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Dynamics of Competitive Stochastic Predator-Prey Model with Holling Type II under Small Random Immigration

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Abstract. In this paper, we introduce a novel competitive stochastic predator-prey model by means of Holling type II with small random immigration. A Series of analytical results are conducted on the model such as the boundedness of the model's solution, stochastic mean square stability, and the conditions of the persistence and extinction of the predator-prey population. All analytical results were validated with numerical simulations by choosing different sets of small random immigration. Conditions on the random immigration under which the system is stochastically stable are specified. In addition, small immigration has been shown to have a great impact on persistence and extinction of the system, which can play an important role in species conservation, especially by controlling the conditions of stochastic small immigration.

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Key Words and Phrases: Stochastic predator-prey model, Persistence, Extinction, Small random immigration

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1. Introduction

One of the most interesting topics in biomathematics is the study of the dynamical behavior of the so-called predator-prey models. Most certainly, the initiative work in this direction is that of Lotka [1] and Voltera [2]. However, some of the flaws with this model are that the interaction between the prey and predator populations is instantaneous, which may not be the case in many natural ecosystems; moreover, the model presupposes that the prey population has infinitely many resources to consume. Lotka-Voltera model has been developed in its structure to display more realistic features of the dynamics

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by many researchers, see, for example, [3–9]. Due to many questions that come from ecological points of view, up to date many researchers are still working to answer such questions, for some examples of predator-prey models include the Allee effect, the fear effect, immigration, prey refuge, and cannibalism see, for example, [10–13].

One of the important and recent external elements that is considered to play a crucial role in the dynamics of predator-prey models is immigration. This factor could lead to stabilizing unstable models as in [14], and also helps in understanding the persistence and extinction of some models that were not clear to explain [15, 16]. In general, many of these elements are described by means of deterministic system of ordinary differential equations, which have it is own shortcomings in considering natural fluctuations that could affect the ecological environment. This constraint has increased interest in stochastic modeling tools [17–21], which capture the natural unpredictability that affects ecological systems. Small random immigration has mostly impacted simpler predator-prey interactions, such as Holling type I models [22]. These studies often focus on specific elements, such as stochastic square stability, without completely addressing wider dynamical behaviors like persistence and extinction. This indicates a significant research gap: understanding how random immigration effects more complicated predator-prey systems, particularly those regulated by Holling type II responses, which better depict predator saturation and are ecologically more realistic.

Our goal in this paper is to fill this gap by investigating a stochastic predator-prey model with Holling type II dynamics and extra ecological interactions, emphasizing how random immigration influences the boundedness of solutions, stochastic stability, persistence, and extinction of both prey and predator populations. To provide a more complete understanding of these complex dynamics, we use both rigorous mathematical analysis and numerical simulation.

2. Problem Statement

We consider a deterministic competitive predator-prey model with Holling type II which has been used in earlier works for various perspectives [23–26]. The model equations are defined below

$$\frac{du}{dt} = \rho u\left(1 - \frac{u}{k}\right) - \frac{\alpha uv}{1+\beta u} \tag{1}$$

$$\frac{dv}{dt} = -\sigma v + \frac{\gamma uv}{1+\beta u} - \lambda v^2 \tag{2}$$

$$u_0 = u(0), v_0 = v(0) \tag{3}$$

where u and v are the prey and predator populations, respectively. Recently, Jawdat et al. [26], studied system (1)-(3) with small immigration in the prey population. The model's stability and coexistence and extinction conditions between prey and predator populations were investigated. However, the effects of small immigration for both populations were not included. Hence, in this paper, we aim to study the stochastic counterpart of system

(1)-(3), more precisely, we consider small random immigration on both prey and predator populations by means of the following system:

$$\begin{cases} du = \left[\rho u\left(1-\dfrac{u}{k}\right) - \dfrac{\alpha uv}{1+\beta u}\right] dt + \delta u dB_1(t) & (4)\\ dv = \left[-\sigma v + \dfrac{\gamma uv}{1+\beta u} - \lambda v^2\right] dt + \epsilon v dB_2(t) & (5)\\ u_0 = u(0), \quad v_0 = v(0), & (6)\end{cases}$$

In general, the mathematical model (4)-(6) represents prey-predator interactions, where the term $\rho u\left(1-\frac{u}{k}\right)$ is the logistic growth rate for the prey population, $\frac{\alpha uv}{1+\beta u}$ is the functional response (which presents the prey consumption rate by the predator population), the term $-\sigma v$ is the loss rate for the predator population, the term $\frac{\gamma uv}{1+\beta u}$ is the response of predator population density through consumption of the prey population, λv^2 is the effect of intraspesific competition for predator population. $\delta u dB_1(t)$ and $\epsilon v dB_2(t)$ refer to low-intensity stochastic processes that represent small random immigration in the populations (prey and predator). For more detailed interpretation on the parameters, we have the following:

- ρ is the rate of growth in the prey population.
- k is the carrying capacity of the model.
- α is the rate of catch prey population by the predator.
- β is the handling rate of the predator.
- σ is the natural death rate in the predator population.
- γ is the effectiveness of transforming ate up prey into predator birth.
- λ is the intraspesific competition between predators.
- δdB_1 and ϵdB_2 are the small random immigration in the prey and predator populations, respectively.

