EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6093
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Implicative Filters of Implicative Negatively Partially Ordered Ternary Semigroups

Kansada Nakwan[1], Panuwat Luangchaisri[1], Thawhat Changphas[1,*]

[1] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Abstract. In this paper, firstly, we give characterizations of filters in implicative negatively partially ordered ternary semigroups and in commutative implicative negatively partially ordered ternary semigroups. Secondly, we introduce the notion of implicative filters of implicative negatively partially ordered ternary semigroups; an example is established. We show that every implicative filter is a filter and give an example to show that the converse is not true in general. Moreover, we state some equivalent conditions for an implicative filter by using a particular set defined by a filter. Finally, we introduce and study a generalization of implicative filters.

2020 Mathematics Subject Classifications: 20M12, 06F99, 06A06, 06A12

Key Words and Phrases: Filter, implicative filter, generalized implicative filter, implicative negatively partially ordered ternary semigroup

1. Introduction

In [1], M. W. Chan and K. P. Shum introduced and studied the concept of implicative negatively partially ordered semigroups; elementary properties are established; the authors obtained the homomorphism theorems using implicative homomorphisms and filters. In [2], Y. B. Yun introduced and studied the notion of implicative filters; some characterizations were given by using a particular set. In [3], M. Sambasiva Rao and K. P. Shum generalized the notion of implicative filters; a sufficient condition is derived for a generalized implicative filter of an implicative negatively partially ordered semigroups to become a filter.

In [4], K. Nakwan, P. Luangchaisri and T. Changphas introduced and studied the notion of implicative negatively partially ordered ternary semigroups (it is abbreviated by implicative n.p.o ternary semigroups) and filters. The authors applied the concept of implicative n.p.o. semigroups to implicative n.p.o. ternary semigroups and gave elementary properties. In this paper, we continue the investigation of implicative n.p.o.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6093

Email addresses: kansada.n@kkumail.com (K. Nakwan),
panulu@kku.ac.th (P. Luangchaisri), thacha@kku.ac.th (T. Changphas)

ternary semigroups and their filters which was started in the general case by K. Nakwan, P. Luangchaisri and T. Changphas in [4]. We also give equivalent conditions of a filter of implicative and commutative implicative n.p.o. ternary semigroups. We show that every implicative filter is a filter and give an example to show that the converse is not true in general. We state some equivalent conditions for an implicative filter by using a particular set defined by a filter. Finally, we introduce and study a generalization of implicative filters.

2. Preliminaries

For information in this section, we refer to the results obtained in [4].

A negatively partially ordered ternary semigroup (it is abbreviated by n.p.o. ternary semigroup) $(T, [\ ], \leq)$ consists of a non-empty set T together with a partial order $\leq$ and a ternary multiplication $[\ ]$ on T such that the following conditions are satisfied: for any $x, y, z, u, v \in T$,

(1) $[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$;

(2) if $x \leq y$, then $[xuv] \leq [yuv]$, $[uxv] \leq [uyv]$, and $[uvx] \leq [uvy]$;

(3) $[xyz] \leq x$, $[xyz] \leq y$, and $[xyz] \leq z$.

An n.p.o. ternary semigroup $(T, [\ ], \leq)$ is said to be *commutative* if

$$[xyz] = [yzx] = [zxy] = [yxz] = [zyx] = [xzy]$$

for all elements $x, y, z \in T$. That is, if for any $x_1, x_2, x_3 \in T$,

$$[x_1x_2x_3] = [x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}]$$

for any permutation σ on $\{1, 2, 3\}$.

An n.p.o. ternary semigroup $(T, [\ ], \leq)$ with an additional ternary multiplication $[\ ]^*$ on T such that

$$u \leq [xyz]^* \iff [uxy] \leq z$$

for any $x, y, z, u \in T$ is called an *implicative n.p.o. ternary semigroup*. The ternary multiplication $[\ ]^*$ is called a *ternary implication*.

An element 1 of an n.p.o. ternary semigroup $(T, [\ ], \leq)$ is an identity of T if $[11x] = [1x1] = [x11] = x$ for any $x \in T$. Not every n.p.o. ternary semigroup with the same identity and greatest element admits the implicative structure, see Example 2 in [4]. Furthermore, the greatest element of implicative n.p.o. ternary semigroup need not be identity, see Example 1 in [4].

Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. Then the following properties hold:

(1) $x \leq [xxx]^*$;

(2) $[xxx]^* = [yyy]^*$;

(3) T contains the greatest element, namely $[xxx]^*$,

for any $x, y \in T$.

Let 1 be the greatest element of an n.p.o. ternary semigroup $(T, [\ ], \leq)$ if exists. It is observed that if 1 is the multiplicative identity then it can be verified that $[xyz] = 1$ if and only if $x = y = z = 1$ for any $x, y, z \in T$.

Throughout the paper, we deal with an implicative n.p.o. ternary semigroup with 1 which is both the greatest element and the identity.

The following theorem collects several properties of elements of implicative n.p.o. ternary semigroups.

Theorem 1. *[4] Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. Then for any $x, y, z, u, v \in T$, the following conditions hold:*

(1) $x \leq 1$, $[xxx]^* = 1$, $x = [11x]^*$;

(2) $x \leq [yz[xyz]]^*$;

(3) $x \leq [xx[xxx]]^*$;

(4) $x \leq [yzx]^*$;

(5) if $x \leq y$, *then* $[xuv]^* \geq [yuv]^*$ *and* $[uvx]^* \leq [uvy]^*$;

(6) $x \leq y \iff [x1y]^* = 1 \iff [1xy]^* = 1$;

(7) $[xy[zuv]^*]^* = [[xyz]uv]^* = [x[yzu]v]^*$.

Definition 1. *[4] Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. A non-empty subset F of T is called a filter of T if the following conditions hold:*

(F1) $[xyz] \in F$ *for any* $x, y, z \in F$, *that is* F *is a ternary subsemigroup of* T;

(F2) for any $x, y \in T$, *if* $x \leq y$ *and* $x \in F$, *then* $y \in F$.

3. Implicative filters

We begin this section with characterizations of filters in implicative and commutative implicative n.p.o. ternary semigroups.

Theorem 2. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. A non-empty subset F of T is a filter if and only if it satisfies the following conditions:*

(F3) $1 \in F$;

(F4) for any $x, y, z \in T$, *if* $[xyz]^* \in F$ *and* $x, y \in F$, *then* $z \in F$.

Proof. Assume that F is a filter of T. Since 1 is the greatest element of T, $1 \in F$. Let $x, y, z \in T$ be such that $[xyz]^* \in F$ and $x, y \in F$. By $[xyz]^* \leq [xyz]^*$, $[[xyz]^*xy] \leq z$. By (F1), $[[xyz]^*xy] \in F$; hence $z \in F$ by (F2).

Conversely, assume that F satisfies (F3) and (F4). Let $x, y \in T$ such that $x \leq y$ and $x \in F$. By Theorem 1 (6), $[1xy]^* = 1$. Then $[1xy]^* \in F$ by (F3). Since $1, x \in F$ and (F4), $y \in F$. Thus F satisfies (F2). To show that F satisfies (F1), let $x, y, z \in F$. By Theorem 1 (2), $x \leq [yz[xyz]]^*$, and so by (F2) we get $[yz[xyz]]^* \in F$. This implies by (F4) that $[xyz] \in F$. Therefore, F is a filter of T.

Now we denote important elementary properties of a commutative implicative n.p.o. ternary semigroup.

Lemma 1. *If $(T, [\ ], \leq, [\ ]^*)$ is a commutative implicative n.p.o. ternary semigroup, then for any $x, y, z, u, v \in T$:*

(1) $[xy[zuv]^*]^* = [zu[xyv]^*]^*$.

(2) $x \leq [[xyz]^*yz]^*$.

Proof. (1) We have

$$[xy[zuv]^*]^* = [[xyz]uv]^* = [[zxy]uv]^* = [z[xyu]v]^* = [z[uxy]v]^* = [zu[xyv]^*]^*.$$

The first equality follows from Theorem 1 (7), and the second from commutativity.

(2) Since $[xyz]^* \leq [xyz]^*$, $[[xyz]^*xy] \leq z$. Using the commutativity, we have $[x[xyz]^*y] = [[xyz]^*xy] \leq z$. Thus $x \leq [[xyz]^*yz]^*$.

