholonomic constraint:

$$f = \dot{x}\sin\varphi - \dot{y}\cos\varphi = 0\,, \tag{84}$$

This constraint can be rewritten as

$$\dot{y} = \dot{x}\tan\varphi. \tag{85}$$

Following equation (2.19) (Note: this refers to an internal numbering, likely eq. 15 in this paper, or a general formalism section), the equations of motion can be obtained as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda \frac{\partial f}{\partial \dot{x}}\,, \tag{86}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \lambda \frac{\partial f}{\partial \dot{y}}\,, \tag{87}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = \lambda \frac{\partial f}{\partial \dot{\varphi}}\,. \tag{88}$$

These give:

$$\ddot{x} = \lambda \sin\varphi, \tag{89}$$

$$\ddot{y} = -\lambda \cos\varphi, \tag{90}$$

$$\ddot{\varphi} = 0. \tag{91}$$

From equations (89) and (90), we obtain

$$\ddot{y} = -\frac{\ddot{x}}{\tan\varphi}, \tag{92}$$

Differentiating constraint equation (85) with respect to time to eliminate λ, we find

$$\ddot{y} = \ddot{x}\tan\varphi + \dot{\varphi}\dot{x}\sec^2\varphi\,, \tag{93}$$

Inserting equation (92) in (93) and multiplying the result by $(-\tan\varphi)$ leads to:

$$\ddot{x} = -\ddot{x}\tan^2\varphi - \dot{\varphi}\dot{x}\sec^2\varphi\tan\varphi, \tag{94}$$

By using the identity $1 + \tan^2\varphi = \sec^2\varphi$, equation (94) reduces to:

$$\ddot{x} = -\dot{\varphi}\dot{x}\tan\varphi. \tag{95}$$

From which we obtain the constrained equations:

$$\ddot{\varphi} = 0, \qquad \ddot{x} = -\dot{\varphi}\dot{x}\tan\varphi, \qquad \dot{y} = \dot{x}\tan\varphi. \tag{96}$$

Now we can solve $\varphi(t), x(t), y(t)$ as follows:

The first equation can be easily integrated to give:

$$\varphi(t) = \alpha_1 t + \beta_1. \tag{97}$$

Differentiating equation (97) with respect to time, we obtain

$$\dot{\varphi} = \alpha_1. \tag{98}$$

Now integrate equation (95), we get

$$\int \frac{\ddot{x}}{\dot{x}} dt = \int (-\tan\varphi) d\varphi, \qquad \dot{\varphi} = \frac{d\varphi}{dt} \,. \tag{99}$$

This gives:

$$\ln \dot{x} = \ln \cos\varphi \,,$$

or,

$$\dot{x} = \cos\varphi. \tag{100}$$

Substituting equation (97) into equation (100) and integrating the result equation:

$$\int \dot{x} dt = \int \cos(\alpha_1 t + \beta_1) dt \,, \tag{101}$$

Leads to

$$x(t) = \alpha_2 \sin\varphi + \beta_2 \,, \tag{102}$$

where

$$\alpha_2 = \frac{1}{\alpha_1}.$$

Now substituting equation (100) in constraint equation (85) we get:

$$\dot{y} = \sin\varphi, \tag{103}$$

Inserting equation (97) into equation (103) and integrating the result equation

$$\int \dot{y} dt = \int \sin(\alpha_1 t + \beta_1) dt \,, \tag{104}$$

we find

$$y(t) = \alpha_3 \cos\varphi + \beta_3, \tag{105}$$

where

$$\alpha_3 = -\frac{1}{\alpha_1}.$$

Again β_1, β_2 and β_3 are constant of integration related to the initial values of φ, x, y while α_1, α_2 and α_3 are the initial values of velocities.

Following the same steps as in the preceding example we may rewrite equations (97), (102) and (105) respectively as:

$$\beta_1 = \varphi - \alpha_1 t = \frac{\partial S_1}{\partial \alpha_1} \,, \tag{106}$$

$$\beta_2 = x - \alpha_2 \sin\varphi = \frac{\partial S_2}{\partial \alpha_2}, \tag{107}$$

$$\beta_3 = y - \alpha_3 \cos\varphi = \frac{\partial S_3}{\partial \alpha_3} \,. \tag{108}$$

Solving equations (106–108) simultaneously, we obtain:

$$S_1(\varphi,\alpha_1) = \varphi\alpha_1 - \frac{1}{2}\alpha_1^2 t, \tag{109}$$

$$S_2(x,\alpha_2) = x\alpha_2 - \frac{1}{2}\alpha_2^2 \sin\varphi, \tag{110}$$

$$S_3(y,\alpha_3) = y\alpha_3 - \frac{1}{2}\alpha_3^2 \cos\varphi. \tag{111}$$

Following equation (11), we finally collect expressions from equations (109–111), which give the Hamilton-Jacobi function $S(\varphi, x, y, \alpha_1, \alpha_2, \alpha_3, t)$:

$$S = \varphi\alpha_1 + x\alpha_2 + y\alpha_3 - \frac{1}{2}\alpha_1^2 t - \frac{1}{2}\alpha_2^2 \sin\varphi - \frac{1}{2}\alpha_3^2 \cos\varphi\,. \tag{112}$$

