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Fixed Points of Mappings Contracting Perimeters of Polygons: A Geometric Generalization of Banach's Principle

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Abstract. In this paper, we investigate fixed points of mappings that contract the perimeters of polygons. Our study is motivated by the idea that the perimeter, as a global geometric measure, provides a more natural and flexible framework than the individual edge lengths when analyzing contraction properties in metric spaces. We extend Petrov's fixed point theorem from triangles to polygons with an arbitrary number of vertices and establish conditions under which such mappings admit unique fixed points. The methodology relies on generalizations of contraction mappings and properties of metric spaces. Our main contributions include a new perimeter-based contraction principle, a demonstration of its application in proving Banach's contraction theorem, and examples that validate the theoretical results. This generalization enriches the existing literature on fixed point theory and opens avenues for further applications in geometric analysis and related areas.

2020 Mathematics Subject Classifications: 47H09, 47H10, 30L15

Key Words and Phrases: Fixed point, perimeter of polygons, contractions, complete metric space

1. Introduction

The choice of considering the perimeter instead of the side lengths individually is motivated by both mathematical and geometric reasons. The perimeter provides a single aggregate measure that captures the overall contraction of a polygon, avoiding the need to track each side separately. This global perspective not only simplifies the analysis but also aligns naturally with contraction principles, which are often defined in terms of distances between objects rather than individual components. Furthermore, by contracting perimeters, we establish a unified framework that generalizes earlier results (such as

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Petrov's theorem for triangles) while offering broader applicability to polygons with arbitrary numbers of vertices. This approach highlights the novelty and significance of our contribution in comparison to other extensions of Banach's fixed point theorem.
We consider a mapping contracting perimeters of polygons and demonstrate the continuity of such mapping. We prove a fixed point theorem for this mapping. Through an example we illustrate that maximum number of the fixed points can be $n-1$. We deduce the well-known Banach fixed point theorem. Our work is generalization of Petrov fixed point theorem [1]. By defining the concept of mappings contracting perimeters of triangles and proving the fixed point theorem for such mappings, Petrov[1] (2023) provided a new generalization of the Banach fixed point theorem (BFPT). The key distinction is that, though the central hypothesis of his research is predicated on the concepts of Banach's classical theorem, the definition of this mapping is based on the mapping of three points of space rather than two. Further a condition is imposed on mapping L, $L(L(y)) \neq y$ for all $y \in Y$ such that $Ly \neq y$. One significant subclass of these mappings is the ordinary contraction mapping, from which we are able to derive the classical Banach theorem as a straightforward corollary right away. Banach developed the contraction principle in 1922 [2]. Though the concept of successive approximations had previously been mentioned in the writings of Chebysher, Picard, and others, it was applied in approximation theory, solutions to differential and integral equations, etc. The first person to correctly formulate this outcome was Banach. Over time, there have been numerous ways to generalize the Banach contraction principle. Additionally, metric extensions are typically verified against three classical fixed point theorems[3]. These are the extension of Banach's theorem to nonexpansive mappings[4], Caristi's theorem[5], and Nadler's well-known set-valued extension of Banach's theorem[6]. In present paper, we extend petrov's idea and prove Banach's theorem as a sub-result. The fundamental distinction is that mentioned contraction mapping is predicated on n points in the space. A mapping contracting perimeter of polygon that is not a contraction mapping is defined with $|Y| = N_0$, where N_0 is the cardinality of set Y. Recent developments in fixed point theory and Ulam-type stabilities include results on dislocated quasi-metric and b-metric settings [7, 8], and on Ulam-type stabilities for integral and integro-differential equations with delays [9–13]. Recent works have also emphasized the importance of stability and qualitative analyses in related contexts. For instance, several studies on Ulam stability have investigated nonlinear integral equations and differential equations with delays [9–13]. These contributions highlight how stability conditions can broaden the applicability of fixed point techniques. Motivated by these directions, our paper proposes a geometric generalization where contraction acts on the perimeters of polygons rather than point-to-point distances.

