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Uni-Soft Ideals in Ordered Semigroups, Delineated by Soft Union Products

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Abstract. In this paper, we consider the concepts of uni-soft left and uni-soft right ideals, uni-soft quasi-ideals, and uni-soft bi-ideals within the context of ordered semigroups. We demonstrate that in ordered semigroups, both uni-soft right and uni-soft left ideals exhibit properties of uni-soft quasi-ideals. Similarly, uni-soft quasi-ideals possess characteristics of uni-soft bi-ideals. Furthermore, our analysis establishes that the definitions of uni-soft quasi-ideals and uni-soft bi-ideals align, indicating their equivalence within this specific class of semigroups. Additionally, we prove that in an ordered semigroup, uni-soft quasi-ideals can be understood simply as the unions of uni-soft right and uni-soft left ideals. This elucidates the relationship between these concepts, shedding light on their fundamental role in the structure of ordered semigroups.

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Key Words and Phrases: Ordered semigroup, soft set, soft union product, uni-soft left (right) ideal, uni-soft quasi-ideal, uni-soft bi-ideal.

1. Introduction

Let U be an initial universe set, E a set of parameters, and $A \subseteq E$. The notation $P(U)$ represents the power set of the set U. A soft set over the set U is defined by a mapping $f_A : E \longrightarrow P(U)$, where $f_A(x) = \emptyset$ for all x not belonging to A. Traditional mathematics often struggle to capture the complex uncertainties found in diverse fields such as economics, engineering, environmental science, medical science, and social science. To address this limitation, Molodtsov [1] developed soft set theory as an innovative mathematical approach to handling uncertainty through a parameterization point of view. A significant advantage of soft set theory is that it circumvents the challenges associated with defining membership functions, making it remarkably versatile and applicable across numerous domains. Recently, the application of soft set theory have been considerable advancement to algebraic structures [2, 3].

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An *ordered semigroup*, denoted by $(S, ., \leq)$, is an algebraic structure consisting of a semigroup $(S, .)$, that is, . is an associative binary operation on S ($(xy)z = x(yz)$ for all $x, y, z \in S$) together with a partial order $\leq$ that is compatible with the semigroup operation, that is,

$$\text{if } x \leq y, \text{ then } xz \leq yz \text{ and } zx \leq zy$$

for all $x, y, z \in S$. The study of ordered semigroups has been significantly advanced and widely expanded, especially through the work of Kehayopulu and colleagues [4]. Concurrently, Jun et al. [5] applied the concept of soft set theory to ordered semigroups. They applied the notion of soft sets by Molodtsov to ordered semigroups and introduced the notions of soft ordered semigroups, soft ordered subsemigroups, soft l-ideals, soft r-ideals, and l-idealistic and r-idealistic soft ordered semigroups. They investigated various related properties. In [6], Jun et al. provided a soft algebraic tool in ordered semigroups in considering many problems that contain uncertainties. They introduced the notions of union-soft semigroups, union-soft l-ideals, and union-soft r-ideals and investigate their properties. Characterizations of them were considered. Khan et al. [7] defined uni-soft ideals and the characterized left, right and completely regular ordered semigroup in term of them. The concept of uni-soft bi-ideals were introduced by Khan et al. [8]. Their algebraic properties were also investigated.

Cagman et al. [9] further expanded the scope by introducing a new variant of soft groups, termed as soft intersection groups (abbreviated as soft int-groups). These soft intersection groups differ from the definition of soft groups proposed by Aktas and Cagman [2]. The concept of soft int-groups is grounded in the principles of inclusion relations and set intersection, amalgamating concepts from soft set theory, set theory, and group theory. Various supplementary properties of soft int-groups and normal soft int-groups, reminiscent of classical group theory and fuzzy group theory, have been explored ([10], [11], [3]).

Furthermore, recent research has extended these concepts into the semigroups, with investigations into ideal theory based on soft int-semigroups ([12]). Additionally, E. Hamouda [13] contributed to the discourse by discussing soft ideals and soft filters within the context of soft ordered groupoids.

