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Domination in Rough m-Polar Fuzzy Digraphs Based on Trade Networking

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Abstract. In graph theory, dominance has been a key idea for examining influence, control, and optimization in various systems. By examining domination in rough m-polar fuzzy digraphs—a hybrid model that combines directed graphs, m-polar fuzzy logic, and rough set theory—this work presents a fresh expansion of this idea. This structure makes it possible to simulate multi-polar decision contexts, uncertainty, and imprecision in a single framework. We justify the requirement for a more sophisticated concept of domination in rough m-polar fuzzy digraphs by first going over the fundamental concepts behind them. After a formal definition of domination in this context is put forward, its basic characteristics and structural ramifications are thoroughly examined. After that, the study concentrates on two crucial operations: the strong product of rough m-polar fuzzy digraphs and the tensor product. We characterize these procedures in the new framework and examine their effects on the resulting graphs' dominating parameters. Under multi-valued and uncertain circumstances, these operations help to generalize the relationships and interactions amongst intricate network topologies. A thorough numerical example is given to illustrate the suggested notions' applicability in real-world situations. Lastly, the use of domination in rough m-polar fuzzy digraphs is examined, emphasizing its potential in practical contexts like information systems, social network analysis, and uncertain decision-making. The study's conclusions open up new possibilities for using domination theory in dynamic and unpredictable contexts and further the theoretical development of fuzzy and rough graph models as process innovation.
2020 Mathematics Subject Classifications: 05C69, 05C72, 03E72, 68R10
Key Words and Phrases: Domination in rough m-polar fuzzy digraphs, Tensor Product of Rough m-Polar Fuzzy Digraphs, application.

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1. Introduction

Due to recent advances in science and technology, traditional mathematical tools are not sufficient for dealing with the complex problems arising in our real world day by day [1–5]. To address these increasing challenges, novel and innovative mathematical tools are needed. One of the greatest issues in the universe uncertainty [6], making it difficult to deal with certain complex problems even with typical crisp methods [7]. The development of graph theory was sparked by the Konigsberg Bridge Problem in 1735. Following his investigation into the Konigsberg problem, Euler developed a structure known as an Eulerian graph, which answered the problem. Zadeh [8] introduced the fuzzy set theory that provides information on the likelihood that an element will be included in the target set based on a given property. Kauffman [9] proposed fuzzy graphs in 1973. In 1975, Rosenfeld described fuzzy graph structure. Later on, Bhattacharya [10] added additional remarks about fuzzy graphs and [11]. Mordeson and Chang Shyh cite [12] defined several operations (union, Cartesian product, composition, and join) on fuzzy graphs in 1994. Mordeson et al. [13] developed a few operations on fuzzy graphs. Nagoorgani et al. [14] talked about a fuzzy graph's complement characteristics. Nagoorgani and Latha [15] talked about fuzzy graph irregularities. Bipolar fuzzy sets and relations were initially introduced by Zhang [16] in 1994. Bipolar fuzzy sets was fuzzy set extensions with membership degrees spanning from [1, 1]. The membership degree of the object is shown as (0, 1) if it meets a specific criterion, and as [1, 0) if it satisfies the counter attribute. Negative data shows what is formally deemed to be unfeasible, whereas positive data shows what is perceived to be achievable. Bipolar, two-sided thinking and judgments on both positive and negative factors cause a wide range of decision-making problems. Recent research by Chen et al. [17] expanded the idea of bipolar fuzzy sets to m-polar fuzzy sets. In an m-polar fuzzy set, a membership value is located between [0, 1] . Data from n-agents (n 2), or multipolar information not well represented by the current graphs (e.g., bipolar fuzzy graphs, which correlate to two-valued logic, or fuzzy graphs that correspond to single-valued logic), was a common occurrence in many real-world scenarios. One method using only one parameter is fuzzy set theory. Rough Set Theory [18, 19] used a general mathematical approach. In situations where we need to manipulate data based on a collection of attributes, we apply rough set theory. The foundation of rough set theory is the idea that each item in the set has a certain property. A rough set is made up of two upper approximations and two lower approximations that are determined by this relation. The target set contains the lower approximation, and it may also contain the upper approximation. According to rough set theory, each element's objective approximation in the target set can be understood as a measure of how much, in terms of the information conveyed by the given relation, the element belongs to the target set. The boundary region of the rough set is the difference between the upper and lower approximations. Studying rough sets and fuzzy sets, Dubois and Prade [20] found that while these are two distinct methods of handling vagueness, they are not mutually exclusive. Akram [21] was the first to suggest bipolar fuzzy graphs. A digraph is a basic graph with directed edges. An arc connecting a vertex w to another vertex z indicates that one can travel [22–26] and other used in fuzzy sets [27–35] from w to z but not from z to w. Arrows on the edges also convey directional information. In 1986, Wu [36] introduced fuzzy digraphs. Akram et al. [24] demonstrated the innovative uses of intuitionistic FDs in decision support systems. In 1990, Dubois and Prade introduced fuzzy rough sets and rough fuzzy sets based on the findings of their work. While fuzzy rough sets use fuzzy relations, rough fuzzy sets use crisp relations to approximate fuzzy sets. In 1996, Pawlak [18] established rough relations. In 2010, Feng et al. [37] presented the concepts of soft rough sets and soft rough fuzzy sets. In 2015, Zhang et al. [6] presented the union and intersection operations on rough sets. Akram et al. [26] covered Menger's theorem for m-polar fuzzy graphs and their applications to road networks. Anitha et al. [38] introduced metric dimensions and associated mathematical features to rough graphs in 2021. Prassanna et al. [39] examined the uncertainty in soft graphs and presented the soft covering-based rough graphs. Ishfaq et al. [40] introduced the rough neutrosophic digraphs, which approximate the neutrosophic set under the effect of a crisp equivalency relation. The operations of fuzzy and rough approximations covered in [41–43]. In relation to conjugacy classes, Talebi and Amiri [44] investigated vertex Pseudo-Cayley graphs and lower and higher approximations of Cayle graphs' edges. El-Atik et al. [45] covered the topological visualization of rough sets by neighbors and a heart application based on graphs, [46] illustrates how rough and fuzzy sets are combined, and how they are directed to graphs that include concepts related to either fuzzy or rough sets. The domination theory of graphs has been a rich subject of study with many practical applications in several domains [9, 12, 34, 36, 47–56]. It solved the problem of whether there conceivable tournament on n(k) vertices in which there is a vertex v that dominates all vertices for any set Q of k vertices. it was first explored in complete digraphs. The proposed concept of dominance theory in graphs. Somasundaram [42] presented the concept of fuzzy graph dominance. Since then, the topic's applicability to actual issues has encouraged researchers to focus on it. In fuzzy graphs, the dominance number is determined using many definitions. In a fuzzy graph, the dominating set was determined by applying. The definition of the dominance number, according to Xaviors et al. [57] based on taking the dominating set (DS) with the fewest elements and adding up all of their individual membership values. Manjusha and Sunitha [49] introduced the notions of strong domination and total domination in fuzzy graphs and changed the definition of dominance number in fuzzy graphs by considering strong arc weights. Nagoorgani et al. [50] explained the intuitionistic fuzzy graphs and double domination. Enriquez et al. [58] examined a few unique directed fuzzy graphs and calculated their dominance numbers. Solving network-related problems has greatly benefited from the use of dominating sets and dominating numbers. Further, the article is structured as in: Section 2 also presents the new basic ideas. The features of the Domination in Rough m-Polar Fuzzy Digraphs are defined in Section 3. The tensor product of Rough m-Polar Fuzzy Digraphs are defined in section 4. The Strong Product of Rough m-Polar Fuzzy Digraphs and its characteristics are defined in Section 5. In Section 6, a numerical example is shown. I've finished section 7

1.1. Motivation

Complex systems in the modern world, including social networks, communication systems, decision-making processes, and biological systems, are inherently ambiguous and unclear. When it comes to dealing with such ambiguities, classical graph theory frequently fails. Rough sets and fuzzy sets have been used separately to fill this gap, but when combined to create rough m-polar fuzzy digraphs, they provide a more potent and adaptable paradigm for concurrently capturing multi-attribute and uncertain data.
An understanding of dominance in rough m-polar fuzzy digraphs is essential for managing or influencing these intricate systems with little funding. Furthermore, by building larger and more complex models from simpler components, the investigation of tensor and strong products of these structures facilitates modular analysis and applications.
Not only is the theoretical analysis of these goods and dominance theories important from an academic standpoint, but it also has real-world implications. For instance, knowing tensor and strong products can assist in describing the propagation of information or influence more precisely in social networks, and recognizing dominating sets can aid in optimal monitoring in network security.

Thus, the goal of this work is to investigate domination and product operations in order to further the theoretical underpinnings of rough m-polar fuzzy digraphs and show how these ideas can be successfully implemented in real-world situations involving uncertainty, complexity, and multi-dimensional relationships.

1.2. Problem statement

Uncertainty, ambiguity, and multifaceted interactions are common in real-world systems like biological structures, communication networks, and decision-making frameworks. Such complexities are beyond the scope of conventional graph theory and even typical fuzzy graph models. By fusing the advantages of directed graphs, m-polar fuzzy sets, and rough sets, rough m-polar fuzzy digraphs become a more sophisticated model that makes it possible to depict uncertain and multifaceted systems more flexibly and realistically.
Several fundamental features of rough m-polar fuzzy digraphs are still poorly understood, despite their potential. Specifically, there hasn't been enough research done on the idea of dominance, which is essential for maximizing control, resource distribution, and impact in networks. Furthermore, the behavior and features of graph products, such as tensor and strong products, within rough m-polar fuzzy digraphs have not yet been properly defined and investigated, even though they are essential for constructing larger networks from simpler ones. Furthermore, there aren't many useful frameworks that directly apply the theory of domination in rough m-polar fuzzy digraphs to actual issues. Both the theoretical development of graph theory and the creation of useful applications that can successfully handle complexity and uncertainty in contemporary systems depend on filling in these gaps.

Therefore, this research seeks to:

Find out what domination in rough m-polar fuzzy digraphs is.

Create and evaluate rough m-polar fuzzy digraphs' tensor and strong products, and Examine how dominance is used in these kinds of structures in the actual world.

1.3. Objective of the paper

This study's main goals are to define and develop the idea of domination within the context of rough m-polar fuzzy digraphs and to identify basic characteristics and attributes.
To build and codify the rough m-polar fuzzy digraph tensor product and examine its structural characteristics in light of network dynamics and dominance.
To build and define the strong product of rough m-polar fuzzy digraphs and investigate how it affects the digraphs' connectedness and complexity.
To investigate and suggest useful uses of domination in rough m-polar fuzzy digraphs in situations involving uncertainty and multiple attributes.
To offer case studies and instructive examples that highlight the applicability and significance of the theoretical findings in real-world contexts.

1.4. Gap in the existing research

Even though fuzzy graph theory, rough set theory, and m-polar fuzzy graphs have all advanced significantly, the combination of these ideas into rough m-polar fuzzy digraphs is still very new and undeveloped.
Specifically, the idea of dominance, which is essential for examining control, influence, and optimization in networks has been fully examined in the context of fuzzy and classical graphs, but not in the context of rough m-polar fuzzy digraphs.
Additionally, while graph products like the tensor product and strong product have been established and applied in both traditional and fuzzy graph contexts, there are no formal definitions, theoretical advancements, or thorough analyses for their application to rough m-polar fuzzy digraphs.
Research examining the behavior of these goods in the rough m-polar fuzzy environment and their effects on domination-related features is conspicuously lacking.
Furthermore, there is little application-driven research linking domination in rough m-polar fuzzy digraphs to real-world issues with uncertainty and multi-dimensional characteristics, like social networks, communication systems, and decision support systems, despite the theoretical significance.
As a result, there is a glaring lack of comprehensive theory about domination and graph products in rough m-polar fuzzy digraphs, as well as evidence of their usefulness in intricate, unpredictable settings.

1.5. Novelty

Introduction of Domination Concepts: This work offers novel theoretical underpinnings and properties that have not been covered in previous research, making it one of the first to define and investigate domination explicitly in the context of rough m-polar fuzzy digraphs.

Development of Tensor and Strong Products: By expanding fuzzy graph and classical product notions into a more intricate and realistic context of multi-polar fuzziness and roughness, the work suggests and formulates tensor product and strong product operations for rough m-polar fuzzy digraphs.
Relationship Between Domination and Graph Products: It provides profound insights into the structure and complexity of integrated networks represented by rough m-polar fuzzy digraphs by investigating, for the first time, how domination acts under these recently established graph products.
Application-Based Methodology: In contrast to strictly theoretical research, this work bridges the gap between theory and practice by applying domination in rough m-polar fuzzy digraphs to real-world uncertain systems in addition to developing new mathematical models.
Case Studies and Illustrative Examples: The research offers thorough case studies and examples to support the suggested notions, improving comprehension and proving the produced theories' applicability in real-world situations.

2. Basic ideas

Definition 2.1. [8] If X is a universe of discourse and x is a particular element of X, then a fuzzy set A defined on X can be written as a collection of ordered pairs:

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

where $\mu_A(x) : X \to [0, 1]$ is called the membership function.
Example 2.2. Let $X = \{S_1, S_2, S_3, S_4, S_5\}$ be the reference set of students. Let $\tilde{A}$ be the fuzzy set of intelligent students, where *intelligent* is a fuzzy term.

$$\tilde{A} = \{(S_1, 0.4),\ (S_2, 0.5),\ (S_3, 1),\ (S_4, 0.9),\ (S_5, 0.8)\}$$

Here, $\tilde{A}$ indicates that the intelligence of S_1 is 0.4, S_2 is 0.5, and so on.

Definition 2.3. [39] Assume that $y, z \in V$. If $\mu_D(x, y)$ is an effective arc, then the vertex $\sigma_D(x)$ dominates in $\sigma_D(y)$ in G_D.
Example 2.4. Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the set of vertices $V = \{a, b, c, d\}$. Also, let the membership values of vertices and edges be given as:

$$\sigma(a) = 0.6, \quad \sigma(b) = 0.4, \quad \sigma(c) = 0.7, \quad \sigma(d) = 0.9$$

$$\mu(ab) = 0.4, \quad \mu(bc) = 0.2, \quad \mu(cd) = 0.6, \quad \mu(da) = 0$$

In this graph, ab and ac are effective arcs because vertex a dominates vertices b and c.

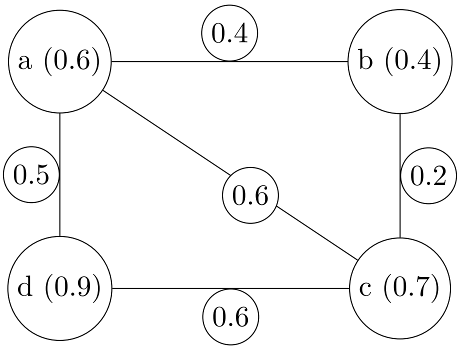


Figure 1: A weighted undirected graph with nodes labeled with values and edges showing weights.

