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Hankel Determinant of Analytical Functions Closely Tied to Bell Polynomials

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Abstract. Applying the state-of-the-art Bell polynomials to the open unit disk, a differential operator $\vartheta^{m}_{\xi,P}$ is produced. In this paper, we shall provide a family of analytic functions related to the differential operator indicated above. The upper bound for the nonlinear functional $|a_2a_4 - a_3^2|$, otherwise known as the Hankel determinant, is our primary finding. Aside from using the Bell polynomial, the differential operator is gained using the Hadamard product. Coefficient equating and other fundamentals of classical calculus will be used in the primary finding of the upper bound.

2020 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Univalent function, Hankel determinant, Bell polynomial, Inclusion relation

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1. preliminaries

Various branches within mathematics, such as complex analysis, differential geometry, and mathematical physics, present opportunities for application in the geometric function theory [1, 2]. Additionally, it provides tools that can be utilized to understand and characterize the geometry of intricate functions and their associated mappings.

Castellares et al. [3] studied the Bell polynomial, which is useful in many areas, such as biology, physics, engineering, and finance. Researchers have successfully used it to model the distribution of stock returns, describe noisy signals, and examine the functioning of biological systems. The Bell curve is used in many areas of statistics, such as testing hypotheses, finding confidence ranges, and performing regression analysis. It is also used

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to describe complicated systems and make predictions based on real-world data, such as in psychology, economics, and finance. The Bell polynomial, which was made to be better than the Bell numbers [4], is defined by a generating function for a discrete random variable X, which can be written in the following way:

$$P\left(X=n\right)=\frac{\xi^{n}e^{e^{\left(-\xi^{2}\right)+1}}\mathbb{F}_{n}}{n!};\quad n=1,2,3,..., \tag{1}$$

where $\mathbb{F}_n=\frac{1}{e}\sum\limits_{b=0}^{\infty}\frac{b^n}{b!}$ is the Bell numbers, $n\geq 1$, and $0<\xi\leq 1$.

The first few terms for Bell numbers are as follows:

$$\mathbb{F}_1=1,\quad \mathbb{F}_2=2,\quad \mathbb{F}_3=5,\quad \mathbb{F}_4=15,\mathbb{F}_5=52.$$

Let us now present a new power series, the coefficients of which will represent the Bell generating function.

$$\mathbb{L}(\xi,z)=z+\sum_{n=2}^{\infty}\frac{\xi^{n-1}e^{e^{\left(-\xi^{2}\right)+1}}\mathbb{F}_{n}}{(n-1)!}z^{n},\qquad z\in U.\ \text{where } 0<\xi\leq 1. \tag{2}$$

Consequently, coefficients can be viewed as probabilities linked to the Bell polynomial. It is possible to confirm the convergence of the previously mentioned series on the unit disk U by applying the ratio test, a widely recognized and proven effective method. Recently, Alnajar and Darus [5], Alnajar et al.,[6–8], Amourah et al., [9], and Illafe et al., [10] employed Bell, Borel and Neutrosophic Poisson polynomials to address specific problems related to complex analysis. The motivation behind this study is to look at the behavior of the polynomials determined by their coefficient values.

Suppose f is defined on the open unit disk, and A represents the categorization of all analytical functions. This is valid only if conditions $U=\{z\in\mathbb{C}:|z|<1\}$ and $f(0)=0$ and $f'(0)-1=0$ are satisfied. For each $f\in A$, we write the following Taylor series:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad (z\in U,\ a_n\in\mathbb{C},\ n\in\mathbb{N}:=\{1,2,3,\dots\}). \tag{3}$$

The examination of inclusion relationships among analytic functions within specific special sets was a topic that was previously incorporated into the realm of geometric function theory. This area held considerable interest for researchers. An investigation conducted by Ruscheweyh [11] centered on the neighborhood and inclusion relationships of univalent functions. Simultaneously, Srivastava et al., [12] delved into a comprehensive study on all inclusion characteristics of multivalent functions. In recent times, scholars in the field of geometric function theory have directed their focus towards diverse sub-classes of univalent functions. Amourah et al., [13], Mahmood et al., [14], Amini et al., [15], Jahangiri et al., [16], and Amini et al., [17] provide additional details that yield a more in-depth understanding, see also [18–22].