The stochastic processes $B_1(t)$ and $B_2(t)$ are Brownian motions with the following properties:

- $B_1(0) = B_2(0) = 0$.
- $B_1(t)$ and $B_2(t)$ are continuous functions with probability 1, for all $t \in [0, T]$.
- $B_1(t)$ and $B_2(t)$ have stationary independent increments.
- $B_1(t+s) - B_1(t)$ and $B_2(t+s) - B_2(t)$ are normally distributed with zero mean and variance t.

3. Model Boundedness

As it is one of main validations of the model, in this section, we use some probabilistic and analytic in inequalities to obtain the boundedness of the model. System (4)-(6) could be written as:

$$dX = M(t,X)dt + N(t,X)dB \tag{7}$$
$$X_0 = X(0) \tag{8}$$

such that

$$M : \left[[0,T] \times \mathbb{R}^+\right]^2 \to \mathbb{R}$$
$$N : \left[[0,T] \times \mathbb{R}^+\right]^2 \to \mathbb{R}$$

$M(t,X)$ and $N(t,X)$ are given by

$$M(t,X) = \begin{bmatrix} \rho u \left(1 - \frac{u}{k}\right) - \frac{\alpha uv}{1+\beta u} \\ -\sigma v + \frac{\gamma uv}{1+\beta u} - \lambda v^2 \end{bmatrix}, \quad N(t,X) = \begin{bmatrix} \delta u \\ \epsilon v \end{bmatrix} \quad X = \begin{bmatrix} u \\ v \end{bmatrix} \quad dB(t) = \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}$$

Lemma 1. *For some positive constant C, the functions $M(t,X)$ and $N(t,X)$ in (7) satisfy the following linear growth condition*

$$\mathbb{E}\|M(t,X)\|^2 + \mathbb{E}\|N(t,X)\|^2 \le C(1 + \mathbb{E}\|X\|^2) \tag{9}$$

Proof. From (4) and the fact that ρ, k, α and β are positive, one sees that

$$u \le u_0 + \rho \int_0^t u ds + \delta \int_0^t u dB_1(s).$$

Taking the expectation on both sides and using Itô's isometry, we have

$$\mathbb{E}|u| \le u_0 + \rho \mathbb{E} \int_0^t |u| ds.$$

From Grownall's inequality, one gets

$$\mathbb{E}|u| \le u_0 e^{\rho t} \tag{10}$$

Now,

$$\begin{aligned} \mathbb{E}\|M(t,X)\|^2 &= \mathbb{E}\|\rho u \left(1 - \frac{u}{k}\right) - \frac{\alpha uv}{1+\beta u}\|^2 \\ &+ \mathbb{E}\| - \sigma v + \frac{\gamma uv}{1+\beta u} - \lambda v^2\|^2 \\ &\le \rho^2 \mathbb{E}\|u\|^2 + \gamma^2 E\|u\|^2 + \|v\|2 \\ &\le C\mathbb{E}\|u\|^2(1 + \|v\|^2) \end{aligned} \tag{11}$$

From (10) and (11) we have

$$\begin{aligned}\mathbb{E}\|M(t,X)\|^2 &\le C(1+\|v\|^2)\\ &\le C(1+\|u\|^2+\|v\|^2)\\ &\le C(1+\|X\|^2). \end{aligned}\tag{12}$$

It is easy to see that

$$\begin{aligned}\mathbb{E}\|N(t,X)\|^2 &= \delta^2\mathbb{E}\|u\|^2+\epsilon^2\mathbb{E}\|v\|^2\\ &\le C(1+\|u\|^2+\|v\|^2)\\ &\le C(1+\|X\|^2). \end{aligned}\tag{13}$$

From (12) and (13) we conclude the proof.

Theorem 1. *For all $p>1$ and some $C>0$, the following condition is true*

$$\mathbb{E}\sup_{0\le t\le T}\|X\|^p \le C. \tag{14}$$

Proof. From (7), we have

$$X = X_0 + \int_0^t M(t,X)ds + \int_0^t N(t,X)dB(t), \tag{15}$$

it is known that see for example [27] for $\alpha_k \in \mathbb{R}$, $r \in [0,\infty)$ and $1 \le p < \infty$, that

$$\sum_{i=1}^r \alpha_i^p \le \left(\sum_{i=1}^r \alpha_i\right)^p \le r^{p-1}\sum_{i=1}^r \alpha_i^p. \tag{16}$$

From (16), the sup over $[0,T]$ and the expectation we have the following:

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|X\|^q \le& 3^{q-1}\Bigg(|X_0|^q + \mathbb{E}\sup_{0\le t\le T}\left|\int_0^t M(t,X)dt\right|^q\\ &+\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t N(t,X)dB(t)\right|^q\Bigg). \end{aligned}\tag{17}$$

Using Hölder inequality for

$$\frac{1}{q}+\frac{1}{\frac{q}{q-1}}=1,$$

we get

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t M(t,X)dt\right|^q &\le \mathbb{E}\left|\int_0^T M(t,X)dt\right|^q\\ &\le \mathbb{E}\left(\left|\int_0^T M(t,X)dt\right|^q\right)^{q\times\frac{1}{q}} \times \left(\int_0^T (1)^{\frac{q}{q-1}}\,dt\right)^{q\times\frac{q-1}{q}}\\ &\le T^{q-1}\mathbb{E}\left(\left|\int_0^T M(t,X)dt\right|^q\right). \end{aligned}\tag{18}$$

