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An Exponential Intuitionistic Fuzzy Framework for Sustainable Supplier Selection in Industrial Management

M. Kaviyarasu[1], Luminita-Ioana Cotîrlă [2], Daniel Breaz[3,*], M. Rajeshwari[4]

[1] *Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R & D Institute of Science and Technology, Chennai, Tamil Nadu 600062, India*
[2] *Department of Mathematics, Technical University of Cluj Napoca, 400114, Cluj-Napoca, Romania*
[3] *Department of Mathematics, "1 Decembrie 1918", University of Alba Iulia, 510009, Alba Iulia, Romania*
[4] *Department of Mathematics, Presidency University, Bangalore 560064, India*

Abstract. Dealing with uncertainty and imprecision is critical in modern decision-making processes, particularly in multi-criteria situations. Exponential intuitionistic fuzzy sets ($\mathcal{EIFS}$) are a sophisticated mathematical framework that extends intuitionistic fuzzy sets by integrating an exponential function, improving its capacity to adequately capture ambiguity. This study investigates several $\mathcal{EIFS}$ operations, such as $\mathcal{EIF}$ intersection and union, which expand conventional set operations to accommodate partial and non-membership $\mathbb{NM}$ degrees more flexibly. Furthermore, we look into crucial operations including simple difference, limited difference, disjoint sum, disjunctive sum, and Cartesian product, all of which play important roles in modelling uncertainty in decision processes. A case study on raw material purchase demonstrates how these processes might be applied. The decision-making system takes into account numerous providers, analysing aspects such as cost, quality, and delivery time with $\mathcal{EIFS}$-based aggregation approaches. Using these activities, decision-makers may improve supplier performance, reduce risks, and optimise procurement strategies in unpredictable circumstances.

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Key Words and Phrases: Fuzzy set, Exponential aggregation operators, Exponential fuzzy set, Exponential intuitionistic fuzzy set, Decision-making methods

*Corresponding author.
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Email addresses: kavitamilm@gmail.com (M. Kaviyarasu), luminita.cotirla@math.utcluj.ro (L.-I. Cotîrlă), dbreaz@uab.ro (D. Breaz), rajeshwari@presidencyuniversity.in (M. Rajeshwari)

1. Introduction

Fuzzy Sets($\mathbb{FS}$s), a groundbreaking mathematical framework for dealing with imprecision and uncertainty, was first presented by Zadeh in [1]. According to conventional (crisp) set theory, an element is either a member of a set or it is not (binary membershiph ($\mathbb{M}$)). By proposing $\mathbb{FS}$s, in which each element has a degree of $\mathbb{M}$ in the interval $[0, 1]$, Zadeh expanded on this concept. The idea of an eventual distribution is defined by Zadeh in [2] to be a fuzzy limitation that serves as an elastic constraint on the values that may be given to a variable. This definition relates to the notion of $\mathbb{FS}$s that are fuzzy. Hiroshi et al suggests a fuzzy decision-making approach for multiobjective choice issues in [3]. Decision makers often prefers fuzzy connectives with the primary characteristics of unclear and unreliable solutions. The Analyst interacts with an algorithm to derive the preference pattern. Here, Zimmermann's γ operation and their expansions are used as fuzzy connectives. Chen et al. In [4] introduce novel methods in the perspective of fuzzy set theory for solving multicriteria ambiguous problem solving. The set of factors phase methods enable the presentation of ambiguous values that stipulate the degree of satisfiability and non-satisfiability of all choice. The decision-maker can also give each criterion a varying level of priority thanks to the procedures.

$\mathbb{IFS}$ is a popular $\mathbb{FS}$ that was developed by Atanassov [5]. In contrast to traditional $\mathbb{FS}$ theory, $\mathbb{NM}$ and indeterminacy degree notations are specified in addition to $\mathbb{M}$ degree. By expanding with several technologies, $\mathbb{IFS}$ may be utilised to solve problems and make decisions by taking use of their capabilities to identify ambiguity and inaccuracy. New approaches to multifaceted decision-making in an intuitionistic fuzzy context Liu presented in [6]. To determine the extent to which options meet or fall short of the decision-maker's requirements, define a measure algorithm. After that, talk about intuitionistic fuzzy point operators. According to the finest supplier in a group decision-making setting, Boran et al. [7] suggest combining the TOPSIS technique with intuitionistic $\mathbb{FS}$s. To rate the significance of conditions and alternative solutions, the unique views within decision makers are aggregated using the intuitionistic fuzzy weighted averaging ($\mathbb{IFWA}$) method. The innovative similarity metric between Atanassov's intuitionistic $\mathbb{FS}$s ($\mathbb{AIFS}$s) established the transformation of unique techniques was introduced by Chen et al. in [8].

The connection metric between $\mathbb{AIFS}$s was then applied to pattern recognition tasks. First, provide a novel similarity metric between Atanassov's intuitionistic fuzzy values ($\mathbb{AIFV}$s) and demonstrate a few of its characteristics. Next, The similarity measure of $\mathbb{AIFV}s$ is extends based on the suggested similarity measure between $\mathbb{AIFV}s$. Its properties are demonstrated, and examples are provided to show how the initiate similarity measure between $\mathbb{AIFV}s$ can overcome the shortcomings of the current similarity measures. Lastly, use the $\mathbb{AIFV}s$ similarity metric to address issues with pattern recognition. The intuitionistic fuzzy social relational network ($\mathbb{IFSRN}$) model, which includes positive, neutral, and negative relationships between the degrees which can be expressed by intuitionistic fuzzy values ($\mathbb{IFV}s$) is the basis for fuzzy query progressing methods in [9] Chen et al. Ejegwa and Agbetayo present a new similarity-distance method with a higher score for efficiency in [10]. To demonstrate the benefits of the unique similarity-distance over comparable current techniques, an evaluation is provided. A few characteristics of the similarity-distance method are shown. Additionally, the innovative similarity-distance technique's applicability in various cases of decision-making are investigated.

In [11], Garg et al. introduce a unique algebraic structure for $\mathbb{C}-\mathbb{IFS}$s that is based on Archimedean t-norm operations, such as division, addition, multiplication, and subtraction. A single rating may be created by combining the preferences of several experts thanks to these methods. An expanded $\mathbb{EDAS}$ (Evaluation Based on Distance from Average Solution) approach is also used, which ranks options using defuzzification methods and weighted aggregation methods. New operators for $\mathbb{GIFSB}$s with characteristics including idempotency, boundedness, monotonicity, and commutativity are introduced by Wasim et al. in [12], producing aggregated values that are in line with $\mathbb{GIFN}$s. The links between these processes are thoroughly examined, providing a deep comprehension of their application. In the world of contemporary decision making especially in uncertainty and imprecision, techniques of multi-criteria decision-making play a critical role. The study by Jawad Ali [13] presents a hybrid MCDM method which uses T-spherical fuzzy sets and Aczel?Alsina prioritized aggregation operators. This holistic modeling strategy increases the complex scenario modeling of decisions, where subjective judgment and objective data are essential. It is effective in the selection of apt medical experts that provide stable and flexible ranking system. Complimenting this, another contribution by Jawad Ali complimented by Suhad Ali Osman Abdallah and N. S. Abd El-Gawaad [14] suggests a state-of-art approximation decision-analysis framework based on SWARA method in conjunction with Frank aggregation and p,q-rung orthopair fuzzy information. This model is tailored for the cases in which the decision-makers have to evaluate a number of criteria with unknown weights for the priority evaluation in the environments of uncertainty. A multi-criteria decision-making approach for assessing startup performance in the IT sector provides a real demonstration of these concepts. [15] Presents the idea of complex intuitionistic fuzzy classes, which are distinguished by their pure complex intuitionistic fuzzy $\mathbb{M}$ grade. Kaviyarasu and team [16] propose exponential fuzzy sets for AI-type investment decisions, considering weighted mean aggregation as a way of capturing non-linear uncertainty better. A compelling example of a pertinent application is given, together with the fundamental terminology and computations on intricate intuitionistic fuzzy classes.

Using quaternion numbers as a basis Ngan et al. in [17] provide new operations and analyse the logic operations and order relations of the complex intuitionistic $\mathbb{FS}$ theory. Two algebraic and polar quaternion distance measurements were also covered, along with an analysis of their characteristics. Akram et al. present the CIF Hamacher weighted averaging ($\mathbb{CIFHWA}$), $\mathbb{CIF}$ Hamacher ordered weighted averaging ($\mathbb{CIFHOWA}$), $\mathbb{CIF}$ Hamacher weighted geometric ($\mathbb{CIFHWG}$), and $\mathbb{CIF}$ Hamacher ordered weighted geometric ($\mathbb{CIFHOWG}$) operators in [18]. In order to determine the phase term, Zeeshan and Khan [19] produced several fundamental conclusions and specific instances for standard complex of fuzzy intersection, union, and complement functions with the same function. Sobhi and Dick [20] a novel neuro-fuzzy infrastructure built for enormous scale learning issues, utilising complex $\mathbb{FS}$s. In [21], Bilal et al. offer numerous unique operations on complex intuitionistic $\mathbb{FS}$s, adding distance measure and σ-equalities.