2. Comparison with other generalizations

The Banach contraction principle has been generalized in many directions, including in b-metric spaces, dislocated quasi-metric spaces, ordered metric spaces, and multivalued mapping settings. For example, Shoaib and Mehmood [7] established fixed point results for mappings satisfying (δ, φ)-domination and continuity in dislocated quasi-metric spaces, while Shoaib and Mir [8] studied interpolative multivalued α^*-dominated contractive mappings in dislocated b-metric spaces. These works extend Banach's theorem by relaxing the underlying distance structure or by considering generalized contraction conditions.

Our approach differs in a fundamental aspect: instead of directly contracting distances between two points, we require contraction of the *perimeter* of polygons with n vertices. This perimeter-based contraction condition naturally generalizes Petrov's theorem (which focused on triangles, $n = 3$) and provides a geometric perspective distinct from the analytic contractive inequalities found in earlier literature. In particular, our framework allows us to re-derive Banach's contraction theorem as a corollary while also offering new geometric insights that are not captured by distance-based or order-based generalizations.

Therefore, while earlier results broadened the applicability of Banach's principle by altering the ambient space or the contractive condition, the present work contributes a complementary direction by showing that geometric contraction of polygons leads to robust fixed point results. This highlights the uniqueness of our contribution relative to other existing generalizations.

3. Mappings contracting perimeters of polygons

In this section, we recall some fundamental notions in metric fixed point theory and give some properties of mappings contracting perimeters of polygons.

Definition 1. *Let Z be any non-empty set. A function $s : Z \times Z \to R$ is a metric on Z if it satisfies the following axioms:*

(1) $s(l_1, l_2) \geq 0$*;*

(2) $s(l_1, l_2) = 0 \iff l_1 = l_2$*;*

(3) $s(l_1, l_2) = s(l_2, l_1)$ *for all* $l_1, l_2 \in Z$*;*

(4) $s(l_1, l_2) \leq s(l_1, l) + s(l, l_2)$ *for all* $l_1, l_2, l \in Z$*,*

the pair (Z, s) is known as metric space.

Definition 2. *Let $|Z| \geq n$ and (Z, s) be a metric space. We refer to $L : Z \to Z$ as a mapping contracting perimeters of polygons on Z, if there exists $\beta \in [0, 1)$ such that the inequality*

$$s(Ll_1, Ll_2) + s(Ll_2, Ll_3) + \cdots + s(Ll_{n-1}, Ll_n) \leq \beta \left(s(l_1, l_2) + s(l_2, l_3) + \cdots + s(l_{n-1}, l_n) \right) \tag{1}$$

holds for every n points $l_1, l_2, l_3, \cdots, l_n \in Z$.

Proposition 3. *Mappings contracting perimeters of polygons are continuous.*

Proof. Given a metric space (Z, s), and $|Z| \geq n$, and $L : Z \to Z$ a mapping contracting perimeters of polygons on Z and consider l_0 as an isolated point in Z. The mapping L is thus obviously continuous at l_0. It can be shown that for each $\epsilon \geq 0$, there exists a $\delta \geq 0$ such that

$$s\left(Ll_0, Ll\right) \leq \epsilon,$$

whenever

$$s\left(l_0, l\right) \leq \delta.$$

Given l_0 as an accumulation point. Therefore, for every $\delta \geq 0$, $\exists\ l \in Z$ such that $s(l_i, l) \leq \delta$ for all $i \geq 0$

$$\begin{aligned} s(Ll_0, Ll) &\leq s(Ll_0, Ll) + s(Ll, Ll_1) + s(Ll_1, Ll_2) + \cdots + s(Ll_{n-1}, Ll_0) \\ &\leq \beta(s(l_o, l) + s(l, l_1) + s(l_1, l_2) + \cdots + s(l_{n-1}, l_0)) \\ &\leq \beta(s(l_0, l) + s(l, l_0) + s(l_0, l_1) + s(l_1, l_0) + s(l_0, l_2) + \cdots + s(l_{n-2}, l_0)) \\ &\leq 2\beta(n-1)\delta. \end{aligned}$$

Set $\delta = \epsilon/2(n-1)\beta$, the desired inequality is obtained.