Definition 2.5 [3] Let U be a non-empty universe. Let $R \subseteq U \times U$ be an indiscernibility relation on U. Further, we assume that R is an equivalence relation. A pair (X, R) is called an *approximation space.*

Let A^* be a subset of U and characterized by the lower and upper approximations, respectively, as follows:

$$\underline{R}_B = \{x \in U \mid [x]_R \subseteq A^*\},$$
$$\overline{R}^B = \{x \in U \mid [x]_R \cap A^* \neq \emptyset\},$$

where $[x]_R$ denotes the equivalence class containing x.

The pair of the lower and upper approximations $(\underline{R}_B, \overline{R}^B)$ is called a *rough set.*

Definition 2.6 [20] Let T be a fuzzy equivalence relation on U and let U be a universe. Consider a fuzzy set A on U. We define:

$$(\underline{T}A)(w) = \bigwedge_{y \in U} \left[(1 - T(w, x)) \vee A(x)\right],$$
$$(\overline{T}A)(w) = \bigvee_{y \in U} \left[T(w, x) \wedge A(x)\right], \quad \forall w \in U,$$

as the upper and lower approximations of A under T, designated as $\underline{T}_A$ and $\overline{T}_A$, respectively. If $\underline{T}_A - \overline{T}_A \neq 0$, the pair $(\underline{T}_A, \overline{T}_A)$ is referred to as a *fuzzy rough set.*

Definition 2.7 [22] A fuzzy rough digraph on a set X that is not empty is a four-ordered tuple $G = (A, T_A, B, H_B)$, where:

(i) T is a fuzzy tolerance relation on X.

(ii) H is a fuzzy tolerance relation on $B^* \subseteq U \times U$.

(iii) $T_A = (\underline{T}_A, \overline{T}_A)$ is a fuzzy rough set on X.

(iv) $H_B = (\underline{H}_B, \overline{H}_B)$ is a fuzzy rough relation on X.

(v) $\underline{G} = (\underline{T}_A, \underline{H}_B)$ and $\overline{G} = (T_A, H_B)$ are fuzzy digraphs, where G represents the lower approximation of G and the upper approximation is represented by G, such that:

(vi) $\underline{G} = (\underline{T}_A, \underline{H}_B)$ and $\overline{G} = (T_A, H_B)$ are fuzzy digraphs, where G represents the lower approximation of G and the upper approximation is represented by G, such that:

(vii) $\underline{G} = (\underline{T}_A, \underline{H}_B)$ and $\overline{G} = (\overline{T}_A, \overline{H}_B)$ are fuzzy digraphs, where G represents the lower approximation of G and the upper approximation is represented by G, such that:

$$(\underline{H}_B)(xy) \leq \min\{(\underline{T}_A)(x), (\underline{T}_A)(y)\}, (\overline{H}_B)(xy) \leq \max\left\{(\overline{T}_A)(x), (\overline{T}_A)(y)\right\}, \quad \forall xy \in B^*.$$

Definition 2.8 [41] The strength of connectedness of rough fuzzy digraph $G = (\underline{G}, \overline{G})$ between two vertices z_0 and z_1 ($\mathrm{CONN}_{\underline{G}(z_0,z_1)}$) is the maximum value of strength of all paths from z_0 to z_1 both in $\underline{G}$ and $\overline{G}$.

Definition 2.9 [2] Let $G = (G, \overline{G})$ and let $w, y \in X$. If there is a strong arc $wy \in SE$ in G, then the vertex w dominates y in G. In other words, an arc in G that runs from w to y such that:

$$\mathrm{CONN}_{\underline{G}^{-wy}(wy) \leq \underline{S}E(wy)}.$$

Similarly, if there is a strong arc $wy \in SE$ in $\overline{G}$, then w dominates y. In other words, an arc in $\overline{G}$ that runs from w to y such that:

$$\mathrm{CONN}_{\overline{G}}^{-wy}(wy) \leq \overline{S}E(wy).$$

If there is a strong arc wy in both G and $\overline{G}$, then the vertex w is said to dominate y in $G = (G, \overline{G})$.

Definition 2.10 [17] An m-polar fuzzy set (or $[0,1]^m$-set) on X is exactly a mapping $A : X \to [0,1]^m$.

Definition 2.11 [54]
Let $G = (V, E)$ be a pair representing an m-polar fuzzy graph on a non-empty set X, where $C : X \to [0,1]^m$ is an m-polar fuzzy set on the set of vertices X and $D : X \times X \to [0,1]^m$ is an m-polar relation such that:

$$p_i D(xy) \leq \inf\{p_i C(x), p_i C(y)\}, \quad \text{and for every } xy \in E, \quad D(xy) = 0,$$

for every $xy \in X \times X - E$, where $0 = (0, 0, \ldots, 0)$ and $E \subseteq X \times X$ is the set of edges.

Example 2.12 Let us consider the graph $G = (V, E)$ where $V = \{K, L, M\}$ and $E = \{KL, LM, MK\}$. A three-polar fuzzy graph G is shown in the figure.

3. Domination in Rough m-Polar Fuzzy Digraphs

We have defined a variety of concepts in this section, such as path, strength, strength of connectedness, strong arc, dominating set, minimal dominating set, cardinality of set and domination number of Rough m-Polar Fuzzy Digraphs.

Definition 3.1 A rough m-polar fuzzy path in a rough m-polar fuzzy digraph $G = (p_i\underline{G}, p_i\overline{G})$ on a nonempty set Y is a directed path $P : y_0 \to y_1 \to y_2 \to \cdots \to y_m$ of strength m from y_0 to y_m both in $\underline{G}$ and $\overline{G}$.

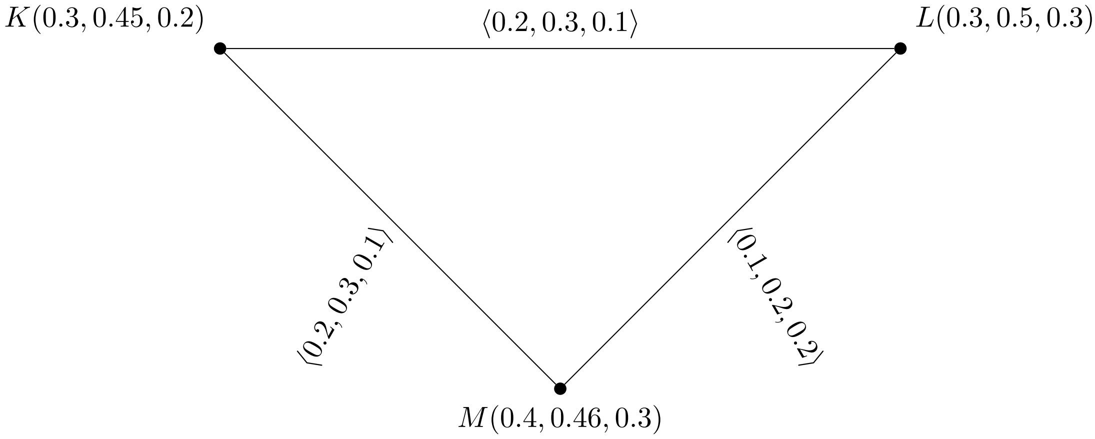


Figure 2: m-polar fuzzy graph.

Definition 3.2 The strength of a rough m-polar fuzzy digraph path P in $(p_i\underline{G}, p_i\overline{G})$ is defined as the membership values of the weakest arc in $(p_i\underline{G}, p_i\overline{G})$, respectively.

Definition 3.3 The strength of connectedness of rough m-polar fuzzy digraph $G = (p_i\underline{G}, p_i\overline{G})$ between two vertices y_0 and y_1, denoted as $\mathrm{CONN}_{\overline{G}(y_0,y_1)}$, is the maximum value of strength of all paths from y_0 to y_1 both in $\underline{G}$ and $\overline{G}$.

Definition 3.4 An arc yz in a rough m-polar fuzzy digraph $G = (p_i\underline{G}, p_i\overline{G})$ is said to be a strong arc in $(p_iG, p_i\overline{G})$ if:

$$\mathrm{CONN}G_y z(yz) \leq SE(yz), \quad \text{and} \quad \mathrm{CONN}_{\overline{G}}^{-yz}(yz) \leq SE(yz),$$

respectively. An arc is said to be a strong arc in $G = (p_iG, p_i\overline{G})$ if it is a strong arc both in $\underline{G}$ and $\overline{G}$.

Definition 3.5 Let $x, y \in Y$ in $G = (p_i\underline{G}, p_i\overline{G})$. The vertex x dominates y in G if there is a strong arc $xy \in SE$ in G, i.e.:

$$\mathrm{CONN}_{G}^{-xy}(xy) \leq SE(xy).$$

Similarly, vertex x dominates y in $\overline{G}$ if there is a strong arc $xy \in SE$ in $\overline{G}$, i.e.:

$$\mathrm{CONN}_{\overline{G}}^{-xy}(xy) \leq SE(xy).$$

The vertex x is said to dominate y in $G = (p_iG, p_i\overline{G})$ if there is a strong arc both in G and $\overline{G}$.

Example 3.6 Let $G = (p_iG, p_i\overline{G})$ be a rough m-polar fuzzy digraph. We compute $\mathrm{CONN}_{G}^{-xy}(xy)$ and $\mathrm{CONN}_{\overline{G}}^{-xy}(xy)$ for each pair of vertices of $(p_iG, p_i\overline{G})$.

$$\mathrm{CONN}_{G}^{-ab}(a,b) = 0 \leq 0.1, 0.3, 0.7 = SE(a,b)$$
$$\mathrm{CONN}_{G}^{-bc}(b,c) = 0 \leq 0.09, 0.08, 0.29 = SE(b,c)$$
$$\mathrm{CONN}_{G}^{-cd}(c,d) = 0 \leq 0.11, 0.22, 0.33 = SE(c,d)$$

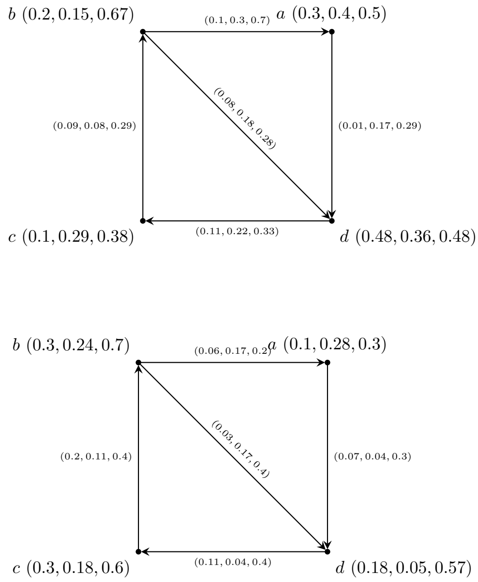


Figure 3: RmPFD $G = (p_iG,\ p_i\bar{G})$

$$\mathrm{CONN}_G^{-da}(d,a) = 0 \leq 0.01, 0.17, 0.27 = SE(d,a)$$
$$\mathrm{CONN}_G^{-ac}(a,c) = 0.1 \wedge 0.09 = 0.09 \quad (\text{which is } \geq 0.08)$$
$$= 0.3 \wedge 0.08 = 0.08 \leq 0.18$$
$$= 0.7 \wedge 0.29 = 0.29 > 0.28$$
$$\mathrm{CONN}_G^{-ac}(a,c) = 0 \leq 0.08, 0.18, 0.28 = SE(a,c)$$

The strong arcs of G are ab, bc, cd, da.

$$\mathrm{CONN}_{\overline{G}}^{-ab}(a,b) = 0 \leq 0.06, 0.17, 0.2 = SE(a,b)$$
$$\mathrm{CONN}_{\overline{G}}^{-bc}(b,c) = 0 \leq 0.2, 0.11, 0.4 = SE(b,c)$$
$$\mathrm{CONN}_{\overline{G}}^{-cd}(c,d) = 0 \leq 0.11, 0.04, 0.4 = SE(c,d)$$
$$\mathrm{CONN}_{\overline{G}}^{-da}(d,a) = 0 \leq 0.07, 0.04, 0.3 = SE(d,a)$$
$$\mathrm{CONN}_{\overline{G}}^{-ac}(a,c) = 0.06 \wedge 0.2 = 0.06 \quad (\text{which is } > 0.03)$$
$$= 0.17 \wedge 0.11 = 0.11 \leq 0.17$$
$$= 0.2 \wedge 0.4 = 0.2 \leq 0.4$$
$$\mathrm{CONN}_{\overline{G}}^{-ac}(a,c) = 0 \leq 0.03, 0.17, 0.4 = SE(a,c)$$

The strong arcs of $\overline{G}$ are ab, bc, cd, da. Thus, these are strong arcs of $(p_iG, p_i\overline{G})$. Therefore, the vertex a dominates b, b dominates c, c dominates d, and vertex d dominates a.

Definition 3.7 Let $G = (p_iG, p_i\overline{G})$, then the cardinality of a set w of vertices in G is defined as the sum of the membership values of $u \in w$ in G and the sum of the membership values of $u \in w$ in $\overline{G}$. Specifically:

$$|W(G)| = \sum_{u\in w} RW(u),$$

$$|W(G)| = \sum_{u\in w} RW(u), \quad i = 1, 2, 3, \ldots, m.$$

Example 3.8 Let the graph G be as defined in the example.

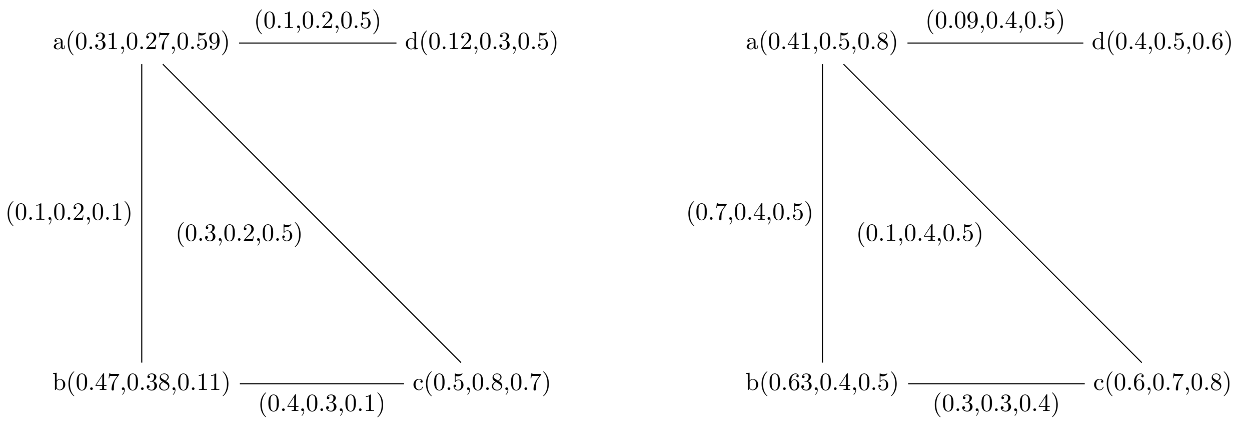


Figure 4: RmPFD $G = (p_iG, p_i\overline{G})$

The cardinality of the set $\{a, c\}$ in G is:

$$|\{a, c\}| = 0.31 + 0.27 + 0.59 + 0.5 + 0.8 + 0.7 = 3.17$$

The cardinality of the set $\{a, c\}$ in $\overline{G}$ is:

$$|\{a, c\}| = 0.41 + 0.5 + 0.8 + 0.6 + 0.7 + 0.8 = 3.81$$

Definition 3.9 Let $G = (p_iG, p_i\overline{G})$ be a rough m-polar fuzzy digraph. A subset p_iC_G of RF is said to be a lower dominating set (DS) of G if for every $w \in p_i(RF - C_G)$, there exists $y \in p_iC_G$ such that y dominates w. A subset p_iC_G of RF is said to be an upper dominating set if for every $w \in p_i(RF - C_G)$, there exists $y \in p_iC_G$ such that y dominates w. A set $D(G)$ of vertices is a dominating set (DS) of $G = (p_iG, p_i\overline{G})$ if it is both an upper and lower DS.