1.1. Motivation

- Traditional fuzzy approaches struggle with contradictory and inadequate data, making decision-making in uncertain situations difficult.
- $\mathcal{EIFS}$ improve upon classic intuitionistic $\mathbb{FS}$s by using exponential functions, resulting in a more sophisticated method to modelling uncertainty.
- $\mathcal{EIFS}$ techniques such as intersection, union, difference, and Cartesian product facilitate the collection and analysis of ambiguous decision criteria.
- $\mathcal{EIFS}$-based methodologies improve supplier evaluation by taking into account unpredictable elements such as cost, quality, and delivery reliability, resulting in more successful purchasing techniques.

1.2. Novelty

- Formalizes new $\mathcal{EIFS}$ operations, such as exponential intersection, union, and some difference operations, to extend the conventional set operations.
- Disjoint and disjunctive sums and an $\mathcal{EIF}$-based Cartesian product, which are specifically designed for uncertainty modeling in decision support systems.
- Illustrates the applicability of $\mathcal{EIFS}$ by using a real-world example of raw material procurement and demonstrates the way EIFS enhances the quality of decisions under uncertainty.
- Emphasizes the way $\mathcal{EIFS}$-based aggregation supports supplier assessment by combining factors such as cost, quality, and delivery time, to enable more practical and risk-averse purchasing decisions.

This paper is organized as: The Preliminaries of $\mathbb{FS}$s and its operations are covered in section 2, The basic definitions and characteristics of $\mathcal{EIFS}$s and some examples are covered in Section 3, The procedure of decision-making methodology under $\mathcal{EIFS}$s are shown in Section 4, application of decision making problems are demonstrated in section 4.1 and the conclusion is in section 5

2. Preliminaries

Definition 1. *A $\mathbb{FS}$ $\mathcal{A}_*$ in a universe of discourse $\mathfrak{X}$ is characterized by a $\mathbb{MF}$ $\mu_{\mathcal{A}_*}$ which takes the value in the unit interval* $[0,1]$

$$\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) = \mathfrak{X} \to [0,1]$$

The value of $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})$ represents the grade of $\mathbb{M}$ of $\mathfrak{X}$ in $\mathcal{A}_*$ and is a point in $[0,1]$.

Definition 2. *$\mathcal{A}_*$ is a $\mathbb{FS}$ on $\mathfrak{X}$, Then its complement $(\mathcal{A}_*{}')$ is*

$$\mu'_{\mathcal{A}_*}(\widetilde{\varsigma_*}) = 1 - \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}).$$

Definition 3. *The union of two $\mathbb{FS}$s $\mathcal{A}_*$ and $\mathcal{B}_*$ is with respective $\mathbb{MF}$s $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})$ and $\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})$ is a $\mathbb{FS}$ $\mathcal{A}_* \cup \mathcal{B}_*$, whose $\mathbb{MF}$ is related to those of $\mathcal{A}_*$ and $\mathcal{B}_*$ by*

$$\mu_{\mathcal{A}_* \cup \mathcal{B}_*}(\widetilde{\varsigma_*}) = \vee \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}), \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\right\}, \forall \widetilde{\varsigma_*} \in \mathfrak{X}.$$

Definition 4. *The intersection of two $\mathbb{FS}$s $\mathcal{A}_*$ and $\mathcal{B}_*$ is with respective memberships $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})$ and $\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})$ is a $\mathbb{FS}$ $\mathcal{A}_* \cap \mathcal{B}_*$, whose $\mathbb{MF}$ is related to those of $\mathcal{A}_*$ and $\mathcal{B}_*$ by*

$$\mu_{\mathcal{A}_* \cap \mathcal{B}_*}(\widetilde{\varsigma_*}) = \wedge \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}), \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\right\}, \forall \widetilde{\varsigma_*} \in \mathfrak{X}.$$

Definition 5. *The usual manner of defining fuzzy simple differences between two fuzzy sets $\mathcal{A}_*$ and $\mathcal{B}_*$ can be expressed as $\mathcal{A}_* - \mathcal{B}_* = \mathcal{A}_* \cap \mathcal{B}_{*c}$.*

Definition 6. *For any two $\mathbb{FS}$s $\mathcal{A}_*$ and $\mathcal{B}_*$, the bounded difference is defined as:*

$$\mu_{\mathcal{A}_* o \mathcal{B}_*}(\widetilde{\varsigma_*}) = \vee \left[o, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}), \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\right]$$

where $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})$ and $\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})$ denotes the $\mathbb{MF}$to which x is a member of $\mathcal{A}_*$ and $\mathcal{B}_*$.

Definition 7. *The disjoint sum of any two $\mathbb{FS}$s $\mathcal{A}_*$ and $\mathcal{B}_*$ on $\mathfrak{X}$*

$$\mu_{\mathcal{A}_* \otimes \mathcal{B}_*}(\widetilde{\varsigma_*}) = \left|\otimes, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}), \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\right|$$

where $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})$ and $\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})$ denotes the $\mathbb{MF}$ to which x is a member of $\mathcal{A}_*$ and $\mathcal{B}_*$.

Definition 8. *Let $\mathcal{A}_*$ and $\mathcal{B}_*$ be any two fuzzy sets of $\mathfrak{X}$ then the disjunctive sum is defined as:*

$$\mu_{\mathcal{A}_* \Delta \mathcal{B}_*}(\widetilde{\varsigma_*}) = \left(\mathcal{A}_* \cap \mathcal{B}_*{}'\right) \cup \left(\mathcal{A}_*{}' \cap \mathcal{B}_*\right) = \left(\mathcal{A}_* * \mathcal{B}_*{}'\right) \oplus \left(\mathcal{A}_*{}' * \mathcal{B}_*\right).$$

Definition 9. *Let $\mathcal{A}_*$ and $\mathcal{B}_*$ be any two $\mathbb{FS}$s of $\mathfrak{X}$ then the equivalence formula is*

$$\left(\mathcal{A}_*{}' \cup \mathcal{B}_*\right) \cap \left(\mathcal{A}_* \cup \mathcal{B}_*{}'\right) = \left(\mathcal{A}_*{}' \cap \mathcal{B}_*{}'\right) \cup \left(\mathcal{A}_* \cap \mathcal{B}_*\right).$$

Definition 10. *Symmetrical difference formula for two $\mathbb{FS}$s $\mathcal{A}_*$ and $\mathcal{B}_*$ is given by*

$$\left(\mathcal{A}_*{}' \cap \mathcal{B}_*\right) \cup \left(\mathcal{A}_* \cap \mathcal{B}_*{}'\right) = \left(\mathcal{A}_*{}' \cup \mathcal{B}_*{}'\right) \cap \left(\mathcal{A}_* \cup \mathcal{B}_*\right).$$

3. Exponential Intuitionstic Fuzzy Sets

Definition 11. *If $\mathfrak{X}$ is a universe discourse and $\widetilde{\varsigma_*}$ be any particular element of $\mathfrak{X}$. The intuitionistic* $\mathbb{FS}$ *$\mathcal{A}_*$ defined on $\mathfrak{X}$ is a collection of ordered pairs, $\mathcal{A}_* = \{(\widetilde{\varsigma_*}, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}), \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})) \, | x \in \mathfrak{X}\}$, where $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) : \mathfrak{X} \to [0,1]$ and $\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) : \mathfrak{X} \to [0,1]$ is called the* $\mathbb{MF}$ *and* $\mathbb{NMF}$. *The degree of* $\mathbb{M}$ *and* $\mathbb{NM}$ *on elements $\mathfrak{X}$. $0 \leq \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) + \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \leq 1$.*

Definition 12. *If $\mathfrak{X}$ is a universe discourse and $\widetilde{\varsigma_*}$ be any particular element of $\mathfrak{X}$. The $\mathcal{EIFS}$ $\mathcal{E}_{\mathcal{A}_*}$ defined on $\mathfrak{X}$ is a collection of ordered pairs,*
$\mathcal{E}_{\mathcal{A}_*} = \left\{ \left(\widetilde{\varsigma_*}, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \right) | \widetilde{\varsigma_*} \in \mathfrak{X}, a > 0 \right\}$, *where* $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} : \mathfrak{X} \to [0,1]$ *is called the* $\mathbb{MF}$ *and* $\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} : \mathfrak{X} \to [0,1]$ *is called the* $\mathbb{NMF}$. *The degree of* $\mathbb{MF}$ *is* $0 \leq \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} + \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \leq 1$.