Using the symbol ϑ_ξ we may express the linear operator, which is defined by the Hadamard product, also known as convolution: $A \to A$

$$\vartheta_\xi f(z) = \mathbb{L}(\xi, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\xi^{n-1} e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_n}{(n-1)!} a_n z^n, \qquad z \in U. \tag{4}$$

we define the operator $\vartheta^m_{\xi,P} f(z) : A \to A$ as

$$\vartheta^0_{\xi,P} f(z) = z + \sum_{n=2}^{\infty} \frac{\xi^{n-1} e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_n}{(n-1)!} a_n z^n,$$

$$\vartheta^1_{\xi,P} f(z) = (1-P)\vartheta^0_{\xi,P} f(z) + Pz\left(\vartheta^0_{\xi,P} f(z)\right)',$$

$$\vartheta^1_{\xi,P} f(z) = z + \sum_{n=2}^{\infty} \frac{\xi^{n-1} e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_n}{(n-1)!} \left[1 + P(n-1)\right] a_n z^n,$$

$$\vartheta^2_{\xi,P} f(z) = (1-P)\vartheta^1_{\xi,P} f(z) + Pz\left(\vartheta^1_{\xi,P} f(z)\right)',$$

$$\vartheta^2_{\xi,P} f(z) = z + \sum_{n=2}^{\infty} \frac{\xi^{n-1} e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_n}{(n-1)!} \left[1 + P(n-1)\right]^2 a_n z^n,$$

$$\vdots$$

$$\vartheta^m_{\xi,P} f(z) = z + \sum_{n=2}^{\infty} \frac{\xi^{n-1} e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_n}{(n-1)!} \left[1 + P(n-1)\right]^m a_n z^n, \tag{5}$$

$$m \in \mathbb{N} \cup \{0\}, \; P \geq 0, \; 0 < \xi \leq 1.$$

Pommerenke [23, 24] defined the Hankel determinant of f for $r \geq 1$ and $n \geq 1$ as

$$H_r(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2(r-1)} \end{vmatrix}.$$

Many authors have also given this problem some thoughts. For example, Noor [25] calculated the growth rate of $H_r(n)$ as $n \to \infty$ with a constrained boundary, Ehrenborg [26] examined the Hankel determinant of exponential polynomials, and Layman [27] and Panigrahi, and Murugusundaramoorthy [28] covered some of its characteristics. Mishra and Gochhayat [29] also investigated the Hankel determinant using fractional operators. Furthermore, this study develops the classes utilised in [30–39] by incorporating the Bell polynomial.

In the present work, we will examine the Hankel determinant when $r = 2$ and $n = 2$, that is:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|.$$

Bear in mind, $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_1 a_3 - a_2^2|$ is the well-known Fekete-Szego functional for $a_1 = 1$.

This study aims to find the upper bound for the functional $|a_2 a_4 - a_3^2|$ of a function f that belongs to the class $T_k(m)$ defined as follows:

Definition 1. *Let f be given by (3). It is said to satisfy the inequality if $f \in T_k(m)$.*

$$\Re\left\{\left(\vartheta_{\xi,P}^{m} f(z)\right)'\right\} > 0, z \in U. \tag{6}$$

We begin by stating a few foundational lemmas that will be utilized in our proof.

Let B stand for the function class

$$B(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} y_n z^n, \tag{7}$$

which are analytic in U and satisfy $Re\ \{B(z)\} > 0$ for any $z \in U$.