Example 3.10 Let $G = (p_iG, p_i\overline{G})$ be a rough m-polar fuzzy digraph. We compute $\mathrm{CONN}_G^{-xy}(xy)$ and $\mathrm{CONN}_{\overline{G}}^{-xy}(xy)$ for each pair of vertices of $(p_iG, p_i\overline{G})$.

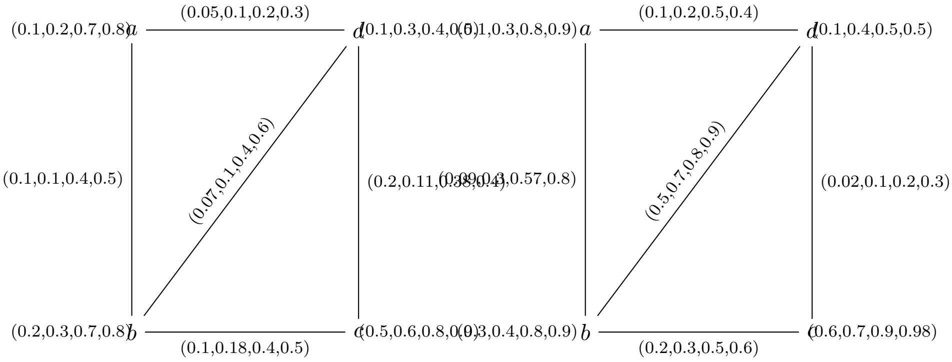


Figure 5: G = $(p_iG,\, p_i\bar{G})$

$$\mathrm{CONN}_G^{-ab}(a,b) = 0 \leq 0.1, 0.1, 0.4, 0.5 = SE(a,b)$$

$$\mathrm{CONN}_G^{-bc}(b,c) = 0 \leq 0.1, 0.18, 0.4, 0.5 = SE(b,c)$$

$$\mathrm{CONN}_G^{-cd}(c,d) = 0 \leq 0.02, 0.11, 0.38, 0.4 = SE(c,d)$$

$$\mathrm{CONN}_G^{-da}(d,a) = 0 \leq 0.05, 0.1, 0.2, 0.3 = SE(d,a)$$

$$\mathrm{CONN}_G^{-ac}(a,c) = 0.1 \wedge 0.02 = 0.02, \quad 0.18 \wedge 0.11 = 0.11, \quad 0.4 \wedge 0.38 = 0.38, \quad 0.5 \wedge 0.4 = 0.4$$

$$\mathrm{CONN}_G^{-ac}(a,c) = 0.1, 0.18, 0.4, 0.5 \nleq 0.07, 0.1, 0.4, 0.6 = SE(a,c)$$

The strong arcs of G are ab, bc, cd, da, while bd is not a strong arc. So, the lower dominating set is $\{a,b,c,d\}$.

$$\mathrm{CONN}_{\overline{G}}^{-ab}(a,b) = 0 \leq 0.09, 0.3, 0.57, 0.8 = SE(a,b)$$

$$\mathrm{CONN}_{\overline{G}}^{-bc}(b,c) = 0 \leq 0.2, 0.3, 0.5, 0.6 = SE(b,c)$$

$$\mathrm{CONN}_{\overline{G}}^{-cd}(c,d) = 0 \leq 0.02, 0.1, 0.2, 0.3 = SE(c,d)$$

$$\mathrm{CONN}_{\overline{G}}^{-da}(d,a) = 0 \leq 0.1, 0.2, 0.5, 0.4 = SE(d,a)$$

$$\mathrm{CONN}_{\overline{G}}^{-bd}(b,d) = 0.2 \wedge 0.02 = 0.02, \quad 0.3 \wedge 0.1 = 0.1, \quad 0.5 \wedge 0.2 = 0.2, \quad 0.6 \wedge 0.3 = 0.3$$

$$\mathrm{CONN}_{\overline{G}}^{-bd}(b,d) = 0.02, 0.1, 0.2, 0.3 \nleq 0.5, 0.7, 0.8, 0.2$$

The strong arcs of $\overline{G}$ are ab, bc, cd, da, while bd is not a strong arc. So, the upper dominating set is $\{a,b,c,d\}$.

The dominating sets in this example are:

$$\{a,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\} \text{ (the entire graph).}$$

Definition 3.11 A dominating set (DS) is called a lower minimal dominating set and upper minimal dominating set of $G = (p_iG, p_i\overline{G})$ if there are no dominating sets that are proper subsets of it in $p_i(G, G)$, respectively. A minimal dominating set which is both a lower and upper minimal DS is called a minimal DS of $G = (p_iG, p_i\overline{G})$. A minimal DS $D(G)$ is a minimal DS of $G = (p_iG, p_i\overline{G})$ for which the sum of fuzzy cardinality is:

$$|C(G)| + |C(G)| = \sum_{w \in C(G)} RV(W) + \sum_{w \in C(G)} RV(W)$$

Example 3.12
Let $G = (p_iG, p_i\overline{G})$ be a rough m-polar fuzzy digraph. We compute $\text{CONN}_G^{-xy}(xy)$ and $\text{CONN}_{\overline{G}}^{-xy}(xy)$ for each pair of vertices of $(p_iG, p_i\overline{G})$.

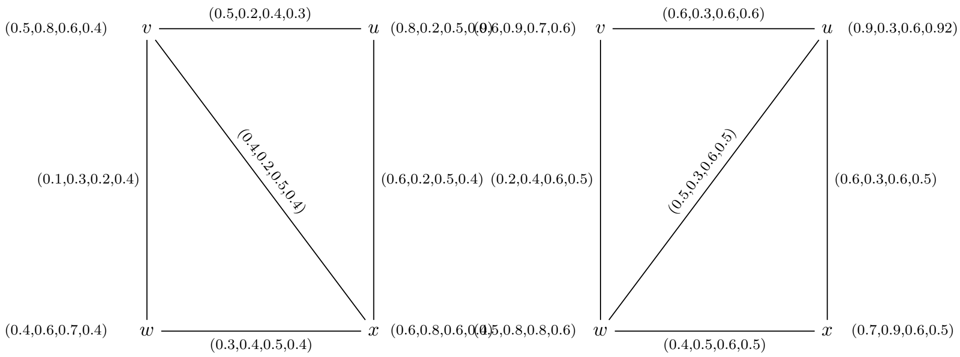


Figure 6: G = $(p_iG,\ p_i\bar{G})$

$$\text{CONN}_G^{-uv}(u, v) = 0 \leq 0.5, 0.2, 0.4, 0.3 = SE(u, v)$$
$$\text{CONN}_G^{-vw}(v, w) = 0 \leq 0.1, 0.3, 0.2, 0.4 = SE(v, w)$$
$$\text{CONN}_G^{-wu}(w, u) = 0 \leq 0.4, 0.2, 0.5, 0.4 = SE(w, u)$$
$$\text{CONN}_G^{-ux}(u, x) = 0.5 \wedge 0.3 = 0.3, \quad 0.2 \wedge 0.4 = 0.2, \quad 0.4 \wedge 0.5 = 0.4, \quad 0.3 \wedge 0.4 = 0.3$$
$$\text{CONN}_G^{-vx}(v, x) = 0.1 \wedge 0.4 \wedge 0.6 = 0.1, \quad 0.3 \wedge 0.2 \wedge 0.2 = 0.2, \quad 0.2 \wedge 0.5 \wedge 0.5 = 0.2, \quad 0.4 \wedge 0.4 \wedge 0.4 = 0.4$$
$$\text{CONN}_G^{-ux}(u, x) = 0.1, 0.2, 0.2, 0.4 \leq 0.3, 0.4, 0.5, 0.4 = SE(u, x)$$

The strong arcs of G are uv, ux, vw, vx, wu. So, the dominating sets are:

$$\{u\}, \{v\}, \{u, v\}, \{u, x\}, \{v, w\}, \{v, x\}, \{u, v, w\}, \{u, v, x\}, \{u, w, x\}, \{v, w, x\}, \{u, v, w, x\}$$

The lower minimal dominating sets are:

$$\{u\}, \{v\}, \{u, x\}, \{v, x\}$$

$$\text{CONN}_G^{-uv}(u, v) = 0 \leq 0.6, 0.3, 0.6, 0.6 = SE(u, v)$$
$$\text{CONN}_G^{-vw}(v, w) = 0 \leq 0.2, 0.4, 0.6, 0.5 = SE(v, w)$$

$$\mathrm{CONN}_G^{-wu}(w,u) = 0 \leq 0.5, 0.3, 0.6, 0.5 = SE(w,u)$$

$$\mathrm{CONN}_G^{-ux}(u,x) = 0.6 \wedge 0.4 = 0.4, \quad 0.3 \wedge 0.5 = 0.3, \quad 0.6 \wedge 0.6 = 0.6, \quad 0.6 \wedge 0.5 = 0.5$$

The strong arcs of G are uv, ux, vw, vx, wu, and the upper minimal dominating sets are:

$$\{u\}, \{v\}, \{u,x\}, \{v,x\}$$

The minimal dominating sets (DS) of G are:

$$\{u\}, \{v\}, \{u,x\}, \{v,x\}$$

The fuzzy cardinality calculations for minimal DS are as follows:

$$|C(G)| + |C(G)| = \sum_{w \in C(G)} RV(W) + \sum_{w \in C(G)} RV(W)$$

For $\{u,x\}$:

$$|u,x| = (0.8+0.2+0.5+0.9+0.6+0.8+0.6+0.4)+(0.9+0.3+0.6+0.92+0.7+0.9+0.6+0.5)$$

$= 4.8 + 5.42$
$= 10.22$

For $\{v,x\}$:

$$|v,x| = (0.5+0.8+0.6+0.4+0.6+0.8+0.6+0.4)+(0.6+0.9+0.7+0.6+0.7+0.9+0.6+0.5)$$

$= 4.7 + 5.5$
$= 10.2$

Thus, the minimum DS is $\{v,x\}$ and the domination number is:

$$\Gamma_D(G) = 10.2$$

Definition 3.13. A *rough m-polar fuzzy digraph* has a lower domination number $\Gamma_D(G)$ and an upper domination number $\Gamma^D(G)$ for each minimal dominating set (DS).

The *lower domination number* $\Gamma_D(G)$ of $G = (p_i G, p_i \overline{G})$ is the fuzzy cardinality of the minimal dominating set D_G in G.

The *upper domination number* $\Gamma^D(G)$ is the fuzzy cardinality of the minimal dominating set D_G in G.

The *domination number* of the rough m-polar fuzzy digraph $G = (p_i G, p_i \overline{G})$ is the sum of the upper and lower domination numbers of the minimum dominating set $D(G)$ of $G = (p_i G, p_i \overline{G})$:

$$\Gamma_D(G) = \Gamma_D(G) + \Gamma^D(G)$$

Example 3.14. In Example 3.11, the cardinality of D_G is given by:

$$|C(G)| + |C(G)| = \sum_{w \in C(G)} RV(w) + \sum_{w \in C(G)} RV(w)$$

For the pair $|u, x|$:

$$\begin{aligned} &= (0.8 + 0.2 + 0.5 + 0.9 + 0.6 + 0.8 + 0.6 + 0.4) \\ &\quad + (0.9 + 0.3 + 0.6 + 0.92 + 0.7 + 0.9 + 0.6 + 0.5) \\ &= 4.8 + 5.42 = 10.22 \end{aligned}$$

For the pair $|v, x|$:

$$\begin{aligned} &= (0.5 + 0.8 + 0.6 + 0.4 + 0.6 + 0.8 + 0.6 + 0.4) \\ &\quad + (0.6 + 0.9 + 0.7 + 0.6 + 0.7 + 0.9 + 0.6 + 0.5) \\ &= 4.7 + 5.5 = 10.2 \end{aligned}$$

So, the minimum dominating set is $\{v, x\}$ and the domination number is:

$$\Gamma_D(G) = 10.2$$

4. Tensor Product of Rough m-Polar Fuzzy Digraphs

This section defines the tensor product of Rough m-polar Fuzzy Digraphs.