Example 1. *Let $\mathfrak{X} = \{1, 2, 3, 4, 5\}$ be the universal set and $\mathcal{IFMV}s$ of $\mathfrak{X}$ is $\mu_A(\widetilde{\varsigma_*}) = \{0.9, 0.7, 0.5, 0.4, 0.3\}$ and $\lambda_A(\widetilde{\varsigma_*}) = \{0.7, 0.5, 0.7, 0.3, 0.2\}$, the decay parameter $a = 0.02$.*
The $\mathcal{EIF}$ $\mathbb{MF}$ *is given by: $\mathcal{E}_{\mathcal{A}_*}(\widetilde{\varsigma_*}) = \left(\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \right)$.*
The $\mathcal{EIF}$ $\mathbb{MV}$*s*
$\mathcal{E}_{\mathcal{A}_*}(\widetilde{\varsigma_*}) = \{(1, 0.884, 0.690), (2, 0.690, 0.495), (3, 0.495, 0.690), (4, 0.397, 0.298), (5, 0.298, 0.199)\}$.

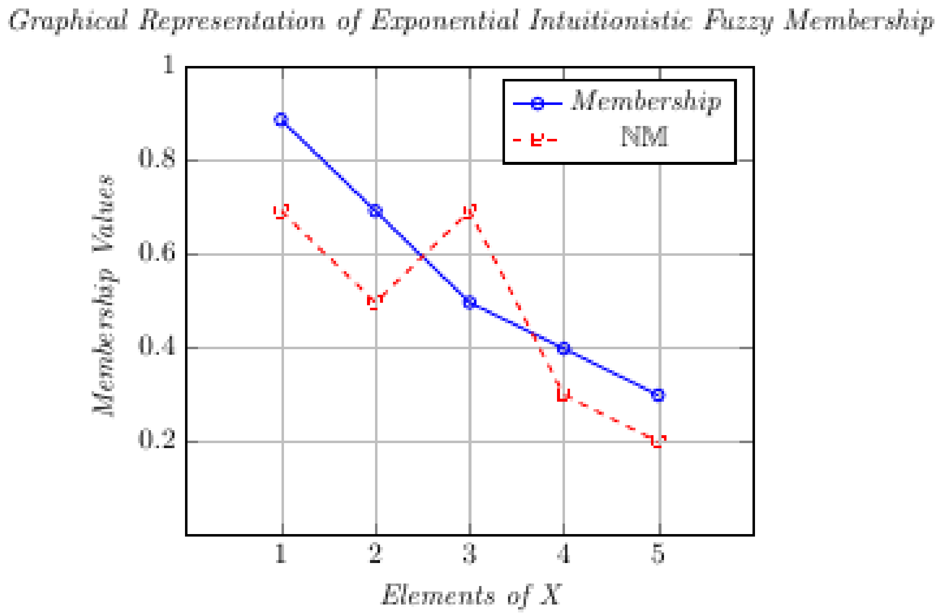

Figure 1: *IEST*.

Definition 13. *Let $\mathcal{E}_{\mathcal{A}_*} = \left\langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}} \right\rangle$ and $\mathcal{E}_{\mathcal{B}_*} = \left\langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}} \right\rangle$ be two $\mathcal{EIFS}s$ on $\mathfrak{X}$ with their grade values given by:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$$

$$\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$$

The $\mathcal{EIF}$ intersection of $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ is defined as:

$$\begin{aligned} \mu_{\mathcal{E}_{\mathcal{A}_*}\cap\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \wedge\left\{\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\} \\ &= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\} \quad (3.1) \\ \lambda_{\mathcal{E}_{\mathcal{A}_*}\cap\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \vee\left\{\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\} \\ &= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}\right\} \quad (3.2) \end{aligned}$$

Similarly, the $\mathcal{EIF}$ union of $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ is defined as:

$$\begin{aligned} \mu_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \vee\left\{\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\} \\ &= \vee\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\} \quad (3.3) \\ \lambda_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \wedge\left\{\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\} \\ &= \wedge\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}\right\} \quad (3.4) \end{aligned}$$

Example 2. *Let $\mathfrak{X} = \{\widetilde{\varsigma_{*1}}, \widetilde{\varsigma_{*2}}, \widetilde{\varsigma_{*3}}, \widetilde{\varsigma_{*4}}, \widetilde{\varsigma_{*5}}\}$ be a finite universe, and let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be two $\mathcal{EIFS}$s defined by the memberships:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$$

$$\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}$$

Now, we compute the corresponding $\mu_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})$ and $\lambda_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})$ $\mathbb{MN}$s:

$\widetilde{\varsigma_*}$	$\widetilde{\varsigma_{*1}}$	$\widetilde{\varsigma_{*2}}$	$\widetilde{\varsigma_{*3}}$	$\widetilde{\varsigma_{*5}}$	$\widetilde{\varsigma_{*1}}$
$\mu_A(\widetilde{\varsigma_*})$	0.8	0.6	0.4	0.2	0.1
$\lambda_A(\widetilde{\varsigma_*})$	0.2	0.3	0.4	0.1	0.1
$\mu_B(\widetilde{\varsigma_*})$	0.3	0.5	0.4	0.2	0.1
$\lambda_B(\widetilde{\varsigma_*})$	0.6	0.5	0.5	0.6	0.7

Definition 14. *Consider two $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*} = \langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}} \rangle$ and $\mathcal{E}_{\mathcal{B}_*} = \langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}} \rangle$ where*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$$

$$\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$$

denotes the $\mathbb{M}$ and non-functions of $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$. The simple difference $\mathcal{E}_{\mathcal{A}_*} - \mathcal{E}_{\mathcal{B}_*}$ of these two $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ is defined as:

$$\begin{aligned} \mathcal{E}_{\mathcal{A}_*} - \mathcal{E}_{\mathcal{B}_*} &= \mathcal{E}_{\mathcal{A}_*} \cap {\mathcal{E}_{\mathcal{B}_*}}' \\ &= \left\{\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*})\right\} * \left\{\mu'_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}), \lambda'_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\} \end{aligned}$$

$$= \left\{\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) * \mu'_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\}, \left\{\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) * \lambda'_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\}$$

$$= \left\{\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}), \lambda'_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right\}$$

$$\mathcal{E}_{\mathcal{A}_*} - \mathcal{E}_{\mathcal{B}_*} = \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda^c_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda^c_B(\widetilde{\varsigma_*})}\right\}$$

Example 3. *Let* $\mathcal{E}_{\mathcal{A}_*} = \left\{\frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*1}} + \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}, \frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*2}} + \frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*3}}\right\}$ *and*
$\mathcal{E}_{\mathcal{B}_*} = \left\{\frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*1}} + \frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*2}} + \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*3}}, \frac{0.55\mathfrak{e}^{-a0.55}}{\widetilde{\varsigma}_{*1}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*2}} + \frac{0.85\mathfrak{e}^{-a0.85}}{\widetilde{\varsigma}_{*3}}\right\}$ *be two two* $\mathcal{EIFS}$*s, the simple difference is*
$\mathcal{E}_{\mathcal{A}_*} - \mathcal{E}_{\mathcal{B}_*} = \mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}'$

$$= \left\{\frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*1}} + \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}, \frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*2}} + \frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*3}}\right\} *$$
$$\left\{\frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.55\mathfrak{e}^{-a0.55}}{\widetilde{\varsigma}_{*2}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*3}}, \frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.85\mathfrak{e}^{-a0.85}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right\}$$
$$= \left\{\left(\frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*1}} + \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right) * \left(\frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.55\mathfrak{e}^{-a0.55}}{\widetilde{\varsigma}_{*2}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*3}}\right)\right\},$$
$$\left\{\left(\frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*2}} + \frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*3}} * \frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.85\mathfrak{e}^{-a0.85}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right)\right\}$$
$$= \left\{\left(\frac{0.25\mathfrak{e}^{-a0.25}}{\widetilde{\varsigma}_{*1}} + \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right), \left(\frac{0.45\mathfrak{e}^{-a0.45}}{\widetilde{\varsigma}_{*1}} + \frac{0.65\mathfrak{e}^{-a0.65}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right)\right\}.$$

Definition 15. *Consider two* $\mathcal{EIFS}$*s* $\mathcal{E}_{\mathcal{A}_*} = \left\langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}} \right\rangle$ *and* $\mathcal{E}_{\mathcal{B}_*} = \left\langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}} \right\rangle$ *where*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$$

$$\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \ \& \ \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$$

denotes the $\mathbb{M}$ *and non-functions of* $\mathcal{E}_{\mathcal{A}_*}$ *and* $\mathcal{E}_{\mathcal{B}_*}$*. The bounded difference of these two* $\mathcal{EIFS}$*s* $\mathcal{E}_{\mathcal{A}_*}$ *and* $\mathcal{E}_{\mathcal{B}_*}$ *is defined as:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\diamond\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \vee\left[0, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right].$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\diamond\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \vee\left[0, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}\right].$$