Theorem 4. *Given a complete metric space $(Z, s), |Z| \geq n$, if the mapping $L : Z \to Z$ satisfy the following conditions.*

(i) $L(L(l)) \neq l\ \forall\ l \in Z$ such that $L(l) \neq l$.

(ii) L is a mapping contracting perimeters of polygons.

Then it admits a fixed point, moreover, maximum number of fixed points is $(n-1)$.

Proof. Let $l_0 \in Z, Ll_0 = l_1, Ll_1 = l_2, \cdots, Ll_n = l_{n+1}$. Let $l_i \neq L(l_i)$ for each $i = 0, 1, 2, ...$ So that, we have $l_i \neq l_{i+1} = Ll_i$. As $l_{i+2} = L(Ll_i) \neq l_i$, by condition (i) and assuming that l_{i+1} is not fixed, we obtain $l_{i+1} \neq l_{i+2} = Ll_{i+1}$. Continuing the process, we have $l_i, l_{i+1}, \cdots, l_{i+n}$ pairwise distinct. Moreover, set

$$\begin{aligned} p_0 &= s(l_0, l_1) + s(l_1, l_2) + s(l_2, l_3) + \cdots + s(l_{n-1}, l_0) \\ p_1 &= s(l_1, l_2) + s(l_2, l_3) + s(l_3, l_4) + \cdots + s(l_n, l_1) \\ &\cdots \\ p_k &= s(l_k, l_{k+1}) + s(l_{k+1}, l_{k+2}) + \cdots + s(l_{n+(k-1)}, l_k), \end{aligned}$$

by (1) we have, $p_1 \leq \beta p_0, p_2 \leq \beta p_1, \cdots p_k \leq \beta p_{k-1}$ and

$$p_0 > p_1 > p_2 \cdots > p_k > \cdots. \tag{2}$$

Assume that $j \geq n$ is such that $l_j = l_i$ for $0 \leq i \leq j - (n-1)$ then $l_{j+1} = l_{i+1}, l_{j+2} = l_{i+2}, l_{j+3} = l_{i+3}, \cdots$. Thus, $p_i = p_j$, which defies (2). Let's verify that the $\{l_i\}$ is a Cauchy sequence. Clearly,

$$\begin{aligned} s(l_1, l_2) &\leq p_0, \\ s(l_2, l_3) &\leq p_1 \leq \beta p_0, \end{aligned}$$

$$\begin{aligned} s(l_3, l_4) &\leq p_2 \leq \beta^2 p_0, \\ &\cdots \\ s(l_k, l_{k+1}) &\leq p_{k-1} \leq \beta^{k-1} p_0, \end{aligned}$$

by triangular inequality,

$$\begin{aligned} s(l_k, l_{k+p}) &\leq s(l_k, l_{k+1}) + s(l_{k+1}, l_{k+2}) + \cdots + s(l_{k+p-1}, l_{k+p}), \\ &\leq \beta^{k-1} p_0 + \beta^k p_0 + \cdots + \beta^{k+p-2} p_0, \\ &= \beta^{k-1}(1 + \beta + \cdots + \beta^{p-1}) p_0, \\ &= \beta^{k-1}\left(\frac{1-\beta^p}{1-\beta}\right) p_0. \end{aligned}$$

Since, $0 \leq \beta < 1$, we have $s(l_k, l_{k+p}) < \beta^{k-1} \frac{1}{1-\beta} p_0$. Hence, $s(l_k, l_{k+p}) \to 0$ as $k \to \infty$ where p is positive. The sequence l_k is therefore Cauchy. According to (Z, s) completeness, this sequence's limit is $l^* \in Z$. Let us establish $Ll^* = l^*$. Using inequality (1) and triangular inequality, we have

$$\begin{aligned} s(l^*, Ll^*) &\leq s(l^*, l_k) + s(l_k, l^*) \\ &= s(l^*, l_k) + s(Ll_{k-1}, l^*) \\ &\leq s(l^*, l_k) + s(Ll_{k-1}, Ll^*) + s(Ll_{k-1}, Ll_k) + \cdots + s(Ll_{k+n-3}, Ll^*) \\ &\leq s(l^*, l_k) + \beta(s(l_{k-1}, l^*) + s(l_{k-1}, l_k) + \cdots + s(l_{k+n-3}, l^*)). \end{aligned}$$