Definition 4.1 The tensor product of G_1 and G_2 in rough m-polar fuzzy digraphs is defined as:

$$G = G_1 \otimes G_2 =$$

$\mathrm{p}_i(\underline{G}_1 \otimes \underline{G}_2, G_1 \otimes G_2)G = G_1 \otimes G_2$
$= \mathrm{p}_i(\underline{G}_1 \otimes \underline{G}_2, G_1 \otimes G_2)G = G_1 \otimes G_2$
$= \mathrm{p}_i(\overline{G}_1 \otimes \overline{G}_2, G_1 \otimes G_2)$
where:
1. $\mathrm{p}_i(\underline{R}V_1 \otimes \underline{R}V_2)(S_1, S_2)$
$= \min\{p_i(\underline{R}V_1)(S_1),\ p_i(\underline{R}V_2)(S_2)\}$

$$p_i(\overline{R}V_1 \otimes \overline{R}V_2)(S_1, S_2)$$

$= \min\left\{p_i(\overline{R}V_1)(S_1), p_i(\overline{R}V_2)(S_2)\right\}$,
$\forall (S_1, S_2) \in (\underline{R}V_1 \otimes \underline{R}V_2)$,
$(\overline{R}V_1 \otimes \overline{R}V_2)$
2. $p_i(\underline{S}E_1 \otimes \underline{S}E_2)((S_1, S_2)(t_1, t_2))$

$= \min \{p_i(\underline{S}E_1)(S_1, t_1), p_i(\underline{S}E_2)(S_2, t_2)\}$,

$$p_i(\overline{S}E_1 \otimes \overline{S}E_2)((S_1, S_2)(t_1, t_2))$$

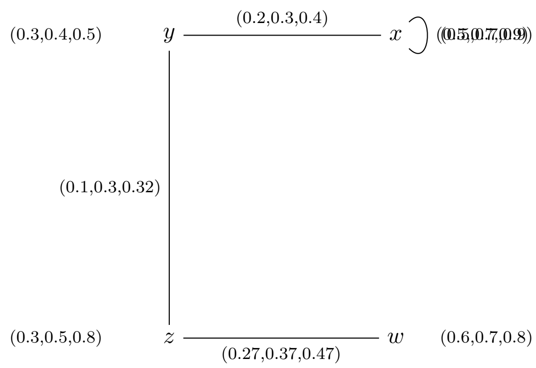
$= \min \left\{p_i(\overline{S}E_1)(S_1, t_1), p_i(\overline{S}E_2)(S_2, t_2)\right\}$,
$\forall (S_1, t_1) \in \overline{S}E_1, (S_2, t_2) \in \overline{S}E_2$

Example 4.2

Let $M = \{x, y, z\}$ be a set. Let $G_1 = p_i(G_1, G_1)$ and $G_2 = p_i(G_2, G_2)$ be two rough m-polar fuzzy digraphs on M, where:

$$G_1$$

$= \mathrm{p}_i(RV_1, SE_1)$ and $G_2 = p_i(RV_2, SE_2)$
are rough m-polar fuzzy digraphs as shown below:



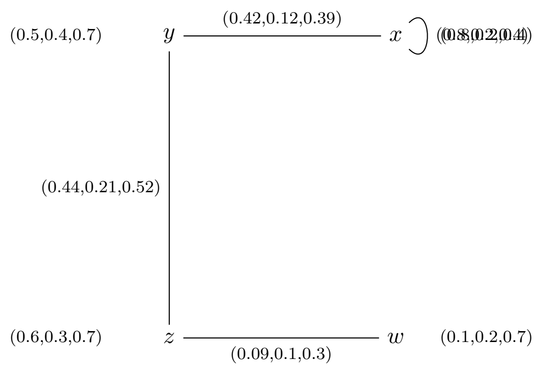


Figure 7: RmPFD $G_1 = p_i(\mathsf{RV}_2, \mathsf{SE}_{E_2})$ and its complement $\overline{G_1} = p_i(\overline{\mathsf{RV}}_2, \overline{\mathsf{SE}}_{E_2})$

Figure 8 is given below

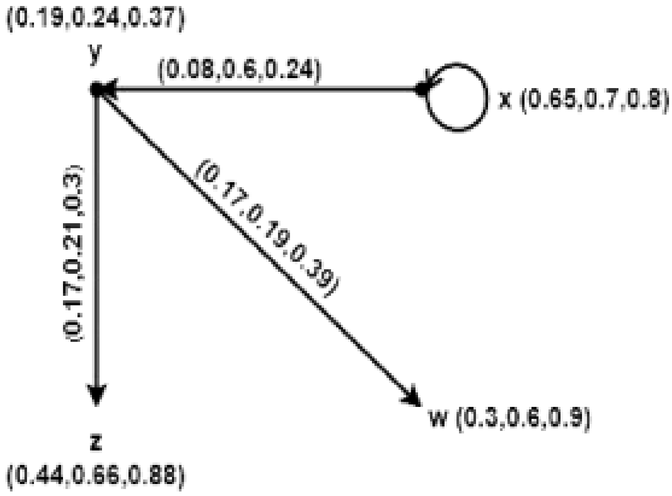


Figure 8 RmPFD $\underline{G_2} = p_i(\underline{R}V_2, \underline{S}E_2)$

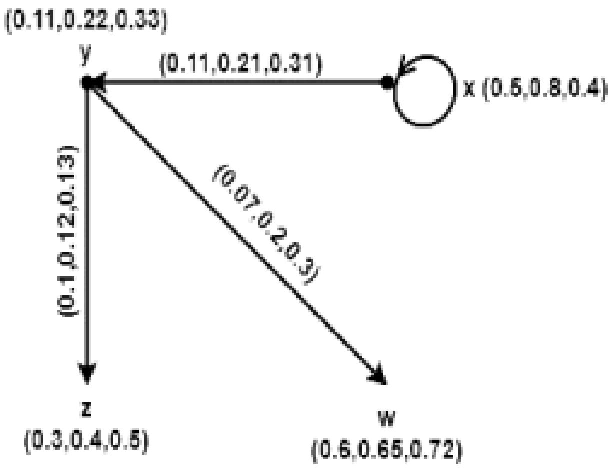


Figure 8. RmPFD $\overline{G_2} = p_i(\overline{R}V_2,\overline{S}E_2)$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(\underline{S}_1, \underline{S}_2)$$
$$= \min\{p_i(\underline{R}V_1)(\underline{S}_1),\ p_i(\underline{R}V_2)(\underline{S}_2)\}$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(x, x)$$
$$= \min\{p_i(\underline{R}V_1)(x),\ p_i(\underline{R}V_2)(x)\}$$
$$= \min\{(0.5, 0.7, 0.9),\ (0.65, 0.7, 0.8)\}$$
$$= (0.5,\ 0.7,\ 0.8)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(x, y)$$
$$= \min\{p_i(\underline{R}V_1)(x),\ p_i(\underline{R}V_2)(y)\}$$
$$= \min\{(0.5, 0.7, 0.9),\ (0.19, 0.24, 0.37)\}$$
$$= (0.19,\ 0.24,\ 0.37)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(x, z)$$
$$= \min\{p_i(\underline{R}V_1)(x),\ p_i(\underline{R}V_2)(z)\}$$
$$= \min\{(0.5, 0.7, 0.9),\ (0.44, 0.66, 0.88)\}$$
$$= (0.44,\ 0.66,\ 0.88)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(x, w)$$
$$= \min\{p_i(\underline{R}V_1)(x),\ p_i(\underline{R}V_2)(w)\}$$
$$= \min\{(0.5, 0.7, 0.9),\ (0.3, 0.6, 0.9)\}$$
$$= (0.3,\ 0.6,\ 0.9)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(y, x)$$
$$= \min\{p_i(\underline{R}V_1)(y),\ p_i(\underline{R}V_2)(x)\}$$

$= \min\{(0.3, 0.4, 0.5),\ (0.65, 0.7, 0.8)\}$

$= (0.3,\ 0.4,\ 0.5)$

$$p_i(RV_1 \otimes RV_2)(y, y)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(y)\}$

$$p_i(RV_1 \otimes RV_2)(y, x)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(x)\}$
$= \min\{(0.3, 0.4, 0.5), (0.19, 0.24, 0.37)\}$
$= (0.19,\ 0.24,\ 0.37)$

$$p_i(RV_1 \otimes RV_2)(y, z)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(z)\}$
$= \min\{(0.3, 0.4, 0.5), (0.44, 0.66, 0.88)\}$
$= (0.3,\ 0.4,\ 0.5)$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(y, w)$$

$= \min\{p_i(\underline{R}V_1)(y),\ p_i(\underline{R}V_2)(w)\}$

$= \min\{(0.3, 0.4, 0.5),\ (0.3, 0.6, 0.9)\}$

$= (0.3,\ 0.4,\ 0.5)$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(z, x)$$

$= \min\{p_i(\underline{R}V_1)(z),\ p_i(\underline{R}V_2)(x)\}$

$= \min\{(0.3, 0.5, 0.8),\ (0.65, 0.7, 0.8)\}$

$= (0.3,\ 0.5,\ 0.8)$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(z, y)$$

$= \min\{p_i(\underline{R}V_1)(z),\ p_i(\underline{R}V_2)(y)\}$

$$= \min\{(0.3, 0.5, 0.8),\ (0.19, 0.24, 0.37)\}$$

$$= (0.19,\ 0.24,\ 0.37)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(z, z)$$

$$= \min\{p_i(\underline{R}V_1)(z),\ p_i(\underline{R}V_2)(z)\}$$

$$= \min\{(0.3, 0.5, 0.8),\ (0.44, 0.66, 0.88)\}$$

$$= (0.3,\ 0.5,\ 0.8)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(z, w)$$

$$= \min\{p_i(\underline{R}V_1)(z),\ p_i(\underline{R}V_2)(w)\}$$

$$= \min\{(0.3, 0.5, 0.8),\ (0.3, 0.6, 0.9)\}$$

$$= (0.3,\ 0.5,\ 0.8)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(w, x)$$

$$= \min\{p_i(\underline{R}V_1)(w),\ p_i(\underline{R}V_2)(x)\}$$
$$= \min\{(0.6, 0.7, 0.8),\ (0.65, 0.7, 0.8)\}$$

$$= (0.6,\ 0.7,\ 0.8)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(w, y)$$

$$= \min\{p_i(\underline{R}V_1)(w),\ p_i(\underline{R}V_2)(y)\}$$

$$= \min\{(0.6, 0.7, 0.8),\ (0.19, 0.24, 0.37)\}$$

$$= (0.19,\ 0.24,\ 0.37)$$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(w, z)$$

$$= \min\{p_i(\underline{R}V_1)(w),\ p_i(\underline{R}V_2)(z)\}$$

$= \min\{(0.6, 0.7, 0.8),\ (0.44, 0.66, 0.88)\}$
$= (0.44, 0.66, 0.88)$

$$p_i(\underline{R}V_1 \otimes \underline{R}V_2)(w, w)$$

$= \min\{p_i(\underline{R}V_1)(w),\ p_i(\underline{R}V_2)(w)\}$

$= \min\{(0.6, 0.7, 0.8),\ (0.3, 0.6, 0.9)\}$
$= (0.3, 0.6, 0.8)$Yes

$$p_i(RV_1 \otimes RV_2)(S_1, S_2)$$

$= \min\{p_i(RV_1)(S_1), p_i(RV_2)(S_2)\}$

$$p_i(RV_1 \otimes RV_2)(x, x)$$

$= \min\{p_i(RV_1)(x), p_i(RV_2)(x)\}$
$= \min\{(0.8, 0.2, 0.4), (0.5, 0.8, 0.4)\} = (0.5, 0.2, 0.4)$

$$p_i(RV_1 \otimes RV_2)(x, y)$$

$= \min\{p_i(RV_1)(x), p_i(RV_2)(y)\}$
$= \min\{(0.8, 0.2, 0.4), (0.11, 0.22, 0.33)\} = (0.11, 0.22, 0.33)$

$$p_i(RV_1 \otimes RV_2)(x, z)$$

$= \min\{p_i(RV_1)(x), p_i(RV_2)(z)\}$
$= \min\{(0.8, 0.2, 0.4), (0.3, 0.4, 0.5)\} = (0.3, 0.2, 0.4)$

$$p_i(RV_1 \otimes RV_2)(x, w)$$

$= \min\{p_i(RV_1)(x), p_i(RV_2)(w)\}$
$= \min\{(0.8, 0.2, 0.4), (0.6, 0.65, 0.72)\}$
$= (0.6,\ 0.2,\ 0.4)$

$$p_i(RV_1 \otimes RV_2)(y, x)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(x)\}$
$= \min\{(0.5, 0.4, 0.7), (0.5, 0.8, 0.4)\} = (0.5, 0.4, 0.4)$

$$p_i(RV_1 \otimes RV_2)(y, y)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(y)\}$
$= \min\{(0.5, 0.4, 0.7), (0.11, 0.22, 0.33)\}$
$= (0.11,\ 0.22,\ 0.33)$

$$p_i(RV_1 \otimes RV_2)(y,z)$$

$= \min\{p_i(RV_1)(y), p_i(RV_2)(z)\}$
$= \min\{(0.5, 0.4, 0.7), (0.3, 0.4, 0.5)\}$
$= (0.3,\ 0.4,\ 0.5)$

$$p_i(RV_1 \otimes RV_2)(y,w) = \min\{p_i(RV_1)(y), p_i(RV_2)(w)\}$$

$= \min\{(0.5, 0.4, 0.7), (0.6, 0.65, 0.72)\} = (0.5, 0.4, 0.7)$

$$p_i(RV_1 \otimes RV_2)(z,x)$$

$= \min\{p_i(RV_1)(z), p_i(RV_2)(x)\}$
$= \min\{(0.6, 0.3, 0.7), (0.5, 0.8, 0.4)\} = (0.5, 0.3, 0.4)$

$$p_i(RV_1 \otimes RV_2)(z,y)$$

$= \min\{p_i(RV_1)(z), p_i(RV_2)(y)\}$
$= \min\{(0.6, 0.3, 0.7), (0.11, 0.22, 0.33)\} = (0.11, 0.22, 0.33)$

$$p_i(RV_1 \otimes RV_2)(z,z)$$

$= \min\{p_i(RV_1)(z), p_i(RV_2)(z)\}$
$= \min\{(0.6, 0.3, 0.7), (0.3, 0.4, 0.5)\} = (0.3, 0.3, 0.5)$

$$p_i(RV_1 \otimes RV_2)(z,w)$$

$= \min\{p_i(RV_1)(z), p_i(RV_2)(w)\} = \min\{(0.6, 0.3, 0.7), (0.6, 0.65, 0.72)\}$
$= (0.6,\ 0.3,\ 0.7)$

$$p_i(RV_1 \otimes RV_2)(w,x)$$

$= \min\{p_i(RV_1)(w), p_i(RV_2)(x)\}$
$= \min\{(0.1, 0.2, 0.7), (0.5, 0.8, 0.4)\} = (0.1, 0.2, 0.4)$

$$p_i(RV_1 \otimes RV_2)(w,y)$$

$= \min\{p_i(RV_1)(w), p_i(RV_2)(y)\} = \min\{(0.1, 0.2, 0.7), (0.11, 0.22, 0.33)\}$
$= (0.1,\ 0.2,\ 0.3)$

$$p_i(RV_1 \otimes RV_2)(w,z)$$

$= \min\{p_i(RV_1)(w), p_i(RV_2)(z)\}$

$$p_i(RV_1 \otimes RV_2)(w, z)$$

$= \min\{(0.1, 0.2, 0.7), (0.3, 0.4, 0.5)\} = (0.1, 0.2, 0.5)$

$$p_i(RV_1 \otimes RV_2)(w, w) =$$

$\min\{p_i(RV_1)(w), p_i(RV_2)(w)\}$
$= \min\{(0.1, 0.2, 0.7), (0.6, 0.65, 0.72)\} = (0.1, 0.2, 0.7)$

$$p_i(SE_1 \otimes SE_2)((S_1, S_2), (t_1, t_2))$$

$= \min\{p_i(SE_1)(S_1, t_1), p_i(SE_2)(S_2, t_2)\}$

$$p_i(SE_1 \otimes SE_2)((x, x), (x, x))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(xx)\}$
$= \min\{(0.1, 0.3, 0.6), (0.2, 0.5, 0.7)\}$
$= (0.1,\ 0.3,\ 0.6)$