Example 4. *Let* $\mathcal{E}_{\mathcal{A}_*} = \left(\frac{0.22\mathfrak{e}^{-a0.22}}{\widetilde{\varsigma}_{*1}} + \frac{0.82\mathfrak{e}^{-a0.82}}{\widetilde{\varsigma}_{*2}} + \frac{0.36\mathfrak{e}^{-a0.36}}{\widetilde{\varsigma}_{*3}}, \ \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.52\mathfrak{e}^{-a0.52}}{\widetilde{\varsigma}_{*2}} + \frac{0.46\mathfrak{e}^{-a0.46}}{\widetilde{\varsigma}_{*3}}\right)$ *and*
$\mathcal{E}_{\mathcal{B}_*} = \left(\frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.43\mathfrak{e}^{-a0.43}}{\widetilde{\varsigma}_{*2}} + \frac{0.05\mathfrak{e}^{-a0.05}}{\widetilde{\varsigma}_{*3}}, \ \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.13\mathfrak{e}^{-a0.13}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right)$ *be two two* $\mathcal{EIFS}$*s. The bounded difference of these two* $\mathcal{EIFS}$*s is:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\diamond\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.07\mathfrak{e}^{-a0.07}}{\widetilde{\varsigma}_{*1}} + \frac{0.39\mathfrak{e}^{-a0.39}}{\widetilde{\varsigma}_{*2}} + \frac{0.31\mathfrak{e}^{-a0.31}}{\widetilde{\varsigma}_{*3}}\right)$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\diamond\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0\mathfrak{e}^{-a0}}{\widetilde{\varsigma}_{*1}} + \frac{0.39\mathfrak{e}^{-a0.39}}{\widetilde{\varsigma}_{*2}} + \frac{0.31\mathfrak{e}^{-a0.31}}{\widetilde{\varsigma}_{*3}}\right).$$

Definition 16. *Let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be any two $\mathcal{EIFS}$s the disjoint sum is defined as:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\otimes\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left|\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} - \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right|$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\otimes\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left|\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} - \lambda_{B}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})}\right|.$$

Example 5. *Let* $\mathcal{E}_{\mathcal{A}_*} = \left(\frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*1}} + \frac{0.43\mathfrak{e}^{-a0.43}}{\widetilde{\varsigma}_{*2}} + \frac{0.05\mathfrak{e}^{-a0.05}}{\widetilde{\varsigma}_{*3}}, \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.13\mathfrak{e}^{-a0.13}}{\widetilde{\varsigma}_{*2}} + \frac{0.15\mathfrak{e}^{-a0.15}}{\widetilde{\varsigma}_{*3}}\right)$ *and*
$\mathcal{E}_{\mathcal{B}_*} = \left(\frac{0.22\mathfrak{e}^{-a0.22}}{\widetilde{\varsigma}_{*1}} + \frac{0.82\mathfrak{e}^{-a0.82}}{\widetilde{\varsigma}_{*2}} + \frac{0.36\mathfrak{e}^{-a0.36}}{\widetilde{\varsigma}_{*3}}, \frac{0.35\mathfrak{e}^{-a0.35}}{\widetilde{\varsigma}_{*1}} + \frac{0.52\mathfrak{e}^{-a0.52}}{\widetilde{\varsigma}_{*2}} + \frac{0.46\mathfrak{e}^{-a0.46}}{\widetilde{\varsigma}_{*3}}\right)$ *be two two $\mathcal{EIFS}$s. Using the max function for calculating the phase term, the disjoint sum of theses two two $\mathcal{EIFS}$s is:*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\otimes\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.07\mathfrak{e}^{-a0.07}}{\widetilde{\varsigma}_{*1}} + \frac{0.39\mathfrak{e}^{-a0.39}}{\widetilde{\varsigma}_{*2}} + \frac{0.31\mathfrak{e}^{-a0.31}}{\widetilde{\varsigma}_{*3}}\right)$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\otimes\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0\mathfrak{e}^{-a0}}{\widetilde{\varsigma}_{*1}} + \frac{0.39\mathfrak{e}^{-a0.39}}{\widetilde{\varsigma}_{*2}} + \frac{0.31\mathfrak{e}^{-a0.31}}{\widetilde{\varsigma}_{*3}}\right).$$

Definition 17. *Let* $\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$ *and* $\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$ *denotes the $\mathbb{M}$ and non membership functions $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$. Let $\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}$ represents the disjunctive sum of $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$, as*

$$\begin{aligned}
\mu_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \left[\mu_{\mathcal{E}_{\mathcal{A}_*}\cap\mathcal{E}_{\mathcal{B}_*}{}'}(\widetilde{\varsigma_*}) \oplus \mu_{\mathcal{E}_{\mathcal{A}_*}{}'\cap\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right] \\
&= \left[\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} * \mu'_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu'_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right] \\
&\oplus \left[\mu'_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu'_{\mathcal{A}_*}(\widetilde{\varsigma_*})} * \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right]. \\
\lambda_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \left[\lambda_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}{}'}(\widetilde{\varsigma_*}) * \lambda_{\mathcal{E}_{\mathcal{A}_*}{}'\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right] \\
&= \left[\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \oplus \lambda'_{B}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu'_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right] \\
&* \left[\lambda'_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda'_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \oplus \lambda_{B}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})}\right].
\end{aligned}$$

Example 6. *Suppose* $\mathcal{E}_{\mathcal{A}_*} = \left(\frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*1}} + \frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}, \frac{0.2\mathfrak{e}^{-a0.2}}{\widetilde{\varsigma}_{*1}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*2}} + \frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*3}}\right)$ *and*
$\mathcal{E}_{\mathcal{B}_*} = \left(\frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*1}} + \frac{0.4\mathfrak{e}^{-a0.4}}{\widetilde{\varsigma}_{*2}} + \frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*3}}, \frac{0.4\mathfrak{e}^{-a0.4}}{\widetilde{\varsigma}_{*1}} + \frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right)$. *Then the disjunctive sum of these two $\mathcal{EIFS}$s is define as*

$$\begin{aligned}
\mu_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \left[\mu_{\mathcal{E}_{\mathcal{A}_*}\cap\mathcal{E}_{\mathcal{B}_*}{}'}(\widetilde{\varsigma_*}) \oplus \mu_{\mathcal{E}_{\mathcal{A}_*}{}'\cap\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right] \\
&= \left[\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} * \mu'_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu'_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right] \oplus \left[\mu'_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu'_{\mathcal{A}_*}(\widetilde{\varsigma_*})} * \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right].
\end{aligned}$$

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*1}} + \frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*2}} + \frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*3}}\right) \oplus \left(\frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*1}} + \frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right).$$

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*1}} + \frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right).$$

$$\begin{aligned}\lambda_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) &= \left[\lambda_{\mathcal{E}_{\mathcal{A}_*}\cup\mathcal{E}_{\mathcal{B}_*}'}(\widetilde{\varsigma_*}) * \lambda_{\mathcal{E}_{\mathcal{A}_*}'\cup\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*})\right] \\ &= \left[\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \oplus \lambda_B'(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}'(\widetilde{\varsigma_*})}\right] * \left[\lambda_{\mathcal{A}_*}'(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}'(\widetilde{\varsigma_*})} \oplus \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}\right].\end{aligned}$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*1}} + \frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right) \oplus \left(\frac{0.8\mathfrak{e}^{-a0.8}}{\widetilde{\varsigma}_{*1}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right).$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\Delta\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \left(\frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*1}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*2}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*3}}\right).$$

Definition 18. *Consider two $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*} = \langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}}\rangle$ and $\mathcal{E}_{\mathcal{B}_*} = \langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}}\rangle$, where*

$$\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \quad \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$$

$$\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \quad \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$$

The Cartesian product of $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ is given by:

$$\mathcal{E}_{\mathcal{A}_*} \times \mathcal{E}_{\mathcal{B}_*} = \left\{ \left((x,y), \mu_{\mathcal{E}_{\mathcal{A}_*}\times\mathcal{E}_{\mathcal{B}_*}}(x,y), \lambda_{\mathcal{E}_{\mathcal{A}_*}\times\mathcal{E}_{\mathcal{B}_*}}(x,y)\right) \mid x \in A, y \in B \right\}$$

where

$$\mu_{\mathcal{E}_{\mathcal{A}_*}\times\mathcal{E}_{\mathcal{B}_*}}(x,y) = \wedge\left(\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(y)\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(y)}\right)$$

$$\lambda_{\mathcal{E}_{\mathcal{A}_*}\times\mathcal{E}_{\mathcal{B}_*}}(x,y) = \vee\left(\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(y)\mathfrak{e}^{-a\lambda_B(y)}\right)$$

Thus, the Cartesian product $\mathcal{E}_{\mathcal{A}_*} \times \mathcal{E}_{\mathcal{B}_*}$ forms an $\mathcal{EIFS}$ with these $\mathbb{M}$ and $\mathbb{NM}$functions.