From Lemma 1 and (16) we have

$$\begin{aligned}\mathbb{E}\int_0^T \left(|M(t,X)|^2\right)^{\frac{q}{2}} dt &\leq \mathbb{E}C^{\frac{q}{2}} \int_0^T \left(1+\|X\|^2\right)^{\frac{q}{2}} dt \\ &\leq C^{\frac{q}{2}} 2^{\frac{q}{2}-1} \mathbb{E}\int_0^T (1+\|X\|^q) dt \\ &\leq C^{\frac{q}{2}} 2^{\frac{q}{2}-1} \left(T + \mathbb{E}\int_0^T \|X\|^q dt\right). \qquad (19)\end{aligned}$$

From the inequality (18) and (19), we have

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t M(t,X)dt\right|^q \leq \left[C^{\frac{q}{2}} 2^{\frac{q}{2}-1} T^q + C^{\frac{q}{2}} 2^{\frac{q}{2}-1} T^{q-1}\right] \mathbb{E}\int_0^T \|X\|^q dt. \qquad (20)$$

To deal with the stochastic term, we employ Burkholder-Davis-Gundyand Hölder inequalities and it yields

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t N(t,X)dB(s)\right|^q &\leq C_B \mathbb{E}\left(\int_0^T |N(t,X)|^2 dt\right)^{\frac{q}{2}} \\ &C_B\left(\mathbb{E}\int_0^T |N(t,X|^q dt\right)\left(\int_0^T (1)^{\frac{q}{q-2}} dt\right)^{\frac{q}{2}\times\frac{q-2}{q}} \\ &\leq C_B T^{\frac{q-2}{2}} \mathbb{E}\int_0^T |N(t,X)|^q dt. \qquad (21)\end{aligned}$$

Again we use Lemma 1 and inequality (16) for $p=\frac{q}{2}$ to get

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t N(s,X)ds\right| &\leq C_B T^{\frac{q-2}{2}} C^{\frac{q}{2}} 2^{\frac{q}{2}-1} \mathbb{E}\int_0^T (1+\|X\|^q) dt \\ &\leq C_B T^{\frac{q-2}{2}} C^{\frac{q}{2}} 2^{\frac{q}{2}-1}\left[T + \mathbb{E}\int_0^T \|X\|^q dt\right] \\ &\leq C_B T^{\frac{q}{2}} C^{\frac{q}{2}} 2^{\frac{q}{2}-1} + C_B T^{\frac{q-2}{2}} C^{\frac{q}{2}} 2^{\frac{q}{2}-1} \mathbb{E}\int_0^T \|X\|^q dt. \qquad (22)\end{aligned}$$

From (17), (21) and (22) one uses Grownall's inequality to complete the proof.

4. Stochastic mean square stability

Here, we investigate the stochastic mean square stability for the system (4)-(6) around the interior equilibrium point of the deterministic version (1)-(3). Note that our system has three non-negative equilibrium points:

1. $P_0 = (0,0)$

2. $P_1 = (k, 0)$
3. $P^* = (u^*, v^*)$, where (u^*, v^*) are the real positive roots of the following system:

$$\begin{cases} \rho u^* \left(1 - \dfrac{u^*}{k}\right) - \dfrac{\alpha u^* v^*}{1 + \beta u^*} = 0, & (23) \\ -\sigma v^* + \dfrac{\gamma u^* v^*}{1 + \beta u^*} - \lambda {v^*}^2 = 0. & (24) \end{cases}$$

The stochastic mean square stability will be studied around the interior positive equilibrium point (u^*, v^*), we have

$$\begin{cases} du = \left(\rho u \left(1 - \dfrac{u}{k}\right) - \dfrac{-\alpha uv}{1 + \beta u}\right) dt + \delta(u - u^*) dB_1, & (25) \\ dv = \left(-\sigma v + \dfrac{\gamma uv}{1 + \beta u}\right) dt - \lambda v^2 + \epsilon(v - v^*) dW_2. & (26) \\ u_0 = u(0), \quad v_0 = v(0). & (27) \end{cases}$$

By setting $x = u - u^*$ and $y = v - v^*$, we transform the model into:

$$dX = f(t, X) dt + g(t, X) dB, \tag{28}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix},$$

$$f(t, X) = \begin{bmatrix} u^* \frac{\partial k_1(u^*, u^*)}{\partial u} + k_1 & u^* \frac{\partial k_1(u^*, u^*)}{\partial v} \\ v^* \frac{\partial k_2(u^*, v^*)}{\partial u} & \frac{v^* \partial k_2(u^*, v^*)}{\partial v} + k_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$k_1 = \rho \left(1 - \frac{u}{k}\right) - \frac{\alpha v}{1 + \beta u},$$
$$k_2 = -\sigma + \frac{\gamma u}{1 + \beta u} \lambda v.$$