$$p_i(SE_1 \otimes SE_2)((x, x), (x, y)) =$$

$\min\{p_i(SE_1)(xx), p_i(SE_2)(xy)\} = \min\{(0.1, 0.3, 0.6), (0.08, 0.6, 0.24)\} = (0.08, 0.3, 0.24)$

$$p_i(SE_1 \otimes SE_2)((x, y), (x, z))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(yz)\}$
$= \min\{(0.1, 0.3, 0.6), (0.17, 0.21, 0.3)\}$
$= (0.1,\ 0.21,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((x, y), (x, w))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(yw)\}$
$= \min\{(0.1, 0.3, 0.6), (0.17, 0.19, 0.39)\} = (0.1, 0.19, 0.39)$

$$p_i(SE_1 \otimes SE_2)((x, x), (y, x))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(xx)\}$
$= \min\{(0.2, 0.3, 0.4), (0.2, 0.5, 0.7)\}$
$= (0.2,\ 0.3,\ 0.4)$

$$p_i(SE_1 \otimes SE_2)((x,x),(y,y))$$

$= \min\{p_i(SE_1)(xy),$
$p_i(SE_2)(xy)$
$= \min\{(0.2, 0.3, 0.4), (0.08, 0.6, 0.24)\}$
$= (0.08,\ 0.3,\ 0.24)$

$$p_i(SE_1 \otimes SE_2)((x,y),$$

(y, z)) =
$\min\{p_i(SE_1)(xy), p_i(SE_2)(yz)\}$
$= \min\{(0.2, 0.3, 0.4), (0.17, 0.21, 0.3)\}$
$= (0.17,\ 0.21,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((x,y),(y,w))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(yw)\}$
$= \min\{(0.2, 0.3, 0.4), (0.17, 0.19, 0.39)\}$
$= (0.17,\ 0.19,\ 0.39)$

$$p_i(SE_1 \otimes SE_2)((y,x),(z,x))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(xx)\}$
$= \min\{(0.1, 0.3, 0.32), (0.2, 0.5, 0.7)\}$
$= (0.1,\ 0.3,\ 0.32)$

$$p_i(SE_1 \otimes SE_2)((y,x),(z,y))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(xy)\}$
$= \min\{(0.1, 0.3, 0.32), (0.08, 0.6, 0.24)\}$
$= (0.08,\ 0.3,\ 0.24)$

$$p_i(SE_1 \otimes SE_2)((y,y),(z,z))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(yz)\}$
$= \min\{(0.1, 0.3, 0.32), (0.17, 0.21, 0.3)\} = (0.1, 0.21, 0.3)$

$$p_i(SE_1 \otimes SE_2)((y,y),(z,w)) = \min\{p_i(SE_1)(yz), p_i(SE_2)(yw)\}$$

$= \min\{(0.1, 0.3, 0.32), (0.17, 0.19, 0.39)\} = (0.1, 0.19, 0.32)$

$$p_i(SE_1 \otimes SE_2)((y, x), (w, x))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(xx)\} = \min\{(0.27, 0.37, 0.47), (0.2, 0.5, 0.7)\}$
$= (0.2,\ 0.37,\ 0.47)$

$$p_i(SE_1 \otimes SE_2)((y, x), (w, y))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(xy)\} = \min\{(0.27, 0.37, 0.47), (0.08, 0.6, 0.24)\}$
$= (0.08,\ 0.37,\ 0.24)$

$$p_i(SE_1 \otimes SE_2)((y, y), (w, z))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(yz)\}$
$= \min\{(0.27, 0.37, 0.47), (0.17, 0.21, 0.3)\}$
$= (0.17,\ 0.21,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((y, y), (w, w))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(yw)\}$
$= \min\{(0.27, 0.37, 0.47), (0.17, 0.19, 0.39)\}$
$= (0.17,\ 0.19,\ 0.39)$

$$p_i(SE_1 \otimes SE_2)((S_1, S_2), (t_1, t_2))$$

$= \min\{p_i(SE_1)(S_1, t_1), p_i(SE_2)(S_2, t_2)\}$

$$p_i(SE_1 \otimes SE_2)((x, x), (x, x))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(xx)\}$
$= \min\{(0.2, 0.4, 0.6), (0.3, 0.1, 0.7)\}$
$= (0.2,\ 0.1,\ 0.6)$

$$p_i(SE_1 \otimes SE_2)((x, x), (x, y))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(xy)\}$
$= \min\{(0.2, 0.4, 0.6), (0.11, 0.21, 0.31)\}$
$= (0.11,\ 0.21,\ 0.31)$

$$p_i(SE_1 \otimes SE_2)((x, y), (x, z))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(yz)\}$
$= \min\{(0.2, 0.4, 0.6), (0.1, 0.12, 0.13)\}$
$= (0.1,\ 0.12,\ 0.13)$

$$p_i(SE_1 \otimes SE_2)((x, y), (x, w))$$

$= \min\{p_i(SE_1)(xx), p_i(SE_2)(yw)\}$
$= \min\{(0.2, 0.4, 0.6), (0.07, 0.2, 0.3)\}$
$= (0.07,\ 0.2,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((x, x), (y, x))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(xx)\}$
$= \min\{(0.42, 0.12, 0.39), (0.3, 0.1, 0.7)\}$
$= (0.3,\ 0.1,\ 0.39)$

$$p_i(SE_1 \otimes SE_2)((x, x), (y, y))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(xy)\}$
$= \min\{(0.42, 0.12, 0.39), (0.11, 0.21, 0.31)\}$
$= (0.11,\ 0.12,\ 0.31)$

$$p_i(SE_1 \otimes SE_2)((x, y), (y, z))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(yz)\}$
$= \min\{(0.42, 0.12, 0.39), (0.1, 0.12, 0.13)\}$
$= (0.1,\ 0.12,\ 0.13)$

$$p_i(SE_1 \otimes SE_2)((x, y), (y, w))$$

$= \min\{p_i(SE_1)(xy), p_i(SE_2)(yw)\}$
$= \min\{(0.42, 0.12, 0.39), (0.07, 0.2, 0.3)\}$
$= (0.07,\ 0.12,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((y, x), (z, x))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(xx)\}$
$= \min\{(0.44, 0.21, 0.62), (0.3, 0.1, 0.7)\}$
$= (0.3,\ 0.1,\ 0.62)$

$$p_i(SE_1 \otimes SE_2)((y, x), (z, y))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(xy)\}$
$= \min\{(0.44, 0.21, 0.62), (0.11, 0.21, 0.31)\}$
$= (0.11,\ 0.21,\ 0.31)$

$$p_i(SE_1 \otimes SE_2)((y, y), (z, z))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(yz)\}$
$= \min\{(0.44, 0.21, 0.62), (0.1, 0.12, 0.13)\}$
$= (0.1,\ 0.12,\ 0.13)$

$$p_i(SE_1 \otimes SE_2)((y, y), (z, w))$$

$= \min\{p_i(SE_1)(yz), p_i(SE_2)(yw)\}$
$= \min\{(0.44, 0.21, 0.62), (0.07, 0.2, 0.3)\}$
$= (0.07,\ 0.2,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((y, x), (w, x))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(xx)\}$
$= \min\{(0.09, 0.1, 0.3), (0.3, 0.1, 0.7)\}$
$= (0.09,\ 0.1,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((y, x), (w, y))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(xy)\}$
$= \min\{(0.09, 0.1, 0.3), (0.11, 0.21, 0.31)\}$
$= (0.09,\ 0.1,\ 0.3)$

$$p_i(SE_1 \otimes SE_2)((y, y), (w, z))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(yz)\}$

$= \min\{(0.09, 0.1, 0.3), (0.1, 0.12, 0.13)\} = (0.09, 0.1, 0.13)$

$$p_i(SE_1 \otimes SE_2)((y, y), (w, w))$$

$= \min\{p_i(SE_1)(yw), p_i(SE_2)(yw)\}$
$= \min\{(0.09, 0.1, 0.3), (0.07, 0.2, 0.3)\}$
$= (0.07,\ 0.1,\ 0.3)$

Figure 9 is given as below

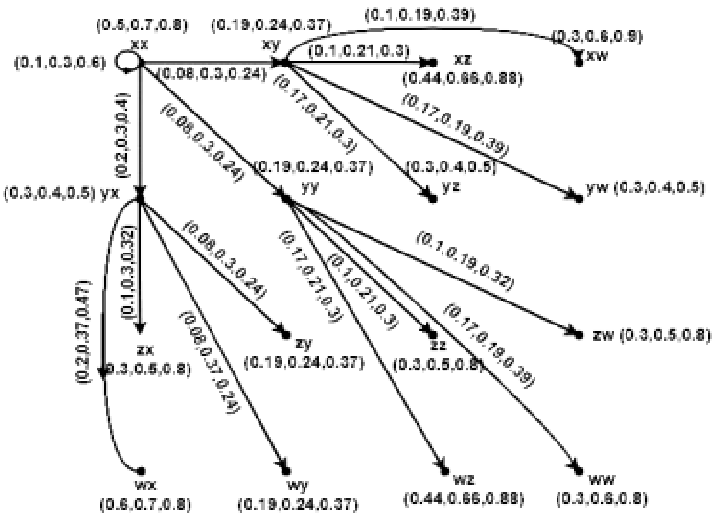


Figure 9: RmPFD $G_1 \otimes G_2 = p_i(\underline{G}_1 \otimes \underline{G}_2)$

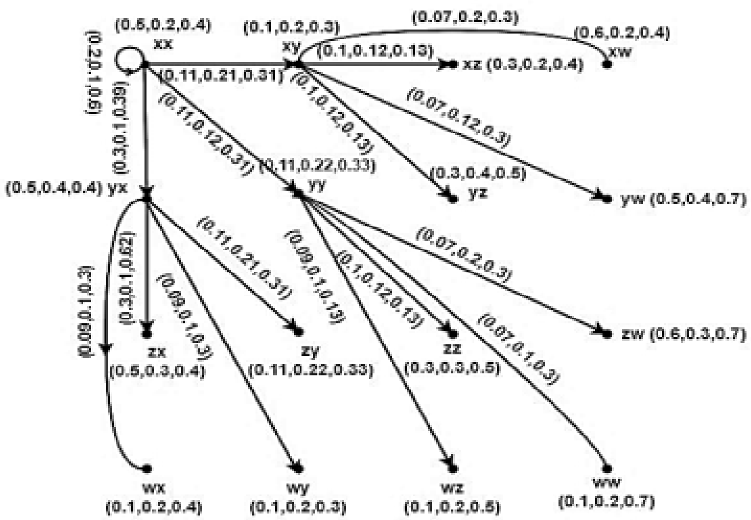


Figure 9: RmPFD $G_1 \otimes G_2 = p_i(\overline{G}_1 \otimes \overline{G}_2)$

Theorem 4.3. The tensor product of two rough m-polar fuzzy digraphs is also a rough m-polar fuzzy digraph.

Proof. Let $G_1 = p_i(G_1, G_1)$ and $G_2 = p_i(G_2, G_2)$ be two rough m-polar fuzzy digraphs. Let the tensor product of G_1 and G_2 be $G = G_1 \otimes G_2 = p_i(G_1 \otimes G_2), p_i(G_1 \otimes G_2)$ where $G_1 = p_i(RV_1, SE_1)$ and $G_1 = p_i(RV_1, SE_1)$ and $G_2 = p_i(RV_2, SE_2), G_2 = p_i(RV_2, SE_2)$. We claim that $G = G_1 \otimes G_2$ is a rough m-polar fuzzy digraph. It is enough to show that $p_i(HB_1 \otimes HB_2)$ and $p_i(HB_1 \otimes HB_2)$ are m-polar fuzzy relations on $p_i(TA_1 \otimes TA_2)$ and $p_i(TA_1 \otimes TA_2)$, respectively.

First, we show that $p_i(HB_1 \otimes HB_2)$ is a fuzzy relation on $p_i(TA_1 \otimes TA_2)$.

If $S_1t_1 \in (SE_1)^*$, $S_2y_2 \in (SE_2)^*$, then

$$p_i(HB_1 \otimes HB_2)((S_1, S_2)(t_1, t_2))$$

$=$

$\mathrm{p}_i((SE_1)(S_1t_1) \wedge p_i(SE_2)(S_2t_2)) \leq p_i((TA_1)(S_1) \wedge p_i(TA_1)(t_1)) \wedge p_i((TA_2)(S_2) \wedge (TA_2)(t_2))$

$= \mathrm{p}_i((TA_1)(S_1) \wedge (TA_2)(S_2)) \wedge p_i((TA_1)(t_1)) \wedge (TA_2)(t_2)$

$= \mathrm{p}_i(TA_1 \otimes TA_2)(S_1, S_2) \wedge p_i(TA_1 \times TA_2)(t_1, t_2)$

$p_i(HB_1 \otimes HB_2)((S_1, S_2)(t_1, t_2)) \leq p_i(TA_1 \otimes TA_2)(S_1, S_2) \wedge p_i(TA_1 \times TA_2)(t_1, t_2)$

Thus, $p_i(HB_1 \otimes HB_2)$ is a fuzzy relation on $p_i(TA_1 \otimes TA_2)$.

Similarly, we can show that $p_i(HB_1 \otimes HB_2)$ is a fuzzy relation on $p_i(TA_1 \otimes TA_2)$. Hence, G is a rough m-polar fuzzy digraph.

5. Strong Product of Rough m-Polar Fuzzy Digraphs

In this section, the definition of the strong product of Rough m-Polar Fuzzy Digraphs is presented.

Definition 5.1. The strong product of G_1 and G_2 in a rough m-polar fuzzy digraph $G = G_1 \sqcap G_2$

$= \mathrm{p}_i(G_1 \sqcap G_2, G_1 \sqcap G_2)$, where $p_i(G_1 \sqcap G_2)$

$= \mathrm{p}_i(RV_1 \sqcap RV_2, SE_1 \sqcap SE_2)$ and $p_i(G_1 \sqcap G_2)$

$= \mathrm{p}_i(RV_1 \sqcap RV_2, SE_1 \sqcap SE_2)$ are rough m-polar fuzzy digraphs, respectively, such that:

1)

$$p_i(RV_1 \sqcap RV_2)(S_1, S_2)$$

$= \min\{p_i(RV_1)(S_1), p_i(RV_2)(S_2)\}, \quad \forall (S_1, S_2) \in (RV_1 \times RV_2)$

2)

$$p_i(SE_1 \sqcap SE_2)((S_1, S_2), (S_1, y_2))$$

$= \min\left\{p_i(RV_1)(S_1), p_i(SE_2)(S_2, y_2)\right\}, \quad \forall S_1 \in (RV_1, RV_1), (S_2, y_2) \in (SE_2, SE_2)$

3)

$$p_i(SE_1 \sqcap SE_2)((S_1, S_2), (y_1, S_2))$$

$= \min\left\{p_i(SE_1)(S_1, y_1), p_i(RV_2)(S_2)\right\}, \quad \forall (S_1, y_1) \in (SE_1, SE_1), S_2 \in (RV_2, RV_2)$

4)

$$p_i(SE_1 \sqcap SE_2)((S_1, S_2), (y_1, y_2))$$

$= \min\left\{p_i(SE_1)(S_1, y_1), p_i(SE_2)(S_2, y_2)\right\}, \quad \forall (S_1, y_1) \in SE_1, (S_2, y_2) \in SE_2$

Example 5.2. Let $G_1 = p_i(G_1, G_1)$ and $G_2 = p_i(G_2, G_2)$ be two rough m-polar fuzzy digraphs on M, where $G_1 = p_i(RV_1, SE_1)$ and $G_1 = p_i(RV_1, SE_1)$ and $G_2 = p_i(RV_2, SE_2)$, $G_2 = p_i(RV_2, SE_2)$ are rough m-polar fuzzy digraphs shown below.