Example 7. *Let the exponential parameter be $a = 1$, $\mathcal{E}_{\mathcal{A}_*} = \left(\frac{0.8\mathfrak{e}^{-a0.8}}{\widetilde{\varsigma}_{*1}} + \frac{0.6\mathfrak{e}^{-a0.6}}{\widetilde{\varsigma}_{*2}}, \frac{0.3\mathfrak{e}^{-a0.3}}{\widetilde{\varsigma}_{*1}} + \frac{0.4\mathfrak{e}^{-a0.4}}{\widetilde{\varsigma}_{*2}}\right)$ and $\mathcal{E}_{\mathcal{B}_*} = \left(\frac{0.7\mathfrak{e}^{-a0.7}}{\widetilde{\varsigma}_{*1}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*2}}, \frac{0.2\mathfrak{e}^{-a0.2}}{\widetilde{\varsigma}_{*1}} + \frac{0.5\mathfrak{e}^{-a0.5}}{\widetilde{\varsigma}_{*2}}\right)$. Then the Cartesian product $\mathcal{E}_{\mathcal{A}_*} \times \mathcal{E}_{\mathcal{B}_*}$ is*

$$\mathcal{E}_{\mathcal{A}} \times \mathcal{E}_{\mathcal{B}} = \left\{ \begin{array}{l} ((\widetilde{\varsigma}_{*1}, \mathfrak{d}_1), 0.3476, 0.2222), \\ ((\widetilde{\varsigma}_{*1}, \mathfrak{d}_2), 0.3033, 0.3033), \\ ((\widetilde{\varsigma}_{*2}, \mathfrak{d}_1), 0.3293, 0.2681), \\ ((\widetilde{\varsigma}_{*2}, \mathfrak{d}_2), 0.3033, 0.3033) \end{array} \right\}$$

Definition 19. *A quasi-triangular norm $\mathfrak{T}$ for $\mathcal{EIFS}$ is a function: $\mathfrak{T} : (0,1] \times (0,1] \to [0,1]$, satisfying the following conditions:*

(i) *Boundary Condition:* $\mathfrak{T}(1,1) = 1$.

(ii) *Commutativity:* $\mathfrak{T}(\rho,\rho') = \mathfrak{T}(\rho',\rho)$.

(iii) Monotonicity: If $\rho \leq \rho''$ *and* $\rho' \leq \rho'''$*, then:* $\mathfrak{T}(\rho, \rho') \leq \mathfrak{T}(\rho'', \rho''')$.

(iv) Associativity: $\mathfrak{T}(\mathfrak{T}(\rho, \rho'), \rho'') = \mathfrak{T}(\rho, \mathfrak{T}(\rho', \rho''))$.

Theorem 1. *For any three* $\mathcal{EIFS}$*s* $\mathcal{E}_{\mathcal{A}_*} = \langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}} \rangle, \mathcal{E}_{\mathcal{B}_*} = \langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}} \rangle$ *and* $\mathcal{E}_{\mathcal{C}} = \langle \mu_{\mathcal{E}_{\mathcal{C}}}, \lambda_{\mathcal{E}_{\mathcal{C}}} \rangle$ *on a universe of discourse* $\mathcal{X}$ *satisfy the following*

i. $\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*} = \mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{A}_*}$ *and* $\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*} = \mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{A}_*}$

ii. $\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cap \mathcal{E}_{\mathcal{C}}$ *and* $\mathcal{E}_{\mathcal{A}_*} \cup (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*}) \cup \mathcal{E}_{\mathcal{C}}$

Proof.

i. Let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be two $\mathcal{EIFS}$s and $\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}$
$\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$, and $\lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$,and represent their $\mathbb{M}$ and non membership function, respectively.

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*} &= \mathcal{E}_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \cap \mathcal{E}_{\mathcal{B}_*}(\widetilde{\varsigma_*}) \\
&= \wedge \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \right\} \\
&= \wedge \left\{ \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \right\} \\
&= \mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{A}_*} \\
\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*} &= \mathcal{E}_{\mathcal{A}_*}(\widetilde{\varsigma_*}) \cap \mathcal{E}_{\mathcal{B}_*}(\widetilde{\varsigma_*}) \\
&= \vee \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})} \right\} \\
&= \vee \left\{ \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \right\} \\
&= \mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{A}_*}
\end{aligned}$$

and similar way we can prove $\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*} = \mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{A}_*}$.

ii. Let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be three $\mathcal{EIFS}$s and $\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) = \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}$,
$\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$,and $\lambda_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) = \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}$ represent their $\mathbb{M}$ and non membership function, respectively. Then

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \cap \left(\mu_{\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})\right) \\
&= \wedge \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \wedge \left\{ \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})} \right\} \right\} \\
&= \wedge \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})} \right\} \\
&= (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cap \mathcal{E}_{\mathcal{C}} \\
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \cap \left(\lambda_{\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})\right) \\
&= \vee \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \vee \left\{ \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})} \right\} \right\}
\end{aligned}$$

$$= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}$$
$$= (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cap \mathcal{E}_{\mathcal{C}}$$

and similar way we can prove $\mathcal{E}_{\mathcal{A}_*} \cup (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*}) \cup \mathcal{E}_{\mathcal{C}}$.

Theorem 2. *For any three $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*} = \langle \mu_{\mathcal{E}_{\mathcal{A}_*}}, \lambda_{\mathcal{E}_{\mathcal{A}_*}}\rangle, \mathcal{E}_{\mathcal{B}_*} = \langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}}\rangle$ and $\mathcal{E}_{\mathcal{C}} = \langle \mu_{\mathcal{E}_{\mathcal{C}}}, \lambda_{\mathcal{E}_{\mathcal{C}}}\rangle$ on a universe of discourse $\mathcal{X}$ satisfy the following*

$\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}})$ *and* $\mathcal{E}_{\mathcal{A}_*} \cup (\mathcal{E}_{\mathcal{B}_*} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*}) \cap (\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{C}})$

Proof.

Let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be three $\mathcal{EIFS}$s and $\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) = \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}, \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$,and $\lambda_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) = \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}$ represent their $\mathbb{M}$ and $\mathbb{NM}$ function, respectively.

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \cap \left(\mu_{\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})\right)\\
&= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \vee\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\}\\
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \vee\left\{\wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\},\right.\\
&\qquad \left.\wedge\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\} \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \cap \left(\lambda_{\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})\right)\\
&= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \wedge\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}\right\}\\
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \wedge\left\{\vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\},\right.\\
&\qquad \left.\vee\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}\right\} \qquad (3.6)
\end{aligned}$$

Case 1. If $\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \le \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) \le \mu_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})$ and $\lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) \le \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) \le \lambda_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*})$

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \vee\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\}\\
&= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\\
&= \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}. \qquad (3.7)\\
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \vee\left\{\wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\},\right. \qquad (3.8)\\
&\qquad \left.\wedge\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\}\\
&= \vee\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\}
\end{aligned}$$

$$= \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \tag{3.9}$$

From equation (3.7) and (3.9)

$$\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}})$$

and

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \wedge\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}\right\} \\
&= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\} \\
&= \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}. \qquad (3.10) \\
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \wedge\left\{\vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}\right\},\right. \qquad (3.11) \\
&\qquad \left.\vee\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}\right\} \\
&= \wedge\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\} \\
&= \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})} \qquad (3.12)
\end{aligned}$$

From equation (3.10) and (3.12)

$$\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}})$$

Case 2. If $\mu_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) \leq \mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) \leq \mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*})$ and $\lambda_{\mathcal{E}_{\mathcal{C}}}(\widetilde{\varsigma_*}) \leq \lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) \leq \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*})$

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \vee\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\} \\
&= \wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\} \\
&= \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}. \qquad (3.13) \\
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) &= \vee\left\{\wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\},\right. \qquad (3.14) \\
&\qquad \left.\wedge\left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}\right\} \\
&= \vee\left\{\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \mu_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\} \\
&= \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \qquad (3.15)
\end{aligned}$$

From equation (3.13) and (3.15)

$$\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}})$$

and

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) &= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \wedge\left\{\lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})}\right\}\right\} \\
&= \vee\left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_C(\widetilde{\varsigma_*})}\right\}
\end{aligned}$$

$$= \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}. \tag{3.16}$$

$$(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}}) = \wedge \left\{ \vee \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})} \right\}, \right. \tag{3.17}$$

$$\left. \vee \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_C(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_C(\widetilde{\varsigma_*})} \right\} \right\}$$

$$= \wedge \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \right\}$$

$$= \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \tag{3.18}$$

From equation (3.16) and (3.18)

$$\mathcal{E}_{\mathcal{A}_*} \cap (\mathcal{E}_{\mathcal{B}_*} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*}) \cup (\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{C}})$$

Theorem 3. *For any two $\mathcal{EIFS}$s $\mathcal{E}_{\mathcal{A}_*} = \langle \mu_{\mathcal{E}_{\mathcal{A}_*}}$ and $\lambda_{\mathcal{E}_{\mathcal{A}_*}} \rangle, \mathcal{E}_{\mathcal{B}_*} = \langle \mu_{\mathcal{E}_{\mathcal{B}_*}}, \lambda_{\mathcal{E}_{\mathcal{B}_*}} \rangle$ on a universe of discourse $\mathcal{X}$ satisfy the following (i) $(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*})^c = (\mathcal{E}_{\mathcal{B}_*})^c \cup (\mathcal{E}_{\mathcal{A}_*})^c$ and (ii) $(\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*})^c = (\mathcal{E}_{\mathcal{B}_*})^c \cap (\mathcal{E}_{\mathcal{A}_*})^c$.*

Proof.