We obtain $s(l^*, Ll^*) = 0$, as every term in the preceding sum tends to zero as $k \to \infty$. Let us assume that there are a minimum of n distinct pairwise fixed points, denoted as $l_1, l_2, \cdots, l_n$. Thus, $Ll_1 = l_1, Ll_2 = l_2, \cdots, Ll_n = l_n$, which is in opposition to (1).

The main result shows that if a mapping always reduces the perimeter of any polygon, then applying the mapping repeatedly forces the polygon to shrink down to a single point. That point is the fixed point of the mapping. This provides a geometric way of understanding Banach's contraction principle and extends Petrov's triangle-based theorem to polygons with any number of vertices.

Remark 5. *Let us assume the following: the mapping L has a fixed point point l^*, a limit of a sequence $\{l_n\}$, consider $l_1 = Ll_0, l_2 = Ll_1 \ldots$ such that $l_n \neq l^*$ for all $n \in N$. Then there is only one fixed point, l^*. Suppose L has an additional fixed point $l^{2*} \neq l^*$. It is evident that $l_n \neq l^{2*}$ $\forall$ $n \geq 1$. Likewise, assume that $l^{3*}, l^{4*}, \cdots, l^{n*}$ are fixed points. Then $l_n \neq l^{3*} \neq l^{4*}, \cdots, \neq l^{n*}$ $\forall$ $n \geq 1$. Thus, $\forall$ $n \geq 1$, we have that the points $l_n, l^*, l^{2*}, \cdots, l^{(n-1)*}$ are pairwise distinct. Consider the ratio*

$$\begin{aligned} R_n &= \frac{s(Ll_n, Ll^*) + s(Ll^*, Ll^{2*}) + \cdots + s(Ll^{(n-1)*}, Ll_n)}{s(l_n, l^*) + s(l^*, l^{2*}) + \cdots + s(l^{(n-1)*}, l_n)} \\ &= \frac{s(l_{n+1}, l^*) + s(l^*, l^{2*}) + \cdots + s(l^{(n-1)*}, l_{n+1})}{s(l_n, l^*) + s(l^*, l^{2*}) + \cdots + s(l^{(n-1)*}, l_n)}. \end{aligned}$$

We obtain $R_n \to 1$ as $n \to \infty$, which does not satisfy 1.

Example 6. *Let's take an example of mapping L having precisely $n-1$ fixed points. Let $Z = \{l_1, l_2, \cdots, l_n\}$, $s(l_1, l_2) = s(l_2, l_3) = s(l_3, l_4) = \cdots = s(l_n, l_1) = 1$ and let $L : Z \to Z$ be such that $L(l_1) = l_1, L(l_2) = l_2, \cdots, L(l_{n-1}) = l_{n-1}, Ll_n = l_1$. It is evident that both of the requirements of Theorem (4) are met.*

Example 7. *We will demonstrate that assumption (i) of Theorem (4) is necessary. Let $Y = \{l_1, l_2, l_3, \cdots, l_n\}$, $s(l_1, l_2) = s(l_2, l_3) = \cdots = s(l_n, l_1)$ and let $L : Z \to Z$ be such that $Ll_1 = l_2, Ll_2 = l_1, Ll_3 = l_4, Ll_4 = l_3, \cdots, Ll_n = l_1$. It is evident that assumption (ii) of Theorem (4) is satisfied, but the mapping L lacks a fixed point.*

Given a metric space (Z, s), a mapping $L : Z \to Z$ is known as a contraction mapping on Z if and only if there is a value for $\beta \in [0, 1)$ such that

$$s(Lx, Ll) \leq \beta s(x, l). \tag{3}$$

where $x, l \in Z$.