Figure 10 is given as below

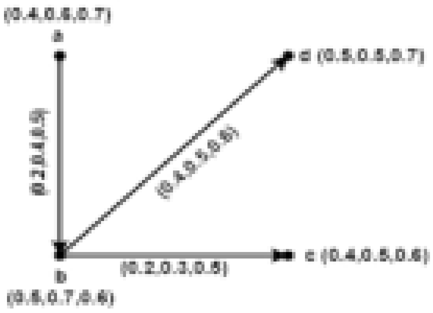


Figure 10: RmPFD $\underline{G_1} = p_l(\underline{R}V_1, \underline{S}E_1)$

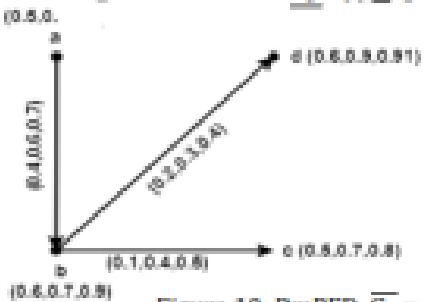


Figure 10: RmPFD $\overline{G_1} = p_l(\overline{R}V_1, \overline{S}\,E_1)$

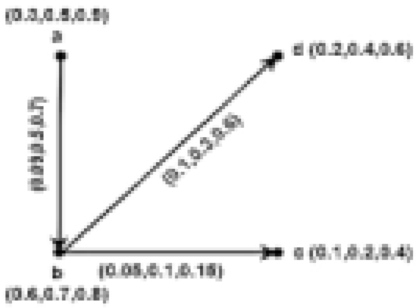


Figure 10: RmPFD $\underline{G_2} = p_l(\underline{R}V_2, \underline{S}E_2)$

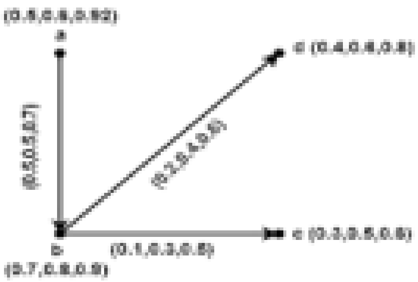


Figure 10: RmPFD $\overline{G_2} = p_l(\overline{R}V_2, \overline{S}E_2)$

$$p_i(RV_1 \Box RV_2)(S_1, S_2)$$
$$= \min\{p_i(RV_1)(S_1), p_i(RV_2)(S_2)\}, \quad \forall (S_1, S_2) \in (RV_1 \times RV_2)$$

$$p_i(RV_1 \Box RV_2)(a, a)$$
$$= \min\{p_i(RV_1)(a), p_i(RV_2)(a)\}$$
$$= 0.4,\ 0.6,\ 0.7\ \wedge 0.3, 0.5, 0.9$$
$$= 0.3,\ 0.5,\ 0.7$$

$$p_i(RV_1 \Box RV_2)(a, b)$$
$$= \min\{p_i(RV_1)(a), p_i(RV_2)(b)\}$$
$$= 0.4,\ 0.6,\ 0.7\ \wedge 0.6, 0.7, 0.8$$
$$= 0.4,\ 0.6,\ 0.7$$

$$p_i(RV_1 \Box RV_2)(a, c)$$
$$= \min\{p_i(RV_1)(a), p_i(RV_2)(c)\}$$
$$= 0.4,\ 0.6,\ 0.7\ \wedge 0.1, 0.2, 0.4$$
$$= 0.1,\ 0.2,\ 0.4$$

$$p_i(RV_1 \Box RV_2)(a, d)$$
$$= \min\{p_i(RV_1)(a), p_i(RV_2)(d)\}$$
$$= 0.4,\ 0.6,\ 0.7\ \wedge 0.2, 0.4, 0.6$$
$$= 0.2,\ 0.4,\ 0.6$$

$$p_i(RV_1 \Box RV_2)(b, a)$$
$$= \min\{p_i(RV_1)(b), p_i(RV_2)(a)\}$$
$$= 0.5,\ 0.7,\ 0.6\ \wedge 0.3, 0.5, 0.9$$
$$= 0.3,\ 0.5,\ 0.6$$

$$p_i(RV_1 \Box RV_2)(b, b)$$
$$= \min\{p_i(RV_1)(b), p_i(RV_2)(b)\}$$
$$= 0.5,\ 0.7,\ 0.6\ \wedge 0.6, 0.7, 0.8$$

$= 0.5,\ 0.7,\ 0.6$

$$p_i(RV_1 \Box RV_2)(b,c)$$

$= \min\{p_i(RV_1)(b), p_i(RV_2)(c)\}$
$= 0.5,\ 0.7,\ 0.6\ \wedge 0.1, 0.2, 0.4$
$= 0.1,\ 0.2,\ 0.4$

$$p_i(RV_1 \Box RV_2)(b,d)$$

$= \min\{p_i(RV_1)(b), p_i(RV_2)(d)\}$
$= 0.5,\ 0.7,\ 0.6\ \wedge 0.2, 0.4, 0.6$
$= 0.2,\ 0.4,\ 0.6$

$$p_i(RV_1 \Box RV_2)(c,a)$$

$= \min\{p_i(RV_1)(c), p_i(RV_2)(a)\}$
$= 0.4,\ 0.5,\ 0.6\ \wedge 0.3, 0.5, 0.9$
$= 0.3,\ 0.5,\ 0.6$

$$p_i(RV_1 \Box RV_2)(c,b)$$

$= \min\{p_i(RV_1)(c), p_i(RV_2)(b)\}$
$= 0.4,\ 0.5,\ 0.6\ \wedge 0.6, 0.7, 0.8$
$= 0.4,\ 0.5,\ 0.6$

$$p_i(RV_1 \Box RV_2)(c,c)$$

$= \min\{p_i(RV_1)(c), p_i(RV_2)(c)\}$
$= 0.4,\ 0.5,\ 0.6\ \wedge 0.1, 0.2, 0.4$
$= 0.1,\ 0.2,\ 0.4$

$$p_i(RV_1 \Box RV_2)(c,d)$$

$= \min\{p_i(RV_1)(c), p_i(RV_2)(d)\}$
$= 0.4,\ 0.5,\ 0.6\ \wedge 0.2, 0.4, 0.6$
$= 0.2,\ 0.4,\ 0.6$

$$p_i(RV_1 \Box RV_2)(d,a)$$

$= \min\{p_i(RV_1)(d), p_i(RV_2)(a)\}$
$= 0.5,\ 0.5,\ 0.7\ \wedge 0.3, 0.5, 0.9$
$= 0.3,\ 0.5,\ 0.7$

$$p_i(RV_1 \Box RV_2)(d,b)$$

$= \min\{p_i(RV_1)(d), p_i(RV_2)(b)\}$
$= 0.5,\ 0.5,\ 0.7\ \wedge 0.6, 0.7, 0.8$
$= 0.5,\ 0.5,\ 0.7$

$$p_i(RV_1 \Box RV_2)(d,c)$$

$= \min\{p_i(RV_1)(d), p_i(RV_2)(c)\}$
$= 0.5,\ 0.5,\ 0.7\ \wedge 0.1, 0.2, 0.4 = 0.1, 0.2, 0.4$

$$p_i(RV_1 \Box RV_2)(d,d)$$

$= \min\{p_i(RV_1)(d), p_i(RV_2)(d)\} = 0.5, 0.5, 0.7 \wedge 0.2, 0.4, 0.6$
$= 0.2,\ 0.4,\ 0.6$

Figure 11 is given as below

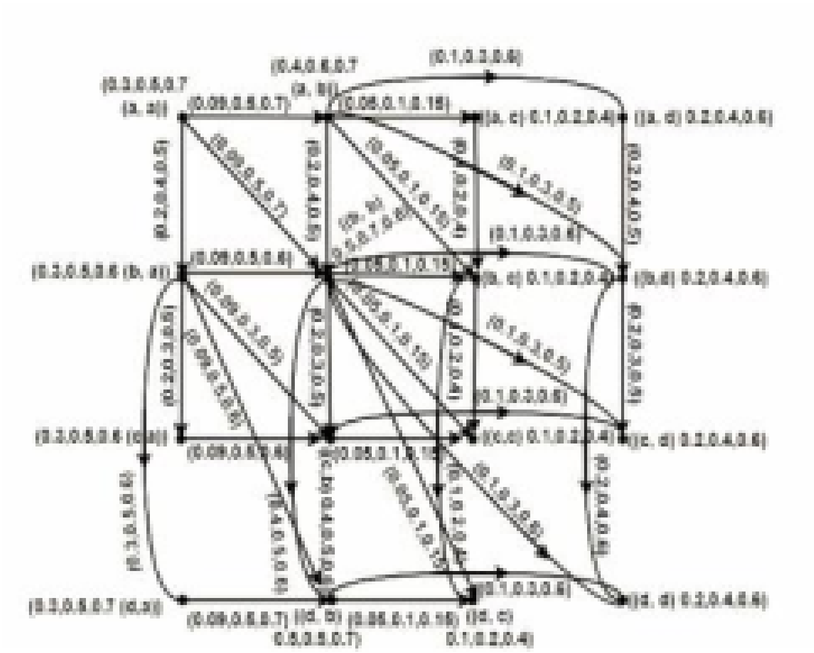


Figure 11: RmPFD $p_l(\underline{G}_1 \boxdot \underline{G}_2)$

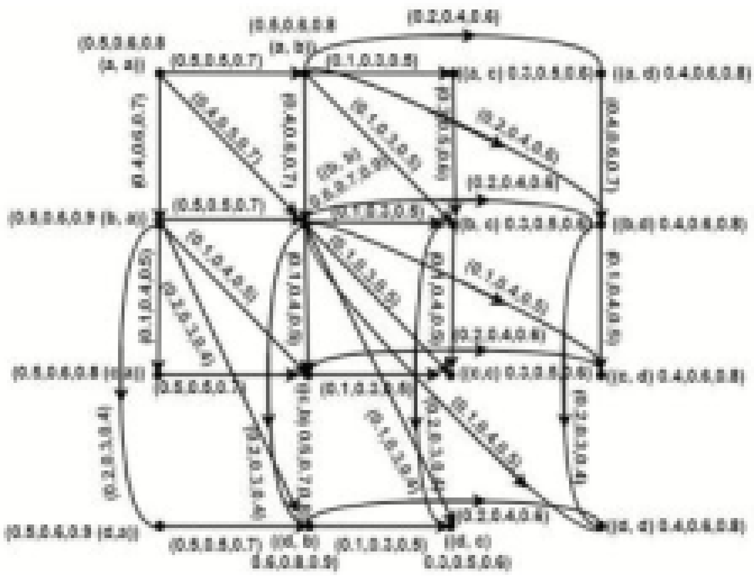


Figure 11: RmPFD $p_l(\overline{G}_1 \boxdot \overline{G}_2)$

Theorem 5.3. The strong product of two rough m-polar fuzzy digraphs is also a rough m-polar fuzzy digraph.

Proof. Let $G_1 = p_i(G_1, G_1)$ and $G_2 = p_i(G_2, G_2)$ be two rough m-polar fuzzy digraphs. Let the strong product of G_1 and G_2 be G
$= \mathrm{G}_1 \odot G_2 = p_i(G_1 \odot G_2), p_i(G_1 \odot G_2)$, where

$$p_i(G_1 \odot G_2)$$

$= \mathrm{p}_i(TA_1 \odot TA_2, HB_1 \odot HB_2)$ and $p_i(G_1 \odot G_2)$
$= \mathrm{p}_i(TA_1 \odot TA_2, HB_1 \odot HB_2)$.We claim that G
$= \mathrm{G}_1 \odot G_2$ is a rough m-polar fuzzy digraph. It is enough to show that $p_i(HB_1 \odot HB_2)$ and $p_i(HB_1 \odot HB_2)$ are m-polar fuzzy relations on $p_i(TA_1 \odot TA_2)$ and $p_i(TA_1 \odot TA_2)$, respectively.

First, we show that $p_i(HB_1 \odot HB_2)$ is a fuzzy relation on $p_i(TA_1 \odot TA_2)$.
If $S_1 y_1 \in (HB_1)$ and $S_2 y_2 \in (HB_2)$, then

$$p_i(HB_1 \odot HB_2)((S_1, S_2)(y_1, y_2))$$

$= \mathrm{p}_i((HB_1)(S_1 y_1)) \wedge p_i(HB_2)(S_2 y_2) \leq p_i((TA_1)(S_1)) \wedge p_i((TA_1)(y_1)) \wedge p_i((TA_2)(S_2)) \wedge (TA_2)(y_2)$

$= \mathrm{p}_i((TA_1)(S_1) \wedge p_i(TA_2)(S_2)) \wedge p_i((TA_1)(y_1) \wedge (TA_2)(y_2))$
$= \mathrm{p}_i(TA_1 \odot TA_2)(S_1, S_2) \wedge p_i(TA_1 \odot TA_2)(y_1, y_2)$.Thus,

$$p_i(HB_1 \odot HB_2)((S_1, S_2)(y_1, y_2)) \leq p_i(TA_1 \odot TA_2)(S_1, S_2)$$

$\wedge p_i(TA_1 \odot TA_2)(y_1, y_2)$.
If $S_1 \in (TA_1)$
and $S_2 y_2 \in (HB_2)$, then

$$p_i(HB_1 \odot HB_2)((S_1, S_2)(S_1, y_2)) =$$

$\mathrm{p}_i(TA_1)(S_1) \wedge p_i(HB_2)(S_2 y_2) \leq p_i(TA_1)(S_1) \wedge ((TA_2)(S_2) \wedge p_i(TA_2)(y_2))$

$= \mathrm{p}_i((TA_1)(S_1) \wedge (TA_2)(S_2)) \wedge p_i((TA_1)(S_1) \wedge (TA_2)(y_2))$
$= \mathrm{p}_i(TA_1 \odot TA_2)(S_1, S_2) \wedge p_i(TA_1 \odot TA_2)(S_1, y_2)$.
If $S_1 y_1 \in (HB_1)$
and $S_2 \in (TA_2)$, then

$$p_i(HB_1 \odot HB_2)((S_1, S_2)(y_1, S_2)) =$$

$\mathrm{p}_i(HB_1)(S_1 y_1) \wedge p_i(TA_2)(S_2) \leq p_i(TA_1)(S_1) \wedge ((TA_1)(y_1) \wedge p_i(TA_2)(S_2))$

$= \mathrm{p}_i((TA_1)(S_1) \wedge (TA_2)(S_2)) \wedge p_i((TA_1)(y_1) \wedge (TA_2)(S_2))$
$= \mathrm{p}_i(TA_1 \odot TA_2)(S_1, S_2) \wedge p_i(TA_1 \odot TA_2)(y_1, S_2)$.
Thus,

$$p_i(HB_1 \odot HB_2)((S_1, S_2)(y_1, S_2))$$

$\leq p_i(TA_1 \odot TA_2)(S_1, S_2) \wedge p_i(TA_1 \odot TA_2)(y_1, S_2)$.
Therefore, $p_i(HB_1 \odot HB_2)$ is a fuzzy relation on $p_i(TA_1 \odot TA_2)$.
Similarly, we can demonstrate that $p_i(HB_1 \odot HB_2)$ is a fuzzy relation on $p_i(TA_1 \odot TA_2)$. Hence, G is a rough m-polar fuzzy digraph.