This gives

$$f(t, X) = \begin{bmatrix} u^* \left(\frac{-\rho}{k} + \frac{\alpha \beta v^*}{(1+\beta u)^2}\right) & u^* \left(\frac{-\alpha}{1+\beta u^*}\right) \\ v^* \left(\frac{(1+\beta u^*)\gamma - \gamma \beta u^*}{(1+\beta u^*)^2}\right) & -\lambda v^* \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and

$$g(t, X) = \begin{bmatrix} \delta x & 0 \\ 0 & \epsilon y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

For strictly positive t_0, we define the cylindrical domain $D = \mathbb{R}^2 \times [t_0, \infty)$. Hence, we introduce the function $\varphi \in C^2(D)$ as $\varphi : D \to \mathbb{R}^+$. As in [28] the generator of Markov process for equation (28) is given by

$$L\varphi(t,X) = \frac{\partial \varphi(t,X)}{\partial t} + f^T(t,X)\frac{\partial \varphi(t,X)}{\partial X} + \frac{1}{2}Trace\left[g^T(t,X)\frac{\partial^2 \varphi(t,X)}{\partial X^2}g(t,X)\right]. \tag{29}$$

The following theorem states the mean-square stability for our system (28)

Theorem 2. *Assume that*

$$\frac{\rho u^*}{k} > \frac{\alpha\beta u^* v^*}{(1+\beta u^*)^2} + \frac{\delta^2}{2}, \tag{30}$$

$$\lambda v^* > \frac{\epsilon^2}{2}, \tag{31}$$

then, system (23)-(27) is mean-square asymptotically stable.

Proof. Let

$$\varphi(t,X) = \frac{1}{2}\left(a_1 x^2 + a_2 y^2\right),$$

be the Markovian process generator, where $a_i (i = 1,2)$ are some parameters to be determined later. Then we have the following

$$\frac{\partial \varphi}{\partial t} = 0, \quad \frac{\partial \varphi}{\partial X} = (a_1 x, a_2 y), \tag{32}$$

$$f^t(t,X)\frac{\partial \varphi}{\partial X} = a_1 u^2\left(\frac{-\rho}{k} + \frac{\alpha\beta v^*}{(1+Bu^*)^2}\right) - \frac{a_1 \alpha u^*}{1+\beta u^*}uv + \frac{a_2 \alpha v^*}{(1+\beta u^*)^2}uv - a_2 \lambda v^* v^*, \tag{33}$$

and

$$\frac{1}{2}Trace\left[g^T(t,X)\frac{\partial^2 \varphi(t,X)}{\partial X_t^2}g(t,X)\right] = \frac{1}{2}a_1 u^2 + \frac{1}{2}a_2 v^2 \epsilon^2. \tag{34}$$

Substituting (32)-(34) into (29), we get

$$\begin{aligned} L(\varphi(t,X)) &= u^*\left(\frac{-\rho}{k} + \frac{\alpha\beta v^*}{(1+Bu^*)^2}\right)u^2 a_1 - \frac{\alpha u^*}{1+\beta u^*}uva_1 \\ &\quad + \frac{\alpha v^*}{(1+\beta u^*)^2}uva_2 - \lambda v^* v^* a_2 + \frac{1}{2}a_1\delta^2 u^2 + \frac{1}{2}a_2\epsilon^2 v^2 \\ &= \left(\frac{-\rho}{k}u^* - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{1}{2}\delta^2\right)a_1 u^2 \\ &\quad - \left(\lambda v^* - \frac{1}{2}\epsilon^2\right)a_2 v^2 - \left(\frac{\alpha u^*}{1+\beta u^*}a_1 - \frac{\alpha v^*}{(1+\beta u^*)^2}a_2\right)uv. \end{aligned} \tag{35}$$

Let $\alpha u^* a_1 = \frac{\gamma v^*}{1+\beta u^*} a_2$, this implies

$$L(\varphi(t,X)) = \left(\frac{-\rho u^*}{k} - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{1}{2}\delta^2\right) a_1 u^2 - \left(\lambda v^* - \frac{1}{2}\epsilon^2\right) a_2 v^2. \tag{36}$$

For $L(\varphi(t,X)) < 0$, then

$$\frac{-\rho u^*}{k} > \frac{\alpha\beta u^* v^*}{(1+\beta u^*)^2} + \frac{1}{2}\delta^2,$$
$$\lambda v^* > \frac{1}{2}\epsilon^2.$$

This completes the proof.

Corollary 1. *When conditions* (30) *and* (31) *are not satisfied, then the system* (4)-(6) *is unstable.*

Remark 1. *The main factors that affect the system's stability are the rate of growth in the prey population, the carrying capacity of the model, the rate of catching prey population by the predator, the handling rate of the predator, intraspecific competition between predators, and the small random immigration in the prey and predator populations.*

5. Persistence and extinction

In this section, we will study the persistence for both prey and predator populations and the extinction for the prey population.

5.1. Persistence of the prey population

The following lemma was collected from [29].