(i) Let $\mathcal{E}_{\mathcal{A}_*}$ and $\mathcal{E}_{\mathcal{B}_*}$ be two $\mathcal{EIFS}$s and $\mu_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$ and $\mu_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}, \lambda_{\mathcal{E}_{\mathcal{A}_*}}(\widetilde{\varsigma_*}) = \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}$ and $\lambda_{\mathcal{E}_{\mathcal{B}_*}}(\widetilde{\varsigma_*}) = \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})}$ represent their $\mathbb{M}$ and $\mathbb{NMF}$, respectively. For $\mathbb{MF}$

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*})^c &= \mu_{\mathcal{A}_* \cap \mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_* \cap \mathcal{B}_*}(\widetilde{\varsigma_*})'} \\
&= \left[1 - \wedge \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \cap \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \right\} \right] \\
&= \vee \left\{ 1 - \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, 1 - \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})} \right\} \\
&= \vee \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})'}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})'} \right\} \\
&= \left\{ \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})'} \cup \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})'} \right\} \\
&= (\mathcal{E}_{\mathcal{B}_*})^c \cup (\mathcal{E}_{\mathcal{A}_*})^c.
\end{aligned}$$

For $\mathbb{N} - \mathbb{MF}$,

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}_*} \cap \mathcal{E}_{\mathcal{B}_*})^c &= \lambda_{\mathcal{A}_* \cap \mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_* \cap \mathcal{B}_*}(\widetilde{\varsigma_*})'} \\
&= \left[1 - \wedge \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \cap \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})} \right\} \right] \\
&= \vee \left\{ 1 - \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, 1 - \lambda_B(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})} \right\} \\
&= \vee \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})'}, \lambda_B(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})'} \right\} \\
&= \left\{ \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})'} \cup \lambda_B(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_B(\widetilde{\varsigma_*})'} \right\} \\
&= (\mathcal{E}_{\mathcal{B}_*})^c \cup (\mathcal{E}_{\mathcal{A}_*})^c.
\end{aligned}$$

(ii) For $\mathbb{M}$ function

$$\begin{aligned}(\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*})^c &= \mu_{\mathcal{A}_* \cup \mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_* \cup \mathcal{B}_*}(\widetilde{\varsigma_*})'} \\ &= \left[1 - \vee \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \cup \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\}\right] \\ &= \wedge \left\{1 - \mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, 1 - \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})}\right\} \\ &= \wedge \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})'}, \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})'}\right\} \\ &= \left\{\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{A}_*}(\widetilde{\varsigma_*})'} \cap \mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\mu_{\mathcal{B}_*}(\widetilde{\varsigma_*})'}\right\} \\ &= (\mathcal{E}_{\mathcal{B}_*})^c \cap (\mathcal{E}_{\mathcal{A}_*})^c.\end{aligned}$$

For $\mathbb{NMF}$,

$$\begin{aligned}(\mathcal{E}_{\mathcal{A}_*} \cup \mathcal{E}_{\mathcal{B}_*})^c &= \lambda_{\mathcal{A}_* \cup \mathcal{B}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_* \cup \mathcal{B}_*}(\widetilde{\varsigma_*})'} \\ &= \left[1 - \vee \left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})} \cap \lambda_{B}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})}\right\}\right] \\ &= \wedge \left\{1 - \lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})}, 1 - \lambda_{B}(\widetilde{\varsigma_*})\mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})}\right\} \\ &= \wedge \left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})'}, \lambda_{B}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})'}\right\} \\ &= \left\{\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{\mathcal{A}_*}(\widetilde{\varsigma_*})'} \cap \lambda_{B}(\widetilde{\varsigma_*})' \mathfrak{e}^{-a\lambda_{B}(\widetilde{\varsigma_*})'}\right\} \\ &= (\mathcal{E}_{\mathcal{B}_*})^c \cap (\mathcal{E}_{\mathcal{A}_*})^c.\end{aligned}$$

4. Methodology for Decision-Making Using Exponential Intuitionstic Fuzzy Sets

$\mathcal{EIFS}$s offer a flexible approach to decision-making, particularly where uncertainty, imprecision, and hesitancy play a role. $\mathcal{EIFS}$ facilitate decision-making by integrating several sources of information using appropriate aggregation techniques.

Sept 1 Suppose you have n distinct possibilities., $\mathfrak{L} = \{\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, ...\mathfrak{L}_n\}$ in a multi-attributes $\mathcal{A}$. The issue and m are various standards from the index set $\mathfrak{A} = \{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ...\mathfrak{A}_m\}$. The alternatives attribute value $\mathfrak{L}_r$ for index $\mathfrak{F}_p$ is $\mathcal{E}_{sr} = \left\langle \mu_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*}))}, \lambda_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*}))}\right\rangle, (s = 1, 2, ...n; r = 1, 2, ...m)$. So the $\mathcal{EIFS}$s $\mathcal{A}$ matrix is

$$\mathfrak{A} = \begin{pmatrix} \mathcal{E}_{11}, ...\mathcal{E}_{12}, ...\mathcal{E}_{1m} \\ \mathcal{E}_{21}, ...\mathcal{E}_{21}, ...\mathcal{E}_{2m} \\ ... \quad ... \quad ... \\ ... \quad ... \quad ... \\ \mathcal{E}_{n1}, ...\mathcal{E}_{n2}, ...\mathcal{E}_{2nm} \end{pmatrix},$$

where, $\mathcal{E}_{sr} = \left\langle \mu_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*}))}, \lambda_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda_{\mathcal{E}_{sr}}(\widetilde{\varsigma_*}))}\right\rangle$.

Sept 2 Assume that

$$\mu_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*}) = \vee \left\{\mu_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*})\ for\ fix\ i\ and\ j = 1,2,3,....m\right\}$$
$$\lambda_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*}) = \wedge \left\{\lambda_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*})\ for\ fix\ i\ and\ j = 1,2,3,....n\right\}$$

and thus

$$\mathcal{A}_{l\ max} = \begin{pmatrix} \mu_{\mathcal{E}_{11}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu_{\mathcal{E}_{11}}(\widetilde{\varsigma_*}))}, \lambda_{\mathcal{E}_{11}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda_{\mathcal{E}_{11}}(\widetilde{\varsigma_*}))} \\ \mu_{\mathcal{E}_{21}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu_{\mathcal{E}_{21}}(\widetilde{\varsigma_*}))}, \lambda_{\mathcal{E}_{21}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda_{E_{21}}(\widetilde{\varsigma_*}))} \\ ... \quad ... \quad ... \\ ... \quad ... \quad ... \\ \mu_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*}))}, \lambda_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*}))} \end{pmatrix}, \tag{4.1}$$

for all $l = 1,2,3...$ is called max $\mathcal{A}$-Matrix.

Sept 3 Assume that

$$\mu'_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*}) = \wedge \left\{\mu'_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*})\ for\ fix\ i\ and\ j = 1,2,3,....m\right\}$$
$$\lambda'_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*}) = \vee \left\{\lambda'_{\mathcal{E}_{ij}}(\widetilde{\varsigma_*})\ for\ fix\ i\ and\ j = 1,2,3,....n\right\}$$

and thus

$$\mathcal{A}_{l\ min} = \begin{pmatrix} \mu'_{\mathcal{E}_{11}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu'_{\mathcal{E}_{11}}(\widetilde{\varsigma_*}))}, \lambda'_{\mathcal{E}_{11}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda'_{\mathcal{E}_{11}}(\widetilde{\varsigma_*}))} \\ \mu'_{\mathcal{E}_{21}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu'_{\mathcal{E}_{21}}(\widetilde{\varsigma_*}))}, \lambda'_{\mathcal{E}_{21}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda'_{\mathcal{E}_{21}}(\widetilde{\varsigma_*}))} \\ ... \quad ... \quad ... \\ ... \quad ... \quad ... \\ \mu'_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu'_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*}))}, \lambda'_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda'_{\mathcal{E}_{n1}}(\widetilde{\varsigma_*}))} \end{pmatrix}, \tag{4.2}$$

for all $l = 1,2,3...$ is called min $\mathcal{A}$-Matrix.