6. Application of Domination in Rough m-Polar Fuzzy Digraphs

A useful tool for simulating trade networks with intricate and unclear links is the rough m-polar fuzzy digraph. These graphs combine the ideas of directed graphs, m-polar fuzzy sets, and rough sets into a single framework to effectively depict the difficulties involved in trade links between nations, businesses, or individuals. This method describes the structure of a trading network by building a crude m-polar fuzzy digraph. The entities engaged in commerce, such as countries, corporations, or individual traders, are represented by the vertices (apexes) in this case, while the trade relations or flows of commodities and services between them are represented by the edges. We employ rough m-polar fuzzy digraphs to address the inherent uncertainty, ambiguity, and incompleteness commonly present in real-world trade exchanges. The assignment of membership values to both vertices and edges based on various attributes makes a complex and multifaceted depiction of trade ties possible. In particular, we create 3-polar fuzzy membership functions for both vertices and edges in our trade network model: An entity's market share or financial strength, which reflects its economic power, is captured by the first polar. The second polar gauges the entity's dependability or credibility in business dealings. The amount or value of trade transactions connected to the entity or relationship is shown by the third polar. Correspondence relations between vertices and edges are created according to their membership values in order to simplify the study further. Two vertices are deemed equivalent if their membership values, which represent comparable trading patterns or financial strength, fall inside a specified range. The clustering of similar entities made possible by the application of equivalence relations reduces the complexity of the trade network and makes analysis easier to handle and more perceptive. To put it simply, rough m-polar fuzzy digraphs offer a strong and adaptable mathematical model for streamlining and improving the analysis of extremely intricate trade networks.

6.1. Application

Global supply chain risk management, a crucial area for multinational businesses (MNCs) functioning in many geopolitical contexts, is an example of how this approach is used in the real world. Global corporations like Apple, Samsung, and Volkswagen depend on extensive networks of distributors and suppliers spread throughout several nations. Each supplier varies in three areas: financial robustness (economic size and stability), reliability (history of on-time deliveries, quality assurance, and ethical standards), and volume/value of transactions (the amount of vital inventory that passes through that supplier). A business can model its worldwide supply chain in a way that concurrently takes into consideration these three crucial characteristics by using a rough 3-polar fuzzy digraph:

Partners in logistics and suppliers are represented as vertices in the model.

Trade operations are represented as edges, such as the supply of raw materials or the shipment of items.

A 3-polar fuzzy membership value is allocated to each vertex and edge according on trade volume, dependability, and financial strength. Using this structure, the business can:

group vendors with comparable risk characteristics,

Determine any possible network bottlenecks or weak points,

Forecast potential hiccups (e.g., a financially fragile supplier going out of business during a recession),

Strengthen ties with dependable and financially stable partners to maximize the flow of goods. Additionally, the business can manage ambiguous or incomplete data (such as lacking information about new suppliers) and yet make logical judgments in the face of uncertainty by using rough set estimates.

In the end, this useful application of rough m-polar fuzzy digraphs in trade networking aids multinational corporations in being more resilient, effective, and competitive in a very unstable global marketplace. Given a set of vertices $V = \{a, b, c, d, e\}$ with a correspondence relation R on V, the vertices are considered equivalent if their membership values are within a quantified threshold, indicating comparable economic strength or market position. The equivalence classes of R are given below in Table 4.1.

R	a	b	c	d	e
a	1	$(0.5, 0.6, 0.7)$	$(0.7, 0.8, 0.9)$	$(0.4, 0.6, 0.8)$	$(0.6, 0.7, 0.8)$
b	$(0.5, 0.6, 0.7)$	1	$(0.3, 0.4, 0.5)$	$(0.2, 0.6, 0.8)$	$(0.7, 0.8, 0.9)$
c	$(0.7, 0.8, 0.9)$	$(0.3, 0.4, 0.5)$	1	$(0.4, 0.6, 0.8)$	$(0.3, 0.5, 0.7)$
d	$(0.4, 0.6, 0.8)$	$(0.2, 0.6, 0.8)$	$(0.4, 0.6, 0.8)$	1	$(0.3, 0.8, 0.7)$
e	$(0.6, 0.7, 0.8)$	$(0.7, 0.8, 0.9)$	$(0.3, 0.5, 0.7)$	$(0.3, 0.8, 0.7)$	1

Let $A = \{(a, 0.3, 0.5, 0.7), (b, 0.4, 0.5, 0.6), (c, 0.6, 0.8, 0.9), (d, 0.6, 0.5, 0.7), (e, 0.5, 0.7, 0.9)\}$ be a fuzzy set on V.

Let $RA = (\underline{R}A, \overline{R}A)$ be an RmPFS, where $\underline{R}A$ and $\overline{R}A$ are the lower and upper approximations of A under the relation R, given as follows:

$$\underline{R}A = \{(a, 0.3, 0.5, 0.6), (b, 0.4, 0.5, 0.6), (c, 0.3, 0.5, 0.6), (d, 0.6, 0.5, 0.6), (e, 0.4, 0.5, 0.3)\}$$

$$\overline{R}A = \{(a, 0.6, 0.8, 0.9), (b, 0.5, 0.7, 0.9), (c, 0.6, 0.8, 0.9), (d, 0.6, 0.7, 0.8), (e, 0.5, 0.7, 0.9)\}$$

Let $B^* = \{ab, bc, cd, de, ea\} \subseteq V \times V$ and S be an equivalence relation on B. The arcs (directed edges) between the vertices (countries) represent the direction and nature of trade relationships. Table 4.2 gives the fuzzy relations between the arcs.

S	ab	bc	cd	de	ea
ab	1	$(0.1, 0.2, 0.3)$	$(0.4, 0.6, 0.8)$	$(0.11, 0.22, 0.33)$	$(0.35, 0.6, 0.7)$
bc	$(0.1, 0.2, 0.3)$	1	$(0.5, 0.8, 0.9)$	$(0.4, 0.2, 0.3)$	$(0.6, 0.7, 0.5)$
cd	$(0.4, 0.6, 0.8)$	$(0.5, 0.8, 0.9)$	1	$(0.2, 0.3, 0.4)$	$(0.8, 0.5, 0.4)$
de	$(0.11, 0.22, 0.33)$	$(0.4, 0.2, 0.3)$	$(0.2, 0.3, 0.4)$	1	$(0.1, 0.7, 0.6)$
ea	$(0.35, 0.6, 0.7)$	$(0.6, 0.7, 0.5)$	$(0.8, 0.5, 0.4)$	$(0.1, 0.7, 0.6)$	1

Let H be a fuzzy set on B defined as:

$$B = \{(ab, 0.1, 0.18, 0.2), (bc, 0.2, 0.5, 0.6), (cd, 0.1, 0.25, 0.8), (de, 0.11, 0.22, 0.5), (ea, 0.5, 0.6, 0.7)\}$$

Let $SB = (\underline{S}B, \overline{S}B)$ be an RmPF relation, where $\underline{S}B$ and $\overline{S}B$ are the lower and upper approximations of B given below:

$$\underline{S}B = \{(ab, 0.1, 0.18, 0.2), (bc, 0.2, 0.25, 0.6), (cd, 0.1, 0.25, 0.2), (de, 0.11, 0.22, 0.5), (ea, 0.4, 0.3, 0.3)\}$$

$$\overline{S}B = \{(ab, 0.35, 0.6, 0.8), (bc, 0.5, 0.6, 0.8), (cd, 0.2, 0.5, 0.8), (de, 0.2, 0.6, 0.6), (ea, 0.5, 0.6, 0.7)\}$$

Thus, $G = (\underline{R}A, \underline{S}B)$ and $\overline{G} = (\overline{R}A, \overline{S}B)$ are rough m-polar fuzzy digraphs show below, Figure 12 is given as below

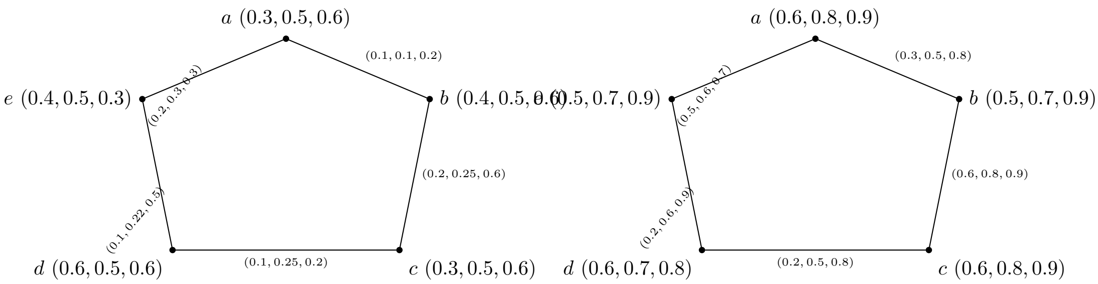


Figure 12 is different strong

The strong arcs in G are determined by the connectedness between each pair of vertices connected by an arc. The connectedness values are given as follows:

$$\text{CONN}_{\underline{G}-ab}(a, b) = \min(0.1, 0.18, 0.2) = \underline{SE}(a, b)$$

$$\text{CONN}_{\underline{G}-bc}(b, c) = \min(0.2, 0.25, 0.6) = \underline{SE}(b, c)$$

$$\text{CONN}_{\underline{G}-cd}(c, d) = \min(0.1, 0.25, 0.2) = \underline{SE}(c, d)$$

$$\text{CONN}_{\underline{G}-de}(d, e) = \min(0.11, 0.22, 0.5) = \underline{SE}(d, e)$$

$$\text{CONN}_{\underline{G}-ea}(e,a) = \min(0.35, 0.6, 0.7) = \underline{SE}(e,a)$$

The strong arcs in $\underline{G}$ are ab, bc, cd, de, and ea.
The strong arcs in $\overline{G}$ are also computed for the upper approximations:

$$\text{CONN}_{\overline{G}-ab}(a,b) = \min(0.35, 0.6, 0.8) = \overline{SE}(a,b)$$
$$\text{CONN}_{\overline{G}-bc}(b,c) = \min(0.5, 0.6, 0.8) = \overline{SE}(b,c)$$
$$\text{CONN}_{\overline{G}-cd}(c,d) = \min(0.2, 0.5, 0.8) = \overline{SE}(c,d)$$
$$\text{CONN}_{\overline{G}-de}(d,e) = \min(0.2, 0.6, 0.6) = \overline{SE}(d,e)$$
$$\text{CONN}_{\overline{G}-ea}(e,a) = \min(0.5, 0.6, 0.7) = \overline{SE}(e,a)$$

Thus, the strong arcs of $\overline{G}$ are also ab, bc, cd, de, and ea.
Dominating Sets
The dominating sets are:

$$D(S) = \{a,d\}, \{b,d\}, \{a,c,e\}, \{b,c,e\}$$

The minimal dominating sets are:

$$\text{Minimal } D(S) = \{a,c,e\}, \{b,d\}$$

Cardinality of the Minimal Dominating Sets
The cardinality of the set $\{a,c,e\}$ is:

$$|(G)| = (0.3+0.5+0.6+0.3+0.5+0.6+0.4+0.5+0.3)+(0.6+0.8+0.9+0.6+0.8+0.9+0.5+0.7+0.9)$$

$$= 4 + 6.7 = 10.7$$

The cardinality of the set $\{b,d\}$ is:

$$|\underline{C}(G)| + |\overline{C}(G)| = \sum_{w\in\underline{C}(G)} \underline{R}V(W) + \sum_{w\in\overline{C}(G)} \overline{R}V(W)$$

$$|C(G)| = (0.4+0.5+0.6+0.6+0.5+0.6)+(0.5+0.7+0.9+0.6+0.7+0.8) = 3.2+4.1 = 7.3$$

Thus, the minimum dominating number is 7.3.

6.2. Comparison with Existing Models and Justification of Superiority

Classical directed graphs, simple fuzzy graphs, and even bipolar fuzzy graphs are examples of traditional trade network modeling techniques that have limited capacity to handle the multifaceted and unpredictable character of actual trade transactions. Graphs that are classically directed: Limitation: They don't take into consideration the strength, unpredictability, or variability of trade ties; instead, they solely depict binary relationships (the existence or lack of a link). The superiority of m-polar rough fuzzy digraphs With the addition of graded membership values, our model provides a more comprehensive depiction of economic scale, trust, and trade intensity. Fundamental Fuzzy Graphs:

Limitation: Although they provide partial membership, these usually employ a one-dimensional membership value that is unable to concurrently account for several influencing elements such as financial capability, trade volume, and reliability.

The benefit of m-polar fuzzy digraphs: Our model incorporates numerous aspects of trade evaluation by leveraging m-polar (particularly 3-polar) membership, allowing for a more thorough and accurate depiction. Intuitionistic or Bipolar Fuzzy Graphs:

Limitation: Although these can feature reluctance or both positive and negative degrees, they are still unable to adequately describe more than two attributes per entity or edge.

The m-polar technique is superior because it allows for several simultaneous qualities, which improves the accuracy of modeling intricate trade interactions.

6.3. Results and discussion

In this section, we have proposed the advantages and Limitations or weaknesses. Domination in rough m-polar fuzzy digraphs was successfully described and characterized for the first time in this paper. Our findings demonstrate that, in contrast to classical or fuzzy digraphs, domination works differently in rough m-polar fuzzy structures. This is mainly because rough approximations and many polarity levels add complexity. We proved several theorems, including existence conditions, bounds, and instances showing minimal dominating sets, that relate dominance numbers to the structural characteristics of the rough m-polar fuzzy digraphs.