Lemma 2.

$$\limsup_{t\to\infty} \sup_{0\le t\le T} \frac{1}{t}\ln u \le 0 \quad a.s.. \tag{37}$$

Theorem 3. *The prey population* u *admits weak persistence in the average with probability almost if*

$$\rho - \frac{\delta^2}{2} > 0.$$

Proof. To prove this, we need to show that $u^l > a > 0$, where

$$u^l := \limsup_{t\to\infty} \frac{1}{t}\int_0^t u ds, \tag{38}$$

we prove this by contradiction. Assume that ϱ_1 be small enough such that

$$-\sigma - \frac{1}{2}\epsilon^2 + \gamma\varrho_1 < 0, \tag{39}$$

and

$$\rho - \frac{1}{2}\delta^2 - \frac{\rho}{k}\varrho_1 > 0,$$

for each $\varrho_1 > 0$, there exists a solution $(\hat{u}, \hat{v})$ where

$$P\{u^l < \varrho_1\}. \tag{40}$$

Therefore

$$d\ln\hat{v} \leq [-\sigma - \frac{\epsilon^2}{2} + \gamma\hat{u}]dt + \epsilon dB_2.$$

Once again, we apply the integration over $(0, t)$ and divide by t, we get

$$\begin{aligned}\frac{\ln\hat{v} - \ln\hat{v}_0}{t} &\leq \frac{1}{t}\int_0^t \left(-\sigma - \frac{\epsilon^2}{2}\right) ds + \frac{1}{t}\int_0^t \gamma\hat{u} ds + \frac{1}{t}\int_0^t \epsilon dB_2(s),\\ &= -\sigma - \frac{\epsilon^2}{2} + \frac{\gamma}{t}\frac{1}{t}\int_0^t \gamma\hat{u} ds + \frac{1}{t}\int_0^t \epsilon dB_2(s).\end{aligned} \tag{41}$$

As before, from the strong law of large number, we get

$$\limsup_{t\to\infty}\frac{\ln\hat{v}}{t} \leq -\sigma - \frac{\epsilon^2}{2} + \gamma\varrho_1 < 0, \tag{42}$$

this implies that $\lim\hat{v} = 0$. Therefore,

$$d\ln(\hat{u}) = \left[\rho - \frac{\rho}{k}\hat{u} - \frac{\alpha\hat{u}\hat{v}}{1+\beta\hat{u}} - \frac{1}{2}\delta^2\right] dt + \delta dB_1(t). \tag{43}$$

Integration followed by division by t, give

$$\begin{aligned}\frac{\ln\hat{u} - \ln\hat{u}_0}{t} &= \frac{1}{t}\int_0^t \left(\rho - \frac{1}{2}\delta^2\right) ds - \frac{\rho}{kt}\int_0^t \hat{u} ds - \frac{\alpha}{t}\int_0^t \frac{\hat{u}\hat{v}}{1+\beta\hat{u}} ds + \frac{\delta}{t}\int_0^t dB_1(s),\\ &= \rho - \frac{1}{2}\delta^2 - \frac{\rho}{kt}\int_0^t \hat{u} ds - \frac{\alpha}{t}\int_0^t \frac{\hat{u}\hat{v}}{1+\beta\hat{u}} ds + \frac{\delta}{t}\int_0^t dB_1(s).\end{aligned}$$

Taking the $\limsup_{t\to\infty}$ in both sides with (39), (40) and the law of large number, we obtain

$$\limsup\frac{1}{t}\ln\hat{u} = \rho - \frac{\delta^2}{2} - \frac{\rho}{k}\varrho_1 > 0. \tag{44}$$

This contradicts the lemma (2). Therefore, $u^l > 0$. This completes the proof.

5.2. Extinction of the prey population

Theorem 4. *The prey population u goes to extinct with probability almost surely when*

$$\rho - \frac{\delta^2}{2} \leq 0. \tag{45}$$

Proof. Let us construct the following comparison system;

$$\begin{aligned} dU &\leq U\left(\rho - \frac{\rho U}{k}\right)dt + \delta U dB_1(t), \\ U_0 &= u_0. \end{aligned}$$

From Itô's Lemma for the function $\varphi(t,U) = \ln U$, we get

$$d\ln U = \left[\rho - \frac{\rho U}{k} - \frac{1}{2}\delta^2\right]dt + \delta dB_1(t). \tag{46}$$

Applying integration on the interval $(0,t)$ followed by division by t to get

$$\frac{\ln U - \ln u_0}{t} = \frac{1}{t}\int_0^t \left(\rho - \frac{\rho u}{k} - \frac{1}{2}\delta^2\right)ds + \frac{1}{t}\int_0^t \delta dB_1(s). \tag{47}$$

Let us note that from the strong law of large number, one gets

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \delta dB_1(s) = 0, \tag{48}$$

and this implies that

$$\limsup_{t\to\infty} \frac{\ln U}{t} \leq \rho - \frac{1}{2}\delta^2 < 0 \quad a.s. \tag{49}$$

Using the comparison theorem for SDEs, we have

$$\limsup_{t\to\infty} \frac{1}{t}\ln u < 0,$$

thus

$$\lim_{t\to\infty} u = 0.$$

5.3. Extinction for the predator population

Theorem 5. *If*

$$\gamma k\left(\rho - \frac{\delta^2}{2}\right) < \rho\left(\delta + \frac{\epsilon^2}{2}\right), \tag{50}$$

then the predator population goes to extinct with probability almost surely.