Corollary 1. *(Banach fixed point theorem) Let (Z, s) be a complete metric space, and let $L : Z \to Z$ be a contraction mapping, then it admits a fixed point.*

Proof. Let $|Z| \geq n$. Assume that $\exists\ l \in Z$ which satisfy $L(L(l)) = l$. As a result, $s(x, Lx) = s(Lx, x) = s(Lx, L(Lx))$, which contradicts to (3). Condition (i) of Theorem(4) is thus met. Assume that $l_1, l_2, l_3 \cdots l_n \in Z$ are distinctive pairs. By using the contraction definition, we have $s(Ll_1, Ll_2) \leq \alpha s(l_1, l_2), s(Ll_2, Ll_3) \leq \alpha s(l_2, l_3), \cdots, s(Ll_{n-1}, Ll_n)) \leq \alpha s(l_{n-1}, l_n)$ and which implies condition (ii) of Theorem(4). This concludes the evidence for the existence of a fixed point. Assume there are $x, y \in Z$ such that $Lx = x$ and $Ly = y$. Hence, $s(Lx, Ly) = s(x, y)$ which contradicts to (3).

Example 8. *Let us define a mapping $L : Z \to Z$ contracting perimeters of polygons for a metric space Z, that is not a contraction mapping with $|Z| = N_0$. Let $Z = \{l^*, l_1, l_2 ...\}$ and let $b \in R^+$. Consider a metric s on Z such that*

$$s(x, l) = \begin{cases} \frac{b}{2^{\lfloor i/(n-1) \rfloor}} & \text{if } x = l_i, l = l_{i+1}, i \in N; \\ s(l_i, l_{i+1}) + \cdots + s(l_{j-1}, l_j) & \text{if } x = l_i, l = l_j, i+1 < j; \\ 2(n-1)b - s(l_0, l_i) & \text{if } x = l_i, l = l^* \\ 0 & \text{if } x = l, \end{cases}$$

where $\lfloor . \rfloor$ is a floor function. Define a mapping $L : Z \to Z$ as $Ll_i = l_{i+1}$ for all $i = 0, 1, ...$ and $Ll^* = l^*$. Since $s(l_{2n}, l_{2n+1}) = s(Ll_{2n}, Ll_{2n+1}), n = 0, 1, ...$ We note that L does not satisfy (1). Let us demonstrate that inequality(1) is true for each of the n distinct pairwise points in the space Z. Consider the points $l_1, l_2, \cdots, l_{n-1}, l^* \in Z$. Using structure of metric s we have

$$s(l_i, l_{i+1}) + s(l_{i+1}, l_{i+2}) \quad + \quad \cdots + s(l^*, l_{i+(n-1)}) + s(l_{i+(n-1)}, l_{i+n})$$

$$\begin{aligned} &= (n-1)s(l_i, l^*) \\ &= (n-1)(2(n-1)a - s(l_i, l^*)), \end{aligned}$$

and

$$s(Ll_i, Ll_{i+1})+s(Ll_{i+1}, Ll_{i+2})+\cdots+s(Ll^*, Ll_{i+(n-1)})+s(Ll_{i+(n-1)}, Ll_{i+n}) = (n-1)s(Ll_{i+1}, Ll^*)$$

$$= (n-1)(2(n-1)b - s(l_{i+1}, l^*)).$$

Moreover,

$$s(l_0, l_i) = \begin{cases} 2(n-1)b(1-\frac{1}{2^n}) & if\ i = n(n-1) \\ 2(n-1)b(1-\frac{1}{2^n}) - \frac{b}{2^{n-1}} & if\ i = n(n-1)-1, \end{cases}$$

n=1,2,..., observe also that $s(t_0, t_{i+1}) = s(t_0, t_i) + \frac{b}{2^{\lfloor i/n-1 \rfloor}}$. *Consider the ratio*