Sept 4 The average similarity measure operator($\mathcal{ASMO}$) S_l is

$$S_l\left(\mathcal{A}_{l\ max}, \mathcal{A}_{l\ min}\right) = \begin{pmatrix} max\left(\frac{\mu'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*}))} + \lambda'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*}))}}{2}\right) \\ min\left(\frac{\mu'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\mu'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*}))} + \lambda'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha(\lambda'_{\mathcal{E}_{i1}}(\widetilde{\varsigma_*}))}}{2}\right) \end{pmatrix}, i = 1,2,3,....n,$$

$$S_l\left(\mathcal{A}_{l\ max}, \mathcal{A}_{l\ min}\right) = \left(\mathbb{D}_{S_{l1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha\mathbb{D}_{S_{l1}}(\widetilde{\varsigma_*})}, \mathbb{D}'_{S_{l1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha\mathbb{D}'_{S_{l1}}(\widetilde{\varsigma_*})}\right)$$

Sept 5 $\mathcal{ASMO}$ for weight $\mathbb{D}^*$ is

$$\mathbb{D}^* = \left[\frac{w_r max \mathbb{D}_{S_{l1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha\mathbb{D}_{S_{l1}}(\widetilde{\varsigma_*})}}{\sqrt{\sum(\mathbb{D}_{S_{l1}}(\widetilde{\varsigma_*}))^2}}, \frac{w_r max \mathbb{D}'_{S_{l1}}(\widetilde{\varsigma_*})\mathfrak{e}^{-\alpha\mathbb{D}'_{S_{l1}}(\widetilde{\varsigma_*})}}{\sqrt{\sum(\mathbb{D}'_{S_{l1}}(\widetilde{\varsigma_*}))^2}}\right], l = 1,2,3..and\ r = 1,2,3....m.$$

where $w_r = (1,2,3...w_m)$ is the index weight vector & $\sum_{i=1}^{n} w_r = 1$.

Step 6 The alternatives are ranked by assessing the $\mathcal{ASMO}$ for weight values, with the most desirable options being chosen.

4.1. Implementation of The Proposed Methodology on a Decision Making Problem

A manufacturing company must select the best supplier for raw materials based on multiple conflicting criteria such as cost ($\mathfrak{L}_1$), quality($\mathfrak{L}_2$), delivery time ($\mathfrak{L}_3$),and reliability($\mathfrak{L}_4$). Experts might be unsure about their judgments due to fluctuating market conditions. Some supplier performance metrics (e.g., delivery time) may have seasonal variations. The phase component in $\mathcal{EIFS}$ helps model such cyclic patterns. The additional exponential-valued information enhances the accuracy of ranking suppliers. By fixing the parameter ($a = 0.2$) for defining the prioritization of the above conflicting criteria.

Sept 1 Consider three matrix $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ of $\mathcal{EIF}$

Matrix 1: Imported Raw materials in 2018:

$$\mathcal{A}_1 = \begin{array}{cc} & \begin{array}{cccc} \mathfrak{F}_1 & \mathfrak{F}_2 & \mathfrak{F}_3 & \mathfrak{F}_4 \end{array} \\ \begin{array}{c} \mathfrak{L}_1 \\ \mathfrak{L}_2 \\ \mathfrak{L}_3 \\ \mathfrak{L}_4 \end{array} & \begin{pmatrix} 0.3\mathfrak{e}^{-0.2(0.3)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.7\mathfrak{e}^{-0.2(0.7)}, 0.2\mathfrak{e}^{-0.2(0.2)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.6\mathfrak{e}^{-0.2(0.6)} \\ 0.2\mathfrak{e}^{-0.2(0.2)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.3\mathfrak{e}^{-0.2(0.3)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.8\mathfrak{e}^{-0.2(0.8)} & 0.1\mathfrak{e}^{-0.2(0.1)}, 0.2\mathfrak{e}^{-0.2(0.2)} \\ 0.1\mathfrak{e}^{-0.2(0.1)}, 0.9\mathfrak{e}^{-0.2(0.9)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.3\mathfrak{e}^{-0.2(0.3)}, 0.6\mathfrak{e}^{-0.2(0.6)} & 0.3\mathfrak{e}^{-0.2(0.3)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.7\mathfrak{e}^{-0.2(0.7)}, 0.1\mathfrak{e}^{-0.2(0.1)} & 0.6\mathfrak{e}^{-0.2(0.6)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.3\mathfrak{e}^{-0.2(0.3)}, 0.2\mathfrak{e}^{-0.2(0.2)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.4\mathfrak{e}^{-0.2(0.4)} \end{pmatrix} \end{array}$$

Matrix 2: Imported Raw materials in 2019:

$$\mathcal{A}_2 = \begin{array}{cc} & \begin{array}{cccc} \mathfrak{F}_1 & \mathfrak{F}_2 & \mathfrak{F}_3 & \mathfrak{F}_4 \end{array} \\ \begin{array}{c} \mathfrak{L}_1 \\ \mathfrak{L}_2 \\ \mathfrak{L}_3 \\ \mathfrak{L}_4 \end{array} & \begin{pmatrix} 0.1\mathfrak{e}^{-0.2(0.1)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.8\mathfrak{e}^{-0.2(0.8)}, 0.2\mathfrak{e}^{-0.2(0.2)} & 0.5\mathfrak{e}^{-0.2(0.5)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.6\mathfrak{e}^{-0.2(0.6)} \\ 0.5\mathfrak{e}^{-0.2(0.5)}, 0.4\mathfrak{e}^{-0.2(0.54)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.6\mathfrak{e}^{-0.2(0.6)} & 0.2\mathfrak{e}^{-0.2(0.2)}, 0.4\mathfrak{e}^{-0.2(0.4)} & 0.1\mathfrak{e}^{-0.2(0.1)}, 0.2\mathfrak{e}^{-0.2(0.2)} \\ 0.1\mathfrak{e}^{-0.2(0.1)}, 0.8\mathfrak{e}^{-0.2(0.8)} & 0.3\mathfrak{e}^{-0.2(0.3)}, 0.6\mathfrak{e}^{-0.2(0.6)} & 0.2\mathfrak{e}^{-0.2(0.2)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.7\mathfrak{e}^{-0.2(0.7)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.1\mathfrak{e}^{-0.2(0.1)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.5\mathfrak{e}^{-0.2(0.5)}, 0.2\mathfrak{e}^{-0.2(0.2)} \end{pmatrix} \end{array}$$

Matrix 3: Imported Raw materials in 2020:

$$\mathcal{A}_3 = \begin{array}{cc} & \begin{array}{cccc} \mathfrak{F}_1 & \mathfrak{F}_2 & \mathfrak{F}_3 & \mathfrak{F}_4 \end{array} \\ \begin{array}{c} \mathfrak{L}_1 \\ \mathfrak{L}_2 \\ \mathfrak{L}_3 \\ \mathfrak{L}_4 \end{array} & \begin{pmatrix} 0.6\mathfrak{e}^{-0.2(0.6)}, 0.1\mathfrak{e}^{-0.2(0.1)} & 0.6\mathfrak{e}^{-0.2(0.6)}, 0.1\mathfrak{e}^{-0.2(0.1)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.5\mathfrak{e}^{-0.2(0.5)}, 0.3\mathfrak{e}^{-0.2(0.3)} \\ 0.3\mathfrak{e}^{-0.2(0.3)}, 0.4\mathfrak{e}^{-0.2(0.4)} & 0.2\mathfrak{e}^{-0.2(0.2)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.5\mathfrak{e}^{-0.2(0.5)}, 0.6\mathfrak{e}^{-0.2(0.6)} & 0.7\mathfrak{e}^{-0.2(0.7)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.4\mathfrak{e}^{-0.2(0.4)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.6\mathfrak{e}^{-0.2(0.6)}, 0.2\mathfrak{e}^{-0.2(0.2)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.7\mathfrak{e}^{-0.2(0.7)} & 0.1\mathfrak{e}^{-0.2(0.1)}, 0.3\mathfrak{e}^{-0.2(0.3)} \\ 0.6\mathfrak{e}^{-0.2(0.6)}, 0.2\mathfrak{e}^{-0.2(0.2)} & 0.4\mathfrak{e}^{-0.2(0.4)}, 0.5\mathfrak{e}^{-0.2(0.5)} & 0.2\mathfrak{e}^{-0.2(0.2)}, 0.3\mathfrak{e}^{-0.2(0.3)} & 0.5\mathfrak{e}^{-0.2(0.5)}, 0.4\mathfrak{e}^{-0.2(0.4)} \end{pmatrix} \end{array}$$

We formulate the $\mathcal{A}_{1max}, \mathcal{A}_{1max}$ and $\mathcal{A}_{1max}$ matrices by using equation (4.1).