Proof. Let us consider the following two cases

(i) $\rho - \frac{\delta^2}{2} < 0$. It is clear that this will imply $u^l < 0$. As in (41), we have

$$\frac{\ln v - \ln v_0}{t} \leq -\sigma - \frac{\epsilon^2}{2} + \frac{\gamma}{t}\int_0^t u ds + \frac{\epsilon B_2(t)}{t}. \tag{51}$$

From this and the fact that $u^l < 0$, then

$$\limsup_{t\to\infty} \frac{\ln v}{t} \leq -\sigma - \frac{\epsilon^2}{2} < 0,$$

then $\lim_{t\to\infty v} = 0$, which means the extinction of predator population $V(t)$.

(ii) $\rho - \frac{\delta^2}{2} > 0$. In this case, one can always $\varrho_2 > 0$ at a time T_1 such that for $t > T_1$ and has $\frac{B_1(t)}{t} < \varrho_2$. Thus, we have

$$\begin{aligned}\ln u - \ln u_0 &\leq \int_0^t \left(\rho - \frac{\delta^2}{2}\right) ds - \frac{\rho}{k}\int_0^t u ds + \delta B_1(t). \\ &\leq \left(\rho - \frac{\delta^2}{2} + \varrho_2\right) t - \frac{\rho}{k}\int_0^t u ds.\end{aligned}$$

From [30, Lemma 2.2], one obtains

$$u^l \leq \frac{k\left(\rho - \frac{\delta^2}{2} + \varrho_2\right)}{\rho}$$

Now, since $\varrho_2 \to 0$ as $t \to \infty$, then we have

$$u^l \leq \frac{k\left(\rho - \frac{\delta^2}{2}\right)}{\rho}.$$

From (17), one sees that

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\ln v &\leq -\sigma - \frac{\epsilon^2}{2} + \gamma u^l \\ &\leq -\sigma - \frac{\epsilon^2}{2} + \frac{k\left(\rho - \frac{\delta^2}{2}\right)}{\rho} < 0.\end{aligned}$$

Thus $\lim_{t\to\infty} v = 0$.

Corollary 2. *The persistence of the predator population is obtained by reversing the condition* (50)*.*

6. Numerical Simulations and discussion

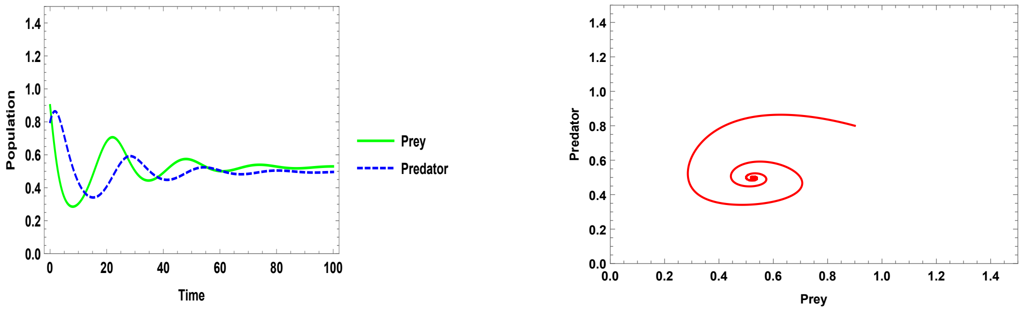
To validate our theoretical results presented in the previous sections, we performed several numerical simulations in this section. We employ the "StochasticRungeKuttaScalarNoise" command in MATHEMATICA 11.1 to perform numerical simulations, as described on the Wolfram website. The initial populations values are chosen to be

$$u_0 = 0.9, v_0 = 0.8.$$

And the set of parameters are chosen such that the stability and coexistence between prey and predator populations hold for the deterministic model (1)-(3) in order to check the effect of the random immigration of the system as follows:

$$\rho = 0.4,\ k = 2.0,\ \alpha = 0.75,\ \beta = 0.5,\ \sigma = 0.2,\ \gamma = 0.6,\ \lambda = 0.1. \tag{52}$$

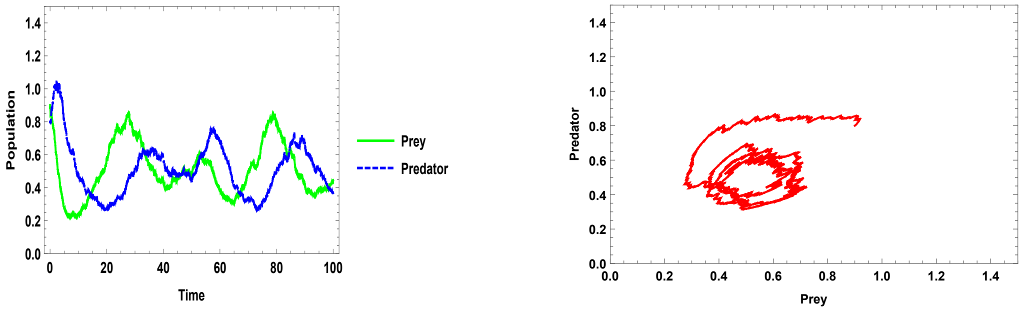
Since the model equations in (4)-(6) are taken to be dimensionless, the parameters were freely chosen. With these selections, our persistence equilibrium point is $(u^*, v^*) \approx (0.5254, 0.4965)$. Now, to study the impact of small random immigration on the prey-predator populations (u, v), several considerations of δ and ϵ were taken into account $(\delta = 0, \epsilon = 0)$, $(\delta = 0.05, \epsilon = 0.05)$, $(\delta = 0.4, \epsilon = 0.4)$, $(\delta = 0.4, \epsilon = 1.2)$ and $(\delta = 1.0, \epsilon = 0.1)$.