Theorem 3. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup and F be a non-empty subset of T. Then F is a filter of T if and only if it satisfies for all $x, y, z \in F$ and $u \in T$:*

(F5) $x \leq [yzu]^*$ *implies* $u \in F$.

Proof. Assume that F is a filter of T. Let $x, y, z \in F$ and $u \in T$ such that $x \leq [yzu]^*$. Then by Theorem 1 (6), $[1x[yzu]^*]^* = 1 \in F$. From $1, x \in F$, it follows by (F4) that $[yzu]^* \in F$. Since $y, z \in F$, $u \in F$.

Conversely, suppose F satisfies (F5). Since $x \leq [xx1]^*$ for all $x \in F$, we have $1 \in F$ by (F5). Let $x, y, z \in T$ such that $[xyz]^* \in F$ and $x, y \in F$. Note that $x \leq [[xyz]^*yz]^*$ by Lemma 1 (2). Applies (F5) with $(x, y, z, u) \mapsto (x, [xyz]^*, y, z)$, we get $z \in F$. Therefore, F is a filter of T.

Following, we give the definition of implicative filters.

Definition 2. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. A non-empty subset F of T is called an implicative filter of T if it satisfies (F3) and*

$$[xy[zuv]^*]^* \in F, [xyz]^* \in F, \text{ and } [xyu]^* \in F \text{ imply } [xyv]^* \in F$$

for all $x, y, z, u, v \in T$.

Example 1. *Let $T = \{1, 2, 3, 5, 7, 9\}$. Let us consider the implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$ with ternary multiplication $[\ ]$, ternary implication $[\ ]^*$, and order relation $\leq$ defined as follows:*

[ ]	1	2	3	5	7	9
11	1	2	3	5	7	9
12	2	3	3	7	9	9
13	3	3	3	9	9	9
15	5	7	9	5	7	9
17	7	9	9	7	9	9
19	9	9	9	9	9	9

[ ]	1	2	3	5	7	9
21	2	3	3	7	9	9
22	3	3	3	9	9	9
23	3	3	3	9	9	9
25	7	9	9	7	9	9
27	9	9	9	9	9	9
29	9	9	9	9	9	9

[ ]	1	2	3	5	7	9
31	3	3	3	9	9	9
32	3	3	3	9	9	9
33	3	3	3	9	9	9
35	9	9	9	9	9	9
37	9	9	9	9	9	9
39	9	9	9	9	9	9

[ ]	1	2	3	5	7	9
51	5	7	9	5	7	9
52	7	9	9	7	9	9
53	9	9	9	9	9	9
55	5	7	9	5	7	9
57	7	9	9	7	9	9
59	9	9	9	9	9	9

[ ]	1	2	3	5	7	9
71	7	9	9	7	9	9
72	9	9	9	9	9	9
73	9	9	9	9	9	9
75	7	9	9	7	9	9
77	9	9	9	9	9	9
79	9	9	9	9	9	9

[ ]	1	2	3	5	7	9
91	9	9	9	9	9	9
92	9	9	9	9	9	9
93	9	9	9	9	9	9
95	9	9	9	9	9	9
97	9	9	9	9	9	9
99	9	9	9	9	9	9

[ ]*	1	2	3	5	7	9
11	1	2	3	5	7	9
12	1	1	2	5	5	7
13	1	1	1	5	5	5
15	1	2	3	1	2	3
17	1	1	2	1	1	2
19	1	1	1	1	1	1

[ ]*	1	2	3	5	7	9
21	1	1	2	5	5	7
22	1	1	1	5	5	5
23	1	1	1	5	5	5
25	1	1	2	1	1	2
27	1	1	1	1	1	1
29	1	1	1	1	1	1

[ ]*	1	2	3	5	7	9
31	1	1	1	5	5	5
32	1	1	1	5	5	5
33	1	1	1	5	5	5
35	1	1	1	1	1	1
37	1	1	1	1	1	1
39	1	1	1	1	1	1

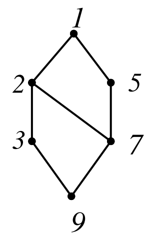
[ ]*	1	2	3	5	7	9
51	1	2	3	1	2	3
52	1	1	2	1	1	2
53	1	1	1	1	1	1
55	1	2	3	1	2	3
57	1	1	2	1	1	2
59	1	1	1	1	1	1