$$\begin{aligned} &\frac{s(Ll_i, Ll_{i+1}) + s(Ll_{i+1}, Ll_{i+2}) + \cdots + s(Ll^*, Ll_{i+n-1}) + s(Ll_{i+n-1}, Ll_{i+n})}{s(l_i, l_{i+1}) + s(l_{i+1}, l_{i+2}) + \cdots + s(l^*, l_{i+n-1}) + s(l_{i+n-1}, l_{i+n}} \\ &= \frac{(n-1)(2(n-1)b - s(l_0, l_{i+1}))}{(n-1)(2(n-1)b - s(l_0, l_i))} \\ &= \frac{2(n-1)b - s(l_0, l_{i+1})}{2(n-1)b - s(l_0, l_i)} \\ &= \frac{2(n-1)b - s(l_0, l_i) - b/2^{\lfloor i/n-1 \rfloor}}{2(n-1)b - s(l_0, l_i)} \\ &= \begin{cases} \frac{2(n-1)b - 2(n-1)b(1-1/2^n) - b/2^{\lfloor i/n-1 \rfloor}}{2(n-1)b - 2b(n-1)(1-1/2^n)} & if\ i = n(n-1); \\ \frac{2(n-1)b - 2(n-1)b(1-1/2^n) + a/2^{n-1} - b/2^{\lfloor i/n-1 \rfloor}}{2(n-1)b - 2(n-1)b(1-1/2^n) - b/2^{n-1}} & if\ i = n(n-1)-1 \end{cases} \\ &= \begin{cases} \frac{2n-3}{2n-2} & if\ i = n(n-1) \\ \frac{n-1}{n-2} & if\ i = n(n-1)-1 \end{cases}. \end{aligned}$$

Let $l_{i+1}, l_{i+2}, \cdots, l_{i+n} \in Z$. *We see that*

$$\begin{aligned} s(l_{i+1}, l_{i+2}) &+ s(l_{i+2}, l_{i+3}) + \cdots + s(l_{i+n-1}, l_{i+n}) \\ &- (s(Ll_{i+1}, Ll_{i+2}) + s(Ll_{i+2}, Ll_{i+3}) + \cdots + s(Ll_{i+n-1}, Ll_{i+n})) \\ &= (n-1)(b/2^{\lfloor i/(n-1) \rfloor} - b/2^{\lfloor m/(n-1) \rfloor}). \end{aligned}$$

Consider the ratio

$$\begin{aligned} R_{i,k} &= \frac{s(Ll_{i+1}, Ll_{i+2}) + s(Ll_{i+2}, Ll_{i+3}) + \cdots + s(Ll_{i+n-1}, Ll_{i+n})}{s(l_{i+1}, l_{i+2}) + s(l_{i+2}, l_{i+3}) + \cdots + s(l_{i+n-1}, l_{i+n})} \\ &= \frac{s(l_{i+1}, l_{i+2}) + s(l_{i+2}, l_{i+3}) + \cdots + s(l_{i+n-1}, l_{i+n}) - (n-1)(b/2^{\lfloor i/(n-1) \rfloor} - b/2^{\lfloor m/(n-1) \rfloor})}{s(l_{i+1}, l_{i+2} + s(l_{i+2}, l_{i+3}) + \cdots + s(l_{i+n-1}, l_{i+n})} \end{aligned}$$

$$= \frac{1-(n-1)(b/2^{\lfloor i/(n-1)\rfloor} - b/2^{\lfloor m/(n-1)\rfloor}}{s(l_{i+1},l_{i+2})+s(l_{i+2},l_{i+3})+\cdots+s(l_{i+n-1},l_{i+n})}.$$

Note that $i+n-2<k$, hence,

$$b/2^{\lfloor k/(n-1)\rfloor} \leq b/2.b/2^{\lfloor i/(n-1)\rfloor}. \tag{4}$$

It can be demonstrated that by using the structure of the space (L,s),

$$s(l_{i+k},l^*) \leq 2(n-1)s(l_{i+1},l_{i+2}).$$

Clearly $s(l_{i+k},l_{i+m}) \leq s(l_{i+k},l^*)$. Hence $s(l_{i+m},l_{i+k}) \leq 2(n-1)s(l_{i+m},l_{i+m+1})$. It can be concluded by using the last inequality

$$\begin{aligned} s(l_{i+1},l_{i+2}) &+ s(l_{i+2},l_{i+3})+\cdots+s(l_{i+n-1},l_{i+n}) = (n-1)s(l_{i+m},l_{i+k}) \\ &\leq 2(n-1)^2 s(l_{i+1},l_{i+2}) \\ &= \frac{2(n-1)^2 b}{2^{\lfloor i/(n-1)\rfloor}}. \end{aligned}$$