Lemma 1. *[40] If $y \in B$, then $|y_n| \leq 2$, for each $n \geq 1$.*

Lemma 2. *[41, 42] If $y \in B$, then*

$$2y_2 = y_1^2 + \left(4 - y_1^2\right) x = y_2 = \frac{1}{2}(y_1^2 + \left(4 - y_1^2\right) x), \tag{8}$$

for some x, $|x| \leq 1$, and

$$\begin{aligned} 4y_3 &= y_1^3 + 2y_1\left(4 - y_1^2\right) x - y_1\left(4 - y_1^2\right) x^2 + 2\left(4 - y_1^2\right)(1 - |x^2||) z = \\ y_3 &= \frac{1}{4}(y_1^3 + 2y_1\left(4 - y_1^2\right) x - y_1\left(4 - y_1^2\right) x^2 + 2\left(4 - y_1^2\right)(1 - |x^2|) z), \end{aligned} \tag{9}$$

for some z, $|z| \leq 1$.

2. Main Result

Our main result as follows:

Theorem 1. *Let $f \in T_k(m)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{16}{9\left(\xi^4 e^{2e^{\left(-\xi^2\right)+1}} \mathbb{F}_3 \left[1+2P\right]^{2m}\right)}.$$

Proof. Since $f \in T_k(m)$, it follows from Eq (5) and Eq (7) that:

$$\left(\vartheta_{\xi,P}^{m} f(z)\right)' = B(z),$$

we can write as follows:

$$1 + 2\xi e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_2 \left[1+P\right]^m a_2 z + \frac{3}{2}\left(\xi^2 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_3 \left[1+2P\right]^m\right) a_3 z^2 +$$
$$\frac{4}{6}\left(\xi^3 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_4 \left[1+3P\right]^m\right) a_4 z^3 = 1 + y_1 z + y_2 z^2 + y_3 z^3.$$

Through coefficient comparison, we obtain

$$\left(2\xi e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_2 \left[1+P\right]^m\right) a_2 = y_1,$$
$$\frac{3}{2}\left(\xi^2 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_3 \left[1+2P\right]^m\right) a_3 = y_2,$$
$$\frac{2}{3}\left(\xi^3 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_4 \left[1+3P\right]^m\right) a_4 = y_3.$$

Therefore,

$$a_2 = \frac{y_1}{\left(2\xi e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_2 \left[1+P\right]^m\right)},\ a_3 = \frac{2y_2}{3\left(\xi^2 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_3 \left[1+2P\right]^m\right)},$$
$$a_4 = \frac{3y_3}{2\left(\xi^3 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_4 \left[1+3P\right]^m\right)}.$$

Then combining $a_2 a_4 - a_3^2$ and take the magnitude based on Hankel determinant, also by applying Lemma (2) and taking common factors we have the following:

$$\left|a_2 a_4 - a_3^2\right| = \frac{1}{\xi e^{e^{\left(-\xi^2\right)+1}}} \left|\frac{3y_1 y_3}{4\xi^2 \mathbb{F}_2 \mathbb{F}_4 \left[1+P\right]^m \left[1+3P\right]^m} - \frac{4y_2^2}{9\xi^3 e^{e^{\left(-\xi^2\right)+1}} \mathbb{F}_3 \left[1+2P\right]^{2m}}\right|. \quad (10)$$

Given that the function $B(z)$ belongs to the class B simultaneously, we can assume that $y_1 = y > 0$ without losing generality. For ease of notation, we will use $y_1 = y (y \in [0, 2])$.

Next, by combining (8) with (9) and substitute in (10), we obtain the following:

$$\left|a_2a_4 - a_3^2\right| = Q(h)\left|\begin{array}{c} \frac{27y^4+54y^2\left(4-y^2\right)x-27y^2\left(4-y^2\right)x^2}{144} \\ +\frac{\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\left(-16y^4-32y^2\left(4-y^2\right)x-16\left(4-y^2\right)^2x^2\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)} \\ +\frac{6y\left(4-y^2\right)\left(1-\left|x^2\right|\right)z}{16} \end{array}\right|$$

where

$$Q(h) = \frac{1}{\left(\xi e^{e^{\left(-\xi^2\right)+1}}\right)\left(\xi^2\mathbb{F}_2\mathbb{F}_4\left[1+P\right]^m\left[1+3P\right]^m\right)}.$$