$[\ ]^*$	1	2	3	5	7	9
71	1	1	2	1	1	2
72	1	1	1	1	1	1
73	1	1	1	1	1	1
75	1	1	2	1	1	2
77	1	1	1	1	1	1
79	1	1	1	1	1	1

$[\ ]^*$	1	2	3	5	7	9
91	1	1	1	1	1	1
92	1	1	1	1	1	1
93	1	1	1	1	1	1
95	1	1	1	1	1	1
97	1	1	1	1	1	1
99	1	1	1	1	1	1

and

$$\begin{aligned}\leq := \{&(1,1),(2,2),(2,1),(3,3),(3,1),(3,2),(5,5),(5,1),(7,7),\\ &(7,1),(7,2),(7,5),(9,9),(9,1),(9,2),(9,3),(9,5),(9,7)\}.\end{aligned}$$



It is easy to verify that $F = \{1, 2, 3\}$ *is an implicative filter of* T.

Theorem 4. *For an implicative n.p.o. ternary semigroup* $(T, [\ ], \leq, [\ ]^*)$*, every implicative filter of* T *is a filter of* T.

Proof. Suppose F is an implicative filter of T. Let $x, y, z \in T$ such that $[xyz]^* \in F$ and $x, y \in F$. By Theorem 1 (1), $[11[xyz]^*]^* \in F$, $[11x]^* \in F$, and $[11y]^* \in F$; so $z = [11z]^* \in F$.

In general, the converse of Theorem 4 is not true.

Example 2. *Consider the implicative n.p.o. ternary semigroup* T *defined in Example 1. It is observed that* $\{1\}$ *is a filter of* T*, whereas* $\{1\}$ *is not an implicative filter of* T*. Indeed,* $[71[719]^*]^* = [712]^* = 1 \in \{1\}$, $[717]^* = 1 \in \{1\}$, *and* $[711]^* = 1 \in \{1\}$, *but* $[719]^* = 2 \notin \{1\}$.

Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. For $a \in T$, define

$$F(a) := \{x \in T : a \leq x\}.$$

Observe that $1, a \in F(a)$. In general, $F(a)$ is not a filter as shown in the following example.

Example 3. *Let us consider the implicative n.p.o. ternary semigroup* T *defined in Example 1. We have* $F(1) = \{1\}$, $F(2) = \{1, 2\}$, $F(3) = \{1, 2, 3\}$, $F(5) = \{1, 5\}$, $F(7) = \{1, 2, 5, 7\}$, *and* $F(9) = \{1, 2, 3, 5, 7, 9\} = T$. *It is observed that* $F(2) = \{1, 2\}$ *is not a filter of* T*, since* $[123]^* = 2 \in F(2)$ *and* $1, 2 \in F(2)$, *but* $3 \notin F(2)$.

Lemma 2. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. Then $F(a)$ is a filter of T for all $a \in T$ if and only if, for all $x, y, z, u \in T$,*

$$u \leq [xyz]^*,\ u \leq x,\ \text{and}\ u \leq y\ \text{imply}\ u \leq z.$$

Proof. Assume that $F(a)$ is a filter of T for all $a \in T$. Let $x, y, z, u \in T$ such that $u \leq [xyz]^*$, $u \leq x$ and $u \leq y$. Then $[xyz]^* \in F(u)$, $x \in F(u)$ and $y \in F(u)$. By assumption, $z \in F(u)$. This means that $u \leq z$.

Conversely, assume that $u \leq [xyz]^*$, $u \leq x$ and $u \leq y$ imply $u \leq z$ for all $x, y, z, u \in T$. Let $a \in T$. Clearly, $1 \in F(a)$. Let $[xyz]^* \in F(a)$ and $x, y \in F(a)$. Then $a \leq [xyz]^*$, $a \leq x$ and $a \leq y$. By assumption, $a \leq z$; so $z \in F(a)$.