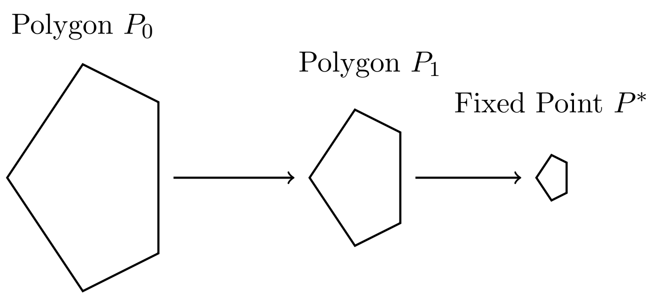
By applying the inequality (4), we get

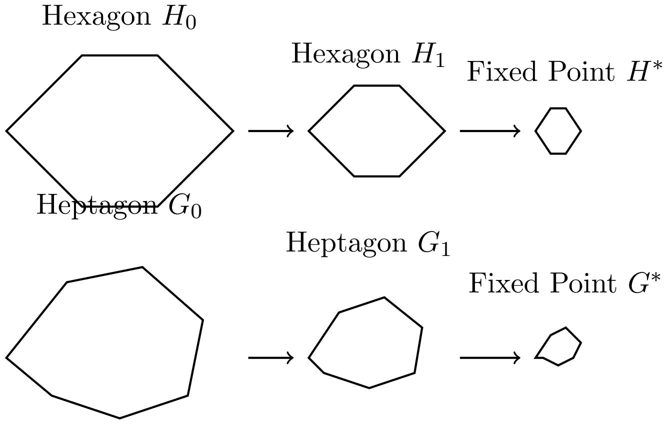
$$\begin{aligned} R_{i,k} &\leq 1-(n-1)\left(\frac{b/2^{\lfloor i/(n-1)\rfloor} - b/2.2^{\lfloor i/(n-1)\rfloor}}{2(n-1)^2 b/2^{\lfloor i/(n-1)\rfloor}}\right) \\ &= \frac{4n-5}{4n-4}. \end{aligned}$$

Hence the inequality (1) holds true.

The following diagrams effectively demonstrate:

1. How a sequence of polygons shrinks under a contracting transformation.
2. The idea that a sequence of transformations converges to a unique fixed point.
3. The concept of the Fixed Point Theorem in action.





4. Conclusion

In this paper, we studied fixed point results for mappings that contract the perimeters of polygons. By extending Petrov's theorem from the case of triangles ($n = 3$) to general polygons with n vertices, we developed a new geometric perspective on contraction mappings. Our results show that requiring the contraction of polygonal perimeters is sufficient to ensure the existence of fixed points, thereby providing a natural generalization of both Petrov's theorem and the Banach contraction principle. We also demonstrated that Banach's contraction theorem can be re-derived as a corollary of our framework, which highlights the strength and versatility of this perimeter-based approach. The comparison with other generalizations of Banach's principle shows that, unlike distance- or order-based extensions, our method provides geometric insights not captured in previous studies. In addition to the theoretical contributions, several illustrative examples were given to confirm the applicability of the main results. The presented framework opens avenues for further work on fixed point theory in geometric and applied settings, such as in dynamical systems, optimization, and problems involving polygonal or polyhedral structures. In summary, the present study contributes a complementary and geometrically motivated extension of fixed point theory. Future work may explore practical applications of perimeter-contraction mappings, connections with Ulam stability problems, and the extension of these ideas to higher-dimensional polyhedra.

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5. Authors' contributions

M.N. tabled the main idea of this paper; U.H. wrote the first draft of this paper; M.N. and M. D. S. reviewed and prepared the second draft; M. D. S. supervised the project. All authors have read and agreed to the published version of the manuscript.

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