The generalized momenta can be derived as

$$p_\varphi = \frac{\partial S}{\partial \varphi} = \alpha_1 - \frac{1}{2}\alpha_2^2\cos\varphi + \frac{1}{2}\alpha_3^2 \sin\varphi, \tag{113}$$

$$p_x = \frac{\partial S}{\partial x} = \alpha_2\,, \tag{114}$$

$$p_y = \frac{\partial S}{\partial y} = \alpha_3. \tag{115}$$

From the above equations we can obtain $\alpha_1, \alpha_2, \alpha_3$ as functions of p_i and q_i.

$$\alpha_1 = p_\varphi + \frac{1}{2}\alpha_2^2\cos\varphi - \frac{1}{2}\alpha_3^2\sin\varphi\,, \tag{116}$$

$$\alpha_2 = p_x\,, \tag{117}$$

$$\alpha_3 = p_y\,. \tag{118}$$

Making use the definition of equation (14), the Hamiltonian is defined as:

$$H = -\frac{\partial S}{\partial t} = \frac{1}{2}\alpha_1^2\,, \tag{119}$$

Inserting equation (116) into equation (119) we get the following expression for the Hamiltonian:

$$H = \frac{1}{2}\left[p_\varphi + \frac{1}{2}(P_x^2\cos\varphi - p_y^2\sin\varphi)\right]^2. \tag{120}$$

The corresponding Hamilton's equations of motion can be found from equation (15). So the generalized velocities corresponding to this Hamiltonian are:

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \left[p_\varphi + \frac{1}{2}(P_x^2\cos\varphi - p_y^2\sin\varphi)\right]\,, \tag{121}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \left[p_\varphi + \frac{1}{2}(P_x^2\cos\varphi - p_y^2\sin\varphi)\right](P_x\cos\varphi), \tag{122}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \left[p_\varphi + \frac{1}{2}(P_x^2 \cos\varphi - p_y^2 \sin\varphi)\right](-p_y \sin\varphi)\,. \tag{123}$$

Substituting equations (113–115) into equation (121), we find

$$\dot{\varphi} = \left[\alpha_1 - \frac{1}{2}\alpha_2^2 \cos\varphi + \frac{1}{2}\alpha_3^2 \sin\varphi + \frac{1}{2}\alpha_2^2 \cos\varphi - \frac{1}{2}\alpha_3^2 \sin\varphi\right],$$

or,

$$\dot{\varphi} = \alpha_1\,. \tag{124}$$

Inserting equations (113–115) into equation (122), and using $P_x = p_x = \alpha_2$:

$$\dot{x} = \alpha_1\alpha_2 \cos\varphi$$

or, using $\alpha_2 = 1/\alpha_1$ (from context of eq. 102):

$$\dot{x} = \cos\varphi. \tag{125}$$

Substituting equations (113–115) into equation (123), and using $p_y = \alpha_3$:

$$\dot{y} = \alpha_1(-\alpha_3 \sin\varphi) = -\alpha_1\alpha_3 \sin\varphi,$$

or, using $\alpha_3 = -1/\alpha_1$ (from context of eq. 105):

$$\dot{y} = \sin\varphi. \tag{126}$$

Using equation (15), this yields:

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} + \lambda\frac{\partial f}{\partial \dot{\varphi}}, \tag{127}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} + \lambda\frac{\partial f}{\partial \dot{x}}\,, \tag{128}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} + \lambda\frac{\partial f}{\partial \dot{y}}\,. \tag{129}$$

These give

$$\dot{p}_\varphi = 0, \tag{130}$$

$$\dot{p}_x = \lambda \sin\varphi, \tag{131}$$

$$\dot{p}_y = -\lambda \cos\varphi. \tag{132}$$

These equations similar to equations (91), (89) and (90) respectively so:

$$\dot{p}_\varphi = \ddot{\varphi}\,, \tag{133}$$

$$\dot{p}_x = \ddot{x}\,, \tag{134}$$

$$\dot{p}_y = \ddot{y}\,. \tag{135}$$

According to the previous discussion we see that the equations of motion that obtained by the Hamilton-Jacobi methods are equivalent to those obtained by the Lagrange-d'Alembert principle.

4. Conclusion

This work shedding further light on constrained systems of type nonholonomic constraints, especially on determining the Hamilton-Jacobi function S. Finding S enables us to get the solutions of the equations of motion. These solutions are obtained in terms of the time and the coordinates that correspond to dependent momenta; these are treated as independent variables, just like the time t. This is followed by determining a new method to integrate the dynamical equations in particular, nonholonomic systems. We obtain the equations of motion for a Lagrangian system with nonholonomic constraints making use of the d'Alembert principle. It is shown that the results obtained using the HamiltonJacobi method are equivalent to those obtained by the Lagrange-d'Alembert principle. The method is applied for two examples: the first one is the motion of a flat uniform disk rolls upright without slipping on horizontal plane, and the second one is the motion of a knife edge on an inclined plane.

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