We showed that the domination number of the final product digraph is typically more than or equal to the product of the domination numbers of the component digraphs for the tensor product of rough m-polar fuzzy digraphs. The roughness and polar levels given to the nodes and edges, however, affect the precise relationship. A number of properties were demonstrated, including the interaction between rough approximations during product operation and the retention of roughness levels. It was demonstrated through examples that the tensor product produces a more "complex" dominating landscape, which is useful for simulating complicated network phenomena such as fault tolerance and redundancy.

The dominating features were discovered to be considerably more dynamic in the case of the powerful product. Under particular rough m-polar conditions, it was found that strong products could, in some configurations, result in a decrease in domination numbers, providing possible optimization techniques for network control. For dominating equivalences

and inequalities in strong products of rough m-polar fuzzy digraphs, our theorems specify sufficient and necessary conditions.
Real-world networks with uncertain, incomplete, and multi-attribute relationships—like social influence networks and multi-agent communication systems—were the focus of the application investigation. It was shown through case studies that domination sets found with rough m-polar fuzzy digraphs may represent crucial agents and control points more precisely than conventional models. This demonstrates the usefulness of our theoretical contributions in domains such as decision support systems, network security, and epidemic control.

6.4. Limitation

- The system of requirements has significant computational assets and periods, which could be a limitation for real submissions.
- The accuracy of the system is highly dependent on the excellence and precision of the exertion data.
- The effort of the cubic Fermatean fuzzy Einstein aggregation operator's strength attitude challenges in application and understanding for experts not familiar with liberal fuzzy set theory.
- The technique's strength meets difficulties when applied to larger and extra compound decision-making states, perhaps cautioning its scalability.

7. Conclusion

We have presented and investigated the idea of domination in rough m-polar fuzzy digraphs in this work. The advantages of directed graph theory, m-polar fuzzy logic, and rough set theory are combined in this innovative framework. This integration makes it possible to describe systems with ambiguity, uncertainty, and multi-valued interactions more effectively. These systems are frequently seen in real-world applications, including complex communication systems, social network analysis, and decision-making. We started by outlining the fundamental concepts that support this novel graph model, offering a solid theoretical framework for more research. Along with a thorough analysis of its essential characteristics, a formal definition of domination in this context was provided. Through the introduction of mechanisms that take into consideration numerous viewpoints, varying degrees of truth, and ambiguous borders, these fundamental ideas broaden the conventional concept of domination. In addition, we examined two significant graph operations in the context of rough m-polar fuzzy digraphs: the tensor product and the strong product. In this context, these operations were reinterpreted, and their effect on dominance was examined. The findings provided a greater understanding of graph behavior under compound operations by illustrating how intricate structures and their relationships affect domination features. To validate the theoretical notions and give a better grasp of how they are implemented, a real-world numerical example was added. Lastly, we talked about

possible uses, highlighting how the suggested model might be used in fields that call for deft handling of granularity and uncertainty. Overall, by applying domination theory to more realistic and broader network models, this work offers up new avenues for its investigation.
Future studies may embrace algorithmic progress for efficient multiplication, evaluation of added graph operations, and application to everyday datasets. This development advances graph science theory as well as its application in intricate, data-rich settings.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Compliance with Ethical Standards: This study is not sup-ported by any source or any organizations.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest The authors declare that they have no conflict of interest.

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References

[1] R. Abu-Gdairi, A. A. El-Atik, and M. K. El-Bably. Topological visualization and graph analysis of rough sets via neighborhoods: A medical application using human heart data. *AIMS Mathematics*, 8(11):26945–26967, 2023.

[2] U. Ahmad and T. Batool. Domination in rough fuzzy digraphs with application. *Soft Computing*, 27(5):2425–2442, March 2023.

[3] R. Biswas. On rough sets and fuzzy rough sets. *Bulletin of the Polish Academy of Sciences-Mathematics*, 42(4):345–350, 1994.

[4] M. K. El-Bably, R. Abu-Gdairi, K. K. Fleifel, and M. A. El-Gayar. Exploring -basic rough sets and their applications in medicine. *European Journal of Pure and Applied Mathematics*, 17(4):3743–3771, 2024.

[5] M. K. El-Bably, R. A. Hosny, and M. A. El-Gayar. Innovative rough set approaches using novel initial-neighborhood systems: Applications in medical diagnosis of covid-19 variants. *Information Sciences*, page 122044, 2025.

[6] W. R. Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In *NAFIPS/IFIS/NASA'94. Proceedings of the First International Joint Conference of The North American Fuzzy*

Information Processing Society Biannual Conference: The Industrial Fuzzy Control and Intelligent Systems, pages 305–309. IEEE, December 1994.

[7] K. T. Atanassov and K. T. Atanassov. *Intuitionistic Fuzzy Sets.* Physica-Verlag HD, 1999.

[8] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[9] A. Kauffman. *Introduction à la théorie des sous-ensembles flous, 1.* 1973.

[10] P. Bhattacharya. Some remarks on fuzzy graphs. *Pattern Recognition Letters*, 6(5):297–302, December 1987.

[11] P. Bhattacharya and N. P. Mukherjee. Fuzzy relations and fuzzy groups. *Information Sciences*, 36(3):267–282, September 1985.

[12] J. N. Mordeson and P. Chang-Shyh. Operations on fuzzy graphs. *Information Sciences*, 79(3-4):159–170, July 1994.

[13] J. N. Mordeson and P. S. Nair. Fuzzy graphs. In *Fuzzy Graphs and Fuzzy Hypergraphs*, pages 19–81. Physica-Verlag HD, Heidelberg, 2000.

[14] A. Nagoorgani, M. Akram, and S. Anupriya. Double domination on intuitionistic fuzzy graphs. *Journal of Applied Mathematics and Computing*, 52:515–528, October 2016.

[15] A. Nagoorgani and S. Latha. Isomorphism on irregular fuzzy graphs. *International Journal of Mathematical Sciences and Engineering Applications*, 6(3):193–208, 2012.

[16] X. Zhang, J. Dai, and Y. Yu. On the union and intersection operations of rough sets based on various approximation spaces. *Information Sciences*, 292:214–229, January 2015.

[17] J. Chen, S. Li, S. Ma, and X. Wang. m-polar fuzzy sets: An extension of bipolar fuzzy sets. *The Scientific World Journal*, pages 1–10, June 2014.

[18] Z. Pawlak. Rough sets. *International Journal of Computer & Information Sciences*, 11:341–356, October 1982.

[19] Z. Pawlak. Rough sets, rough relations and rough functions. *Fundamenta Informaticae*, 27(2-3):103–108, January 1996.

[20] Didier Dubois and Henri Prade. Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*, 17(2-3):191–209, June 1990.

[21] M. Akram. Bipolar fuzzy graphs. *Information Sciences*, 181(24):5548–5564, December 2011.

[22] M. Akram, M. Arshad, and Shumaiza. Fuzzy rough graph theory with applications. *International Journal of Computational Intelligence Systems*, 12(1):90–107, 2018.

[23] Muhammad Akram, Uzma Fatima, and José Carlos Rodríguez Alcantud. Group decision-making method based on pythagorean fuzzy rough numbers. *Journal of Applied Mathematics and Computing*, 71(2):2179–2210, 2025.

[24] M. Akram, A. Ashraf, and M. Sarwar. Novel applications of intuitionistic fuzzy digraphs in decision support systems. *The Scientific World Journal*, 2014:1–12, 2014.

[25] M. Akram, S. Siddique, and U. Ahmad. Menger's theorem for m-polar fuzzy graphs and application of m-polar fuzzy edges to road network. *Journal of Intelligent & Fuzzy Systems*, 41(1):1553–1574, January 2021.

[26] M. Akram and N. Waseem. Certain metrics in m-polar fuzzy graphs. *New Mathe-*

matics and Natural Computation, 12(2):135–155, July 2016.

[27] A. Fahmi, S. Abdullah, F. Amin, and A. Ali. Weighted average rating (war) method for solving group decision-making problems using a triangular cubic fuzzy hybrid aggregation (tcfha) operator. *Punjab University Journal of Mathematics*, 50(1), 2020.

[28] A. Fahmi, R. Ahmed, M. Aslam, T. Abdeljawad, and A. Khan. Disaster decision-making with a mixing regret philosophy ddas method in fermatean fuzzy numbers. *AIMS Mathematics*, 8(2):3860–3884, 2023.

[29] A. Fahmi, F. Amin, S. M. Eldin, M. Shutaywi, W. Deebani, and S. Al Sulaie. Multiple attribute decision-making based on fermatean fuzzy numbers. *AIMS Mathematics*, 8(5):10835–10863, 2023.

[30] A. Fahmi, M. Aslam, and R. Ahmed. Decision-making problem based on a generalized interval-valued bipolar neutrosophic einstein fuzzy aggregation operator. *Soft Computing*, 27(20):14533–14551, 2023.

[31] A. Fahmi, M. A. S. Hassan, A. Khan, T. Abdeljawad, and D. K. Almutairi. A bipolar fermatean fuzzy hamacher approach to group decision-making for electric waste. *European Journal of Pure and Applied Mathematics*, 18(1):5691–5691, 2025.

[32] A. Fahmi, A. Khan, T. Abdeljawad, and M. A. Alqudah. Natural gas based on combined fuzzy topsis technique and entropy. *Heliyon*, 10(1), 2024.

[33] A. Fahmi, A. Khan, Z. Maqbool, and T. Abdeljawad. Circular intuitionistic fuzzy hamacher aggregation operators for multi-attribute decision-making. *Scientific Reports*, 15(1):5618, 2025.

[34] R. A. Hosny, R. Abu-Gdairi, and M. K. El-Bably. Enhancing dengue fever diagnosis with generalized rough sets: Utilizing initial-neighborhoods and ideals. *Alexandria Engineering Journal*, 94:68–79, 2024.

[35] R. A. Hosny, M. K. El-Bably, and M. A. El-Gayar. Primal approximation spaces by -neighborhoods with applications. *European Journal of Pure and Applied Mathematics*, 18(1):5827–5827, 2025.

[36] S. Y. Wu and Y. M. Kao. The compositions of fuzzy digraphs. *Journal of Research in Education Sciences*, 31:603–628, 1986.

[37] F. Feng, C. Li, B. Davvaz, and M. I. Ali. Soft sets combined with fuzzy sets and rough sets: a tentative approach. *Soft Computing*, 14:899–911, July 2010.

[38] K. Anitha, R. Aruna Devi, M. Munir, and K. S. Nisar. Metric dimension of rough graphs. *International Journal of Nonlinear Analysis and Applications*, 12:1793–1806, December 2021.

[39] D. J. Prassanna. Domination in fuzzy graphs. *International Journal of Advanced Research in Engineering and Technology (IJARET)*, 10(6):442–447, November 2019.

[40] N. Ishfaq, S. Sayed, M. Akram, and F. Smarandache. Notions of rough neutrosophic digraphs. *Mathematics*, 6(2):18, January 2018.

[41] M. Shokry. Fuzzy and rough approximations operations on graphs. *IOSR Journal of Mathematics (IOSR-JM)*, 11(3):66–72, May 2015.

[42] A. Somasundaram. Domination in products of fuzzy graphs. *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, 13(2), April 2005.

[43] D. I. Taher, R. Abu-Gdairi, M. K. El-Bably, and M. A. El-Gayar. Decision-making

in diagnosing heart failure problems using basic rough sets. *AIMS Mathematics*, 9(8):21816–21847, 2024.

[44] A. A. Talebi and S. O. Amiri. On a rough cayley graph related to conjugacy classes. *Journal of the Indonesian Mathematical Society*, 26(3):275–285, November 2020.

[45] A. E. El-Atik, M. K. El-Bably, and M. A. El-Gayar. Topological visualization of rough sets by neighborhoods and a heart application based graphs, 2022. Preprint.

[46] D. Chitcharoen and P. Pattaraintakorn. Towards theories of fuzzy set and rough set to flow graphs. In *2008 IEEE International Conference on Fuzzy Systems (IEEE World Congress on Computational Intelligence)*, pages 1675–1682. IEEE, June 2008.

[47] H. Khan, W. K. Alqurashi, J. Alzabut, D. K. Almutairi, and M. A. Azim. Artificial intelligence and neural networking for an analysis of fractal-fractional zika virus model. *Fractals*, 2025.

[48] H. Khan, J. Alzabut, M. Tounsi, and D. K. Almutairi. Ai-based data analysis of contaminant transportation with regression of oxygen and nutrients measurement. *Fractal & Fractional*, 9(2), 2025.

[49] O. T. Manjusha and M. S. Sunitha. Strong domination in fuzzy graphs. *Fuzzy Information and Engineering*, 7(3):1–9, September 2015.

[50] A. Nagoorgani and V. T. Chandrasekaran. Domination in fuzzy graph. *Advances in Fuzzy Sets and Systems*, 1(1):17–26, June 2006.

[51] A. Nagoorgani and J. Malarvizhi. Properties of µ-complement of a fuzzy graph. *International Journal of Algorithms, Computing and Mathematics*, 2(3):73–83, 2009.

[52] A. S. Nawar, R. Abu-Gdairi, M. K. El-Bably, and H. M. Atallah. Enhancing rheumatic fever analysis via tritopological approximation spaces for data reduction. *Malaysian Journal of Mathematical Sciences*, 18(2):321–341, 2024.

[53] L. Rolka and A. Mieszkowicz-Rolka. Labeled fuzzy rough sets versus fuzzy flow graphs. In *International Conference on Fuzzy Computation Theory and Applications*, volume 3, pages 115–120. SCITEPRESS, November 2016.

[54] M. Sarwar and M. Akram. Representation of graphs using m-polar fuzzy environment. *Italian Journal of Pure and Applied Mathematics*, 38:291–312, July 2017.

[55] S. I. Mohammad, N. Yogeesh, N. Raja, R. Chetana, M. S. Ramesha, and A. Vasudevan. Optimizing mimo antenna performance using fuzzy logic algorithms. *Applied Mathematics*, 19(2):349–364, 2025.

[56] D. A. Dewi, S. Surono, R. Thinakaran, and A. Nurraihan. Hybrid fuzzy k-medoids and cat and mouse-based optimizer for markov weighted fuzzy time series. *Symmetry*, 15(8):1477, 2023.

[57] D. A. Xavior, F. Isido, and V. M. Chitra. On domination in fuzzy graphs. *International Journal of Computing Algorithm*, 2(2):81–82, 2013.

[58] E. Enriquez, G. Estrada, C. Loquias, R. J. Bacalso, and L. Ocampo. Domination in fuzzy directed graphs. *Mathematics*, 9(17):2143, September 2021.