Proposition 1. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. If $\{1\}$ is an implicative filter of T, then $F(a)$ is a filter of T for all $a \in T$.*

Proof. Suppose $\{1\}$ is an implicative filter of T. Let $a, x, y, z \in T$ such that $[xyz]^*, x, y \in F(a)$. Then $a \leq [xyz]^*$, $a \leq x$, and $a \leq y$. Thus, by Theorem 1 (6), $[a1[xyz]^*]^* = 1 \in \{1\}$, $[a1x]^* = 1 \in \{1\}$, and $[a1y]^* = 1 \in \{1\}$. By assumption and applying Definition 2 with $(x, y, z, u, v) \mapsto (a, 1, x, y, z)$, we obtain $[a1z]^* \in \{1\}$. Therefore, $[a1z]^* = 1$, that is, $a \leq z$ by Theorem 1 (6). Hence $z \in F(a)$.

Let F be a filter of an implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$. For $a, b \in T$, define

$$F_{ab} := \{x \in T : [abx]^* \in F\}.$$

By Theorem 1 (1), $F_{11} = F$.

The set F_{ab} may not be a filter.

Example 4. *Again, let us consider the implicative n.p.o. ternary semigroup T defined in Example 1. We have $F = \{1, 5\}$ is a filter of T. Note that $F_{11} = F_{15} = F_{51} = F_{55} = \{1, 5\}$, $F_{12} = F_{17} = F_{21} = F_{25} = F_{52} = F_{57} = F_{71} = F_{75} = \{1, 2, 5, 7\}$, and $F_{13} = F_{19} = F_{22} = F_{23} = F_{27} = F_{29} = F_{31} = F_{32} = F_{33} = F_{35} = F_{37} = F_{39} = F_{53} = F_{59} = F_{72} = F_{73} = F_{77} = F_{79} = F_{91} = F_{92} = F_{93} = F_{95} = F_{97} = F_{99} = \{1, 2, 3, 5, 7, 9\} = T$. It is observed that $F_{25} = \{1, 2, 5, 7\}$ is not a filter of T. In fact, $[253]^* = 2 \in F_{25}$ and $2, 5 \in F_{25}$, but $3 \notin F_{25}$.*

Theorem 5. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup, and let F be a filter of T. Then F is an implicative filter of T if and only if for any $a, b \in T$, the set F_{ab} is a filter of T.*

Proof. Assume that F is an implicative filter of T. Let $a, b \in T$. By Theorem 1 (4), $1 \leq [ab1]^* \leq 1$. Thus $[ab1]^* = 1 \in F$, that is, $1 \in F_{ab}$. Let $x, y, z \in T$ such that $[xyz]^*, x, y \in F_{ab}$. Then $[ab[xyz]^*]^* \in F$, $[abx]^* \in F$, and $[aby]^* \in F$. By assumption, $[abz]^* \in F$, so $z \in F_{ab}$. Hence F_{ab} is a filter of T.

Conversely, suppose that F_{ab} is a filter for all $a, b \in T$. Let $x, y, z, u, v \in T$ such that $[xy[zuv]^*]^* \in F, [xyz]^* \in F,$ and $[xyu]^* \in F$. Then $[zuv]^* \in F_{xy}$, $z \in F_{xy}$, and $u \in F_{xy}$. By assumption, $v \in F_{xy}$, that is, $[xyv]^* \in F$. Thus F is an implicative filter of T.

Example 5. *Consider the implicative n.p.o. ternary semigroup T defined in Example 1. We have $F = \{1, 2, 3\}$ is a filter of T. Furthermore, we have $F_{11} = F_{12} = F_{13} = F_{21} = F_{22} = F_{23} = F_{31} = F_{32} = F_{33} = \{1, 2, 3\}$ and $F_{15} = F_{17} = F_{19} = F_{25} = F_{27} = F_{29} = F_{35} = F_{37} = F_{39} = F_{51} = F_{52} = F_{53} = F_{55} = F_{57} = F_{59} = F_{71} = F_{72} = F_{73} = F_{75} = F_{77} = F_{79} = F_{91} = F_{92} = F_{93} = F_{95} = F_{97} = F_{99} = \{1, 2, 3, 5, 7, 9\} = T$. From $\{1, 2, 3\}$ and T are filters of T, it follows by Theorem 5 that $F = \{1, 2, 3\}$ is an implicative filter of T.*

4. Generalized implicative filters

We generalize the notion of implicative filter as follows.