(a) Time series (b) Phase plane

Figure 1: Dynamical behaviour for the deterministic model (1)-(3) with initial values $(u_0 = 0.9, v_0 = 0.8.)$ and parameters values are given in (52).

Figure1 represents the dynamical behavior of the deterministic model (1)-(3) at the equilibrium point $(u^*, v^*) \approx (0.5254, 0.4965)$, in which the system is stable and persist. For more detailed analysis of this model, we refer to [25].



(a) Time series (b) Phase plane

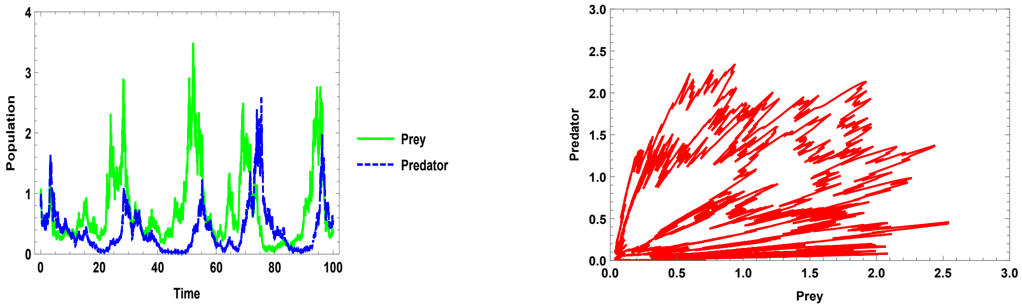
Figure 2: Dynamical behaviour for the stochastic model (4)-(6) with initial values $(u_0 = 0.9, v_0 = 0.8.)$, $(\delta = 0.05, \epsilon = 0.05)$ and parameters values are given in (52).

Figure 2 represents the stochastic model (4)-(6) with $(\delta = 0.05, \epsilon = 0.05)$, in which the calculations give the following:

- $\frac{\rho u^*}{k} - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{\delta^2}{2} \approx 0.0425 > 0.$

- $\lambda v^* - \frac{\epsilon^2}{2} \approx 0.0484 > 0.$
- $\rho - \frac{\delta^2}{2} \approx 0.3988 > 0.$
- $\gamma k\left(\rho - \frac{\delta^2}{2}\right) - \rho\left(\delta + \frac{\epsilon^2}{2}\right) \approx 0.4478 > 0.$

Thus, the conditions of Theorems 2 and 3 are satisfied, which means the system (4)-(6) is stochastically stable and the prey population u is weakly persistence in average with probability almost surely and the predator population v achieved the persistence through simulation under the same condition.



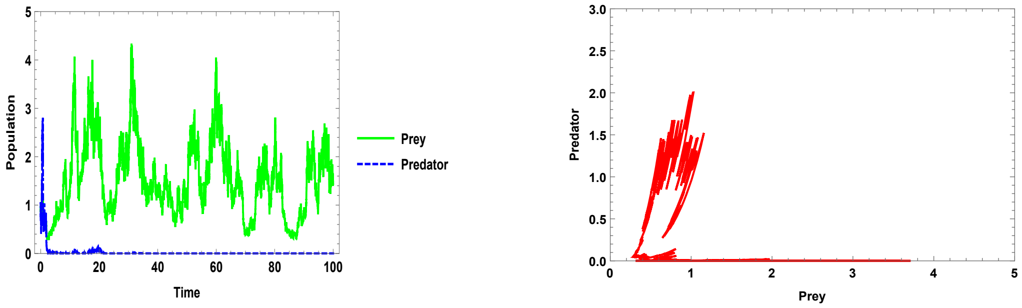
(a) Time series (b) Phase plane

Figure 3: Dynamical behaviour for the stochastic model (4)-(6) with initial values ($u_0 = 0.9, v_0 = 0.8.$), ($\delta = 0.4, \epsilon = 0.4$) and parameters values are given in (52).

Figure 3 indicates the stochastic model (4)-(6) with ($\delta = 0.4, \epsilon = 0.4$), in which the calculations give the following:

- $\frac{\rho u^*}{k} - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{\delta^2}{2} \approx -0.0363 < 0.$
- $\lambda v^* - \frac{\epsilon^2}{2} \approx 0 - 0.0305 < 0.$
- $\rho - \frac{\delta^2}{2} \approx 0.32 > 0.$
- $\gamma k\left(\rho - \frac{\delta^2}{2}\right) - \rho\left(\delta + \frac{\epsilon^2}{2}\right) \approx 0.096 > 0.$

From the above calculations, one can see that the conditions of Theorem 2 are not met, and therefore the system is unstable. The condition of Theorem 3 is satisfied, therefore the prey population is weakly persistent in average with probability almost surely. In the meanwhile, condition of Theorem 5 does not satisfied, which means that the predator population also persist. However, one can see that the destabilization of the system has the potential to affect its persistence.