When $|k|$ is substituted with v and the triangle inequality is applied, we have

$$\left|a_2a_4 - a_3^2\right| \leq Q(h)\left[\begin{array}{c} \left(\frac{27}{144} - \frac{16\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^4 \\ +\left(\frac{54}{144} - \frac{32\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^2(4-y^2)v \\ +\left(\frac{27y^2}{144} + \frac{16\left(4-y^2\right)\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)(4-y^2)v^2 \\ +\frac{6y(4-y^2)(1-v^2)}{16} \end{array}\right] \tag{11}$$

$$= Q(h)\left[\begin{array}{c} \left(\frac{27}{144} - \frac{16\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^4 \\ +\left(\frac{54}{144} - \frac{32\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^2(4-y^2)v \\ +\left(\frac{27y^2}{144} + \frac{16\left(4-y^2\right)\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)} - \frac{6y}{16}\right)(4-y^2)v^2 \\ +\frac{6y(4-y^2)}{16} \end{array}\right]$$

$$= K(y,v) \tag{12}$$

where $0 \leq y \leq 2$ and $0 \leq v \leq 1$. The function $K(y,v)$ is then maximized on the closed square $[0,2]\times[0,1]$. Differentiating $K(y,v)$ with respect to v, we get

$$\frac{dK}{dv} = Q(h)\left[\begin{array}{c} \left(\frac{54}{144} - \frac{32\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^2(4-y^2) \\ +\left(\frac{3y(y-2)}{8} + \frac{16\left(4-y^2\right)\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{72\left(\xi^3e^{e^{\left(-\xi^2\right)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)(4-y^2)v \end{array}\right].$$

For $0 < v < 1$, and for fixed y with $0 < y < 2$, and $\frac{\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)} < \frac{27}{16}$, we observe that $\frac{dK}{dv} > 0$. Consequently, a maximum of $K(y, v)$ cannot exist inside the closed square $[0, 2] \times [0, 1]$. Additionally, for fixed $y \in [0, 2]$, we have $\max\limits_{0 \le v \le 1} K(y, v) = K(y, 1) = G(y)$.

$$G(y) = Q(h)\left[\begin{array}{c} \left(\frac{27}{144} - \frac{16\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^4 \\ +\left(\frac{54}{144} - \frac{32\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right)y^2(4-y^2) \\ +\left(\frac{27y^2}{144} + \frac{16\left(4-y^2\right)\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{144\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)} - \frac{6y}{16}\right)(4-y^2) \\ +\frac{6y(4-y^2)}{16} \end{array}\right].$$

Next

$$G^{'}(y) = Q(h)\left[\frac{3y(3-y^2)}{2} - \frac{8y(4-y^2)\left(\xi^2\mathbb{F}_2\mathbb{F}_4\left[1+P\right]^m\left[1+3P\right]^m\right)}{9\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3\left[1+2P\right]^{2m}\right)}\right] = 0,$$

implies $y = 0$. Further, we observe that $G''(y) =$
$Q(h)\left[\frac{9}{2} - \frac{27y^2}{4} - \frac{8(4-3y^2)\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{9\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)}\right] < 0$, and
$\frac{81}{64} < \frac{\left(\xi^2\mathbb{F}_2\mathbb{F}_4[1+P]^m[1+3P]^m\right)}{\left(\xi^3 e^{e^{(-\xi^2)+1}}\mathbb{F}_3[1+2P]^{2m}\right)} < \frac{27}{16}$.
We also note that $G(y) > G(2)$. Thus, $\max\limits_{0 \le y \le 2} G(y)$ happens at $y = 0$, that is when $v = 1$ and $y = 0$ we obtain the bound of Eq. (11):

$$\left|a_2a_4 - a_3^2\right| \le \frac{16}{9\left(\xi^4 e^{2e^{(-\xi^2)+1}}\mathbb{F}_3\left[1+2P\right]^{2m}\right)}.$$

The proof is completed.

Corollary 1. *Let $f \in T_k(0)$, for $m = 0$. Then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{16}{9\left(\xi^4 e^{2e^{(-\xi^2)+1}}\mathbb{F}_3\right)}.$$

3. Conclusions

The bound of the second Hankel determinant is our main focus. It is established within a class of univalent functions associated with a new operator linked to the Bell polynomial. The result is not sharp and may be improved in future work.

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