Definition 3. *Let $(T, [\ ], \leq, [\ ]^*)$ be an implicative n.p.o. ternary semigroup. A non-empty subset F of T is called a generalized implicative filter of T if it satisfies the following conditions:*

(1) $[xyz] \in F$ for any $x, y, z \in F$, that is F is a ternary subsemigroup of T;

(2) if $x, y \in T$ and $z \in F$, then $[xyz]^* \in F$.

Example 6. *Let $T = \{1, 2, 3, 4\}$. Let us consider the implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$ with ternary multiplication $[\ ]$, ternary implication $[\ ]^*$, and order relation $\leq$ defined as follows:*

[ ]	1	2	3	4
11	1	2	3	4
12	2	4	4	4
13	3	4	4	4
14	4	4	4	4

[ ]	1	2	3	4
21	2	4	4	4
22	4	4	4	4
23	4	4	4	4
24	4	4	4	4

[ ]	1	2	3	4
31	3	4	4	4
32	4	4	4	4
33	4	4	4	4
34	4	4	4	4

[ ]	1	2	3	4
41	4	4	4	4
42	4	4	4	4
43	4	4	4	4
44	4	4	4	4

[ ]*	1	2	3	4
11	1	2	3	4
12	1	1	2	2
13	1	1	1	2
14	1	1	1	4

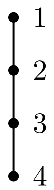
[ ]*	1	2	3	4
21	1	1	2	2
22	1	1	1	1
23	1	1	1	1
24	1	1	1	1

[ ]*	1	2	3	4
31	1	1	1	2
32	1	1	1	1
33	1	1	1	1
34	1	1	1	1

[ ]*	1	2	3	4
41	1	1	1	1
42	1	1	1	1
43	1	1	1	1
44	1	1	1	1

and

$$\leq = \{(1,1),(2,2),(2,1),(3,3),(3,1),(3,2)(4,4),(4,1),(4,2),(4,3)\}.$$



Then clearly $\{1,2,4\}$ *is a generalized implicative filter of* T *but not a filter of* T*, because* $4 \leq 3$ *and* $4 \in \{1,2,4\}$ *but* $3 \notin \{1,2,4\}$*.*

We have that generalized implicative filters need not be implicative filters.

Example 7. *Let us consider the implicative n.p.o. ternary semigroup* T *defined in Example 1. We have* $\{1\}$ *and* $\{1,5\}$ *are generalized implicative filters of* T *whereas the sets* $\{1\}$ *and* $\{1,5\}$ *are not implicative filters of* T*. As in Example 2,* $\{1\}$ *is not an implicative filter of* T*. By* $[21[123]^*]^* = [212]^* = 1 \in \{1,5\}$*,* $[211]^* = 1 \in \{1,5\}$*,* $[212]^* = 1 \in \{1,5\}$*, and* $[213]^* = 2 \notin \{1,5\}$*, it follows that* $\{1,5\}$ *is not an implicative filter of* T*.*

Proposition 2. *If* F *is a generalized implicative filter of an implicative n.p.o. ternary semigroup* $(T,[\ ],\leq,[\ ]^*)$*, then* $1 \in F$*.*

Proof. Let $x \in F$, by F is a generalized implicative filter of T, we have $1 = [xxx]^* \in F$.

Theorem 6. *For an implicative n.p.o. ternary semigroup* $(T,[\ ],\leq,[\ ]^*)$*, every filter is a generalized implicative filter of* T*.*

Proof. Assume that F is a filter of T; then F is a ternary subsemigroup of T. Let $x,y \in T$ and $z \in F$. Since $[zxy] \leq z$, $z \leq [xyz]^*$. By assumption, $[xyz]^* \in F$.

Corollary 1. *For an implicative n.p.o. ternary semigroup* $(T,[\ ],\leq,[\ ]^*)$*, every implicative filter of* T *is a generalized implicative filter of* T*.*

Proof. The assertion follows by Theorem 4 and Theorem 6.

In the following, a sufficient condition is derived for a generalized implicative filter to become a filter.

Theorem 7. *Let* $(T,[\ ],\leq,[\ ]^*)$ *be an implicative n.p.o. ternary semigroup. Then every generalized implicative filter* F *of* T *is a filter if the following condition satisfies: For all* $x,y \in T$*,*

$$x \in F \quad \textit{and} \quad [[xxx]1y]^* \in F \quad \textit{or} \quad [[xx1]xy]^* \in F \implies y \in F.$$

Proof. Let F be a generalized implicative filter of T. There are two cases to consider.