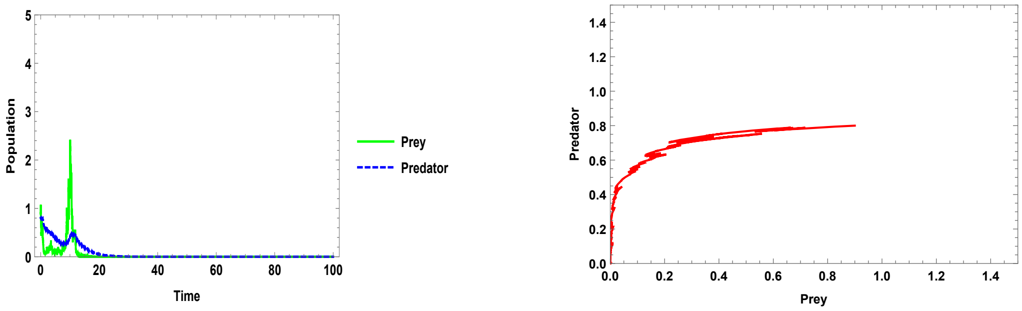
(a) Time series (b) Phase plane

Figure 4: Dynamical behaviour for the stochastic model (4)-(6) with initial values ($u_0 = 0.9, v_0 = 0.8.$), ($\delta = 0.4, \epsilon = 1.2$) and parameters values are given in (52).

Figure 4 represents the stochastic model (4)-(6) with ($\delta = 0.4, \epsilon = 1.2$), in which the calculations shows the following:

- $\frac{\rho u^*}{k} - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{\delta^2}{2} \approx -0.0363 < 0.$
- $\lambda v^* - \frac{\epsilon^2}{2} \approx 0 - 0.6704 < 0.$
- $\rho - \frac{\delta^2}{2} \approx 0.32 > 0.$
- $\gamma k \left(\rho - \frac{\delta^2}{2}\right) - \rho\left(\delta + \frac{\epsilon^2}{2}\right) \approx -0.288 < 0.$

From the above calculations, one can see that the conditions of the mean square stochastic stability Theorem 2 are not satisfied, so the system is unstable. Also, conditions of Theorems 3 and 5 are satisfied, which imply that the prey population u is weakly persist in average, while the predator population v go to extinct.



(a) Time series (b) Phase plane

Figure 5: Dynamical behaviour for the stochastic model (4)-(6) with initial values ($u_0 = 0.9, v_0 = 0.8.$), ($\delta = 1.0, \epsilon = 0.1$) and parameters values are given in (52).

Figure 5 represents the stochastic model (4)-(6) with ($\delta = 1.0, \epsilon = 0.1$), in which the calculations shows the following:

- $\frac{\rho u^*}{k} - \frac{\alpha\beta u^* v^*}{(1+Bu^*)^2} - \frac{\delta^2}{2} \approx -0.4563 < 0.$
- $\lambda v^* - \frac{\epsilon^2}{2} \approx 00.0447 > 0.$
- $\rho - \frac{\delta^2}{2} \approx -0.1 < 0.$
- $\gamma k \left(\rho - \frac{\delta^2}{2}\right) - \rho\left(\delta + \frac{\epsilon^2}{2}\right) \approx -0.723 < 0.$

Here, the calculations show that the conditions of Theorems 2 and 3 are not satisfied, while the condition of Theorem 5 is satisfied, which means that neither stability nor persistence of the stochastic system took place, but the system collapsed for both prey and predator populations. In this figure, high prey immigration results in species extinction, which contradicts the immigration argument. This phenomenon illustrates a case of destabilization within the ecological system due to the stochastic intensity of immigration, which prevents the system from self-regulating. In practice, this is referred to as the "paradox of enrichment".

7. Conclusion

In this paper, we investigated the effects of small random immigration in competitive predator- prey model with Holling type II. We obtained boundedness of the model's solution using both probabilistic and analytics inequalities. The conditions of stochastic mean square stability were derived (Theorem 2 and Corollary 1). A series of results on persistence and extinction for both prey and predator populations was obtained (Theorems 3 - 5 and Corollary 2). All theoretical results were numerically verified in Figures 1-5. From this work, we conclude the following:

- For fixed values of the parameters set (52), the stochastic stability of the model depends on the intensities of random immigration. When random immigration is small enough, the system becomes stochastically stable. However, as random immigration increases, the system is destabilized with high oscillations.
- The persistence of the prey population u clearly affected by δ; where δ was small enough, that is, $\rho - \frac{\delta^2}{2}$, the prey population u would persist with probability almost surely. As in condition (50) when ϵ was large enough, the predator population will extinct with probability almost surely. However, conditions (45) and (50)together emphasize that when δ was large enough (the immigration on the prey population was large enough), the whole system would extinct. This case is demonstrated in Figure 5.

To the best of the authors' knowledge, the first study to examine a competitive stochastic predator-prey model under the influence of small random immigration was [22], where the analysis focused on stochastic stability using a simpler model with a Holling type I functional response. In this paper, we present a comprehensive study of a more general and realistic competitive stochastic predator-prey model incorporating a Holling type II functional response, also subject to small random immigration. We investigate stochastic mean square stability and establish conditions for persistence and extinction of the species. In addition, our theoretical results are reinforced through numerical simulations, which illustrate and validate the analytical findings.

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