$$\mathcal{A}_{1max} = \begin{pmatrix} 0.7\mathfrak{e}^{-0.2(0.7)}, 0.2\mathfrak{e}^{-0.2(0.2)} \\ 0.5\mathfrak{e}^{-0.2(0.5)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.5\mathfrak{e}^{-0.2(0.5)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.7\mathfrak{e}^{-0.2(0.7)}, 0.3\mathfrak{e}^{-0.2(0.3)} \end{pmatrix} \mathcal{A}_{2max} = \begin{pmatrix} 0.8\mathfrak{e}^{-0.2(0.8)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.5\mathfrak{e}^{-0.2(0.5)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.4\mathfrak{e}^{-0.2(0.4)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.7\mathfrak{e}^{-0.2(0.7)}, 0.1\mathfrak{e}^{-0.2(0.1)} \end{pmatrix} \&$$

$$\mathcal{A}_{3max} = \begin{pmatrix} 0.7\mathfrak{e}^{-0.2(0.7)}, 0.3\mathfrak{e}^{-0.2(0.3)} \\ 0.7\mathfrak{e}^{-0.2(0.7)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.6\mathfrak{e}^{-0.2(0.6)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.6\mathfrak{e}^{-0.2(0.6)}, 0.2\mathfrak{e}^{-0.2(0.2)} \end{pmatrix}$$

We formulate the $\mathcal{A}_{1min}, \mathcal{A}_{1min}$ and $\mathcal{A}_{1min}$ matrices by using equation (4.2).

$$\mathcal{A}_{1min} = \begin{pmatrix} 0.3\mathfrak{e}^{-0.2(0.3)}, 0.7\mathfrak{e}^{-0.2(0.7)} \\ 0.1\mathfrak{e}^{-0.2(0.1)}, 0.8\mathfrak{e}^{-0.2(0.8)} \\ 0.1\mathfrak{e}^{-0.2(0.1)}, 0.9\mathfrak{e}^{-0.2(0.9)} \\ 0.3\mathfrak{e}^{-0.2(0.3)}, 0.7\mathfrak{e}^{-0.2(0.7)} \end{pmatrix} \mathcal{A}_{2min} = \begin{pmatrix} 0.1\mathfrak{e}^{-0.2(0.8)}, 0.4\mathfrak{e}^{-0.2(0.3)} \\ 0.1\mathfrak{e}^{-0.2(0.6)}, 0.2\mathfrak{e}^{-0.2(0.6)} \\ 0.1\mathfrak{e}^{-0.2(0.8)}, 0.1\mathfrak{e}^{-0.2(0.7)} \\ 0.1\mathfrak{e}^{-0.2(0.7)}, 0.2\mathfrak{e}^{-0.2(0.5)} \end{pmatrix} \&$$

$$\mathcal{A}_{3min} = \begin{pmatrix} 0.4\mathfrak{e}^{-0.2(0.4)}, 0.3\mathfrak{e}^{-0.2(0.3)} \\ 0.2\mathfrak{e}^{-0.2(0.2)}, 0.6\mathfrak{e}^{-0.2(0.6)} \\ 0.1\mathfrak{e}^{-0.2(0.1)}, 0.7\mathfrak{e}^{-0.2(0.7)} \\ 0.2\mathfrak{e}^{-0.2(0.2)}, 0.5\mathfrak{e}^{-0.2(0.5)} \end{pmatrix}$$

The $\mathcal{ASMO}$ $S_t : t = 1, 2, 3,$

$$S_1(\mathcal{A}_{1max}, \mathcal{A}_{1min}) = \begin{pmatrix} 0.5\mathfrak{e}^{-0.2(0.5)}, 0.35\mathfrak{e}^{-0.2(0.35)} \\ 0.3\mathfrak{e}^{-0.2(0.3)}, 0.45\mathfrak{e}^{-0.2(0.45)} \\ 0.3\mathfrak{e}^{-0.2(0.3)}, 0.5\mathfrak{e}^{-0.2(0.5)} \\ 0.5\mathfrak{e}^{-0.2(0.5)}, 0.5\mathfrak{e}^{-0.2(0.5)} \end{pmatrix},$$

$$S_2(\mathcal{A}_{2max}, \mathcal{A}_{2min}) = \begin{pmatrix} 0.45\mathfrak{e}^{-0.2(0.5)}, 0.35\mathfrak{e}^{-0.2(0.35)} \\ 0.3\mathfrak{e}^{-0.2(0.3)}, 0.15\mathfrak{e}^{-0.2(0.15)} \\ 0.25\mathfrak{e}^{-0.2(0.25)}, 0.1\mathfrak{e}^{-0.2(0.1)} \\ 0.4\mathfrak{e}^{-0.2(0.4)}, 0.15\mathfrak{e}^{-0.2(0.15)} \end{pmatrix} \quad and$$

$$S_3(\mathcal{A}_{3max}, \mathcal{A}_{3min}) = \begin{pmatrix} 0.5\mathfrak{e}^{-0.2(0.5)}, 0.3\mathfrak{e}^{-0.2(0.3)} \\ 0.45\mathfrak{e}^{-0.2(0.45)}, 0.35\mathfrak{e}^{-0.2(0.35)} \\ 0.35\mathfrak{e}^{-0.2(0.4)}, 0.4\mathfrak{e}^{-0.2(0.4)} \\ 0.4\mathfrak{e}^{-0.2(0.35)}, 0.35\mathfrak{e}^{-0.2(0.35)} \end{pmatrix}$$

Sept 5 We find $\mathcal{ASMO}$ of weight $\mathbb{D}^*$ for the vector $w = (0.2, 0.2, 0.2, 0.2)$

$$\mathbb{D}^* = \begin{pmatrix} 0.10\mathfrak{e}^{-2(0.5)}, 0.10\mathfrak{e}^{-2(0.3)} \\ 0.15\mathfrak{e}^{-2(0.45)}, 0.05\mathfrak{e}^{-2(0.15)} \\ 0.14\mathfrak{e}^{-2(0.4)}, 0.02\mathfrak{e}^{-2(0.1)} \\ 0.12\mathfrak{e}^{-2(0.5)}, 0.04\mathfrak{e}^{-2(0.15)} \end{pmatrix}$$

Sept 6 Based on $\mathbb{D}^*$ values, We find that the $\mathcal{NMS}$ of Quality is more than the $\mathcal{MS}$ of all other competing criteria, and that the $\mathcal{NMS}$ of Quality is less than that of all other conflicting criteria. So, $\mathfrak{L}_2 =$ Quality is the important criteria for selection of raw material in a manufacturing company. By changing the fixation value ($a = 0.3, 0.4, ...$) we can get various results that close to 1 which means we can be able get the better solution enhanced with absolute accuracy.

4.2. Comparative Analysis

$\mathcal{EFS}$ and $\mathcal{EIFS}$are two new tools in mathematics introduced in an attempt to add up new ideas on modeling of uncertainty in real-life cases particularly when it comes to decision-making and information processing. Using exponential membership functions, $\mathcal{EFS}$ controls the range of the belongingness of elements to set where the smoother way of the change of the set and the better handling of non-linear fluctuations of data is provided. This methodology ensures that membership values mildly change as input changes are made in small intervals, thus increasing the decision-making sensitivity. However, the $\mathcal{EFS}$ is reliant only on a degree of membership and is not able to measure directly the non-membership or hesitation, thus, limiting the precision of the $\mathcal{EFS}$ in a higher environment with incomplete or inconsistent data. On the other hand, $\mathcal{EFS}$s compliments $\mathcal{EFS}$ infra structure with exponential membership function/non membership function that brings to the argument hesitation degree as well. This three-component framework can assist $\mathcal{EIFSs}$ to describe uncertainty more satisfactorily, as it can offer a sharper and more robust model for cases when decision-making is marred within imprecision, vagueness, and the discordant criteria. The $\mathcal{EIFS}s$ facilitates a more effective handling of the fuzzy data due to differentiation of the unknown information and unfavorable data. In addition, operations such as intersection, union, ordinary difference, and disjoint sum are also expanded in $\mathcal{EIFS}s$ model in order to preserve its expanded structure. It is notwithstanding the fact that $\mathcal{EIFS}s$ computations are more formidable that the system potentiality for accommodating larger uncertainties and inconsistencies and hence the resulting decisions being in the capacity of making more profound dichotomies advances its case as a very powerful tool in scenarios such as multi-criteria decision-making, the appraisal, supplier selection, and resource allocation.

5. Conclusion

$\mathcal{EIFS}$s are a strong expansion of traditional intuitionistic $\mathbb{FS}$s that improves the capacity to express uncertainty by using exponential membership and nonmembership functions. The operations investigated, such as intersection, union, difference measures, sum operations, and Cartesian products, show their usefulness in dealing with complicated decision-making issues. The use of these processes in raw material procurement demonstrates their usefulness in assessing various providers in unpredictable settings. Organisations that use $\mathcal{EIFS}$-based decision-making procedures may make more informed and balanced decisions, taking into account critical criteria such as cost, quality, and delivery time. The findings show that $\mathcal{EIFS}$ not only improves decision accuracy, but also provides an organised way for dealing with contradictory and insufficient information in procurement procedures.
Future Works: The concept exponential intuitionistic fuzzy sets we can applied in Future Works in graph theory and decision making problems.

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