Case 1: Assume that $x \in F$ and $[[xxx]1y]^* \in F$ imply $y \in F$ for all $x, y \in T$. Clearly, F is a ternary subsemigroup of T. Let $x, y \in T$ such that $x \in F$ and $x \leq y$. Then by Theorem 1 (6), $[x1y]^* = 1 \in F$. Since $x \in F$, $[x1y]^* \in F$, and F is a generalized implicative filter of T, we get that $[xx[x1y]^*]^* \in F$. Then by Theorem 1 (7),

$$[xx[x1y]^*]^* = [[xxx]1y]^*,$$

and from assumption, we have $y \in F$.

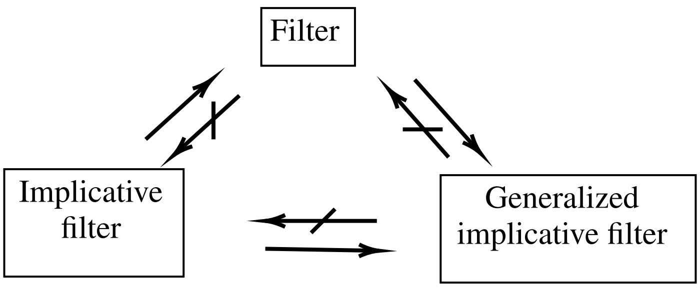
Case 2: Assume that $x \in F$ and $[[xx1]xy]^* \in F$ imply $y \in F$ for all $x, y \in T$. Clearly, F is a ternary subsemigroup of T. Let $x, y \in T$ such that $x \in F$ and $x \leq y$. Then by Theorem 1 (6), $[1xy]^* = 1 \in F$. Since $x \in F$, $[1xy]^* \in F$, and F is a generalized implicative filter of T, we get that $[xx[1xy]^*]^* \in F$. Then by Theorem 1 (7),

$$[xx[1xy]^*]^* = [[xx1]xy]^*,$$

and from assumption, we have $y \in F$.

5. Conclusions

In this paper, we consider filters, implicative filters and generalized implicative filters on implicative n.p.o ternary semigroups. In Section 3, we give characterizations of filters in implicative and commutative implicative n.p.o. ternary semigroups (see Theorem 2 and Theorem 3). We introduce the notion of implicative filters of implicative n.p.o. ternary semigroups (see Definition 2). An example of implicative filter is also established. Then we show that every implicative filter is a filter and give an example to show that the converse is not true in general. Finally, we state some equivalent conditions for an implicative filter by using a particular set defined by a filter. Indeed, let F be a filter of an implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$. For $a, b \in T$, define $F_{ab} := \{x \in T : [abx]^* \in F\}$. We obtain that a filter F is an implicative filter if and only if for any $a, b \in T$, the set F_{ab} is a filter of T. In Section 4, we define a generalized implicative filters on implicative n.p.o ternary semigroups. Then we consider relationships among filters, implicative filters and generalized implicative filters. We have that every filter is a generalized implicative filter and every implicative filter is also a generalized implicative filter. The converse of these statement is not generally true. Now, we conclude the connections of filters, implicative filters, and generalized implicative filters as the picture.



Acknowledgements

The Research on "Implicative filters of implicative negatively partially ordered ternary semigroups" by Khon Kaen University has received funding support from the National Science, Research and Innovation Fund (NSRF).

References

[1] M. W. Chan and K. P. Shum. Homomorphisms of implicative semigroups. *Semigroup Forum*, 46:7–15, 1993.

[2] Y.B. Jun. Implicative ordered filters of implicative semigroups. *Communications of the Korean Mathematical Society*, 14(1):47–55, 1999.

[3] M. Sambasiva Rao and K. P. Shum. On filters of implicative npo semigroups. *Asian-European Journal of Mathematics*, 5(03):1250044, 2012.

[4] K. Nakwan, P. Luangchaisri, and T. Changphas. Implicative negatively partially ordered ternary semigroups. *European Journal of Pure and Applied Mathematics.*, 17(4):4180–4194, 2024.