In this paper, we consider the concepts of soft left and soft right ideals, soft quasi-ideals, and soft bi-ideals within the framework of ordered semigroups. We demonstrate that in ordered semigroups, soft right and soft left ideals can be regarded as soft quasi-ideals. Similarly, in ordered semigroups, soft quasi-ideals exhibit characteristics of soft bi-ideals. Furthermore, our analysis reveals that in specific ordered semigroups, the notions of soft quasi-ideals and soft bi-ideals coincide. Additionally, we establish that in an ordered semigroup, soft quasi-ideals can be understood as intersections of soft right and soft left ideals. This elucidates the interconnectedness and equivalence of these concepts within the context of ordered semigroups, providing valuable insights into the algebraic structures underlying soft set theory in this setting.

2. Preliminaries

Soft set theory was introduced by Molodtsov [1]. Then Cagman and Enginoglu [14] provided new definitions and various results on soft set theory. From now on, let U be an *initial universe set*, E a set of *parameters*, and $A, B, C \subseteq E$. The notation $P(U)$ represents the power set of the set U.

Definition 1. *[1, 14] A soft set f_A over U is defined as $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. A soft set f_A over U can be represented by the set of ordered pairs*

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\},$$

where $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ for all $x \notin A$. Note that the set of all soft sets over U is denoted by $S(U)$.

Definition 2. *[14] Let $f_A, f_B \in S(U)$. Then*

(i) f_A is a soft subset *of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

(ii) f_A is equal *to f_B, denoted by $f_A = f_B$, if $f_A(x) = f_B(x)$ for all $x \in E$.*

(iii) The union *of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is a soft set over U such that*

$$(f_A \tilde{\cup} f_B)(x) = f_A(x) \cup f_B(x)$$

for all $x \in E$.

(iv) The intersection *of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is a soft set over U such that*

$$(f_A \tilde{\cap} f_B)(x) = f_A(x) \cap f_B(x)$$

for all $x \in E$.

Lemma 1. *([14]) Let $f_A, f_B, f_C \in S(U)$. The following statements are noted:*

(i) $f_A \tilde{\cup} f_A = f_A$, $f_A \tilde{\cap} f_A = f_A$.

(ii) $f_A \tilde{\cup} f_B = f_B \tilde{\cup} f_A$, $f_A \tilde{\cap} f_B = f_B \tilde{\cap} f_A$.

(iii) $f_A \tilde{\subseteq} f_A \tilde{\cup} f_B$, $f_A \tilde{\cap} f_B \tilde{\subseteq} f_A$.

(iv) $f_A \tilde{\cup} (f_B \tilde{\cap} f_C) = (f_A \tilde{\cup} f_B) \tilde{\cap} (f_A \tilde{\cup} f_C)$, $f_A \tilde{\cap} (f_B \tilde{\cup} f_C) = (f_A \tilde{\cap} f_B) \tilde{\cup} (f_A \tilde{\cap} f_C)$.

(v) $f_A \tilde{\subseteq} f_B \iff f_A \tilde{\cap} f_B = f_A \iff f_A \tilde{\cup} f_B = f_B$.

From now on, let $(S, \cdot, \leq)$ be an ordered semigroup. For each $a \in S$, let

$$A_a = \{(x, y) \in S \times S \mid xy \leq a\}.$$

It can be observed that A_a may be empty.

Definition 3. *[15] Let $f_S, g_S \in S(U)$. The* soft union product *of f_S and g_S, denoted by $f_S \diamond g_S$, is a soft set over U such that*

$$(f_S \diamond g_S)(a) = \begin{cases} \bigcap\limits_{(x,y)\in A_a} \{f_S(x) \cup g_S(y)\} & , \text{if } A_a \neq \emptyset; \\ U & , \text{if } A_a = \emptyset \end{cases}$$

for all $a \in S$.

Example 1. *Consider an ordered semigroup $(S, \cdot, \leq)$ with $S = \{a, b, c, d\}$, the multiplication and the ordered relation $\leq$ are defined as follows:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq = \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

Then $A_a = (S \times S) \setminus \{(c,c), (d,c), (d,d)\}$, $A_b = S \times S$, $A_c = A_d = \emptyset$. Let $U = \{e, x, y, z\}$ be the universal set. Let f_S and g_S be soft sets over U such that

$$f_S(a) = \{e, y, z\}, f_S(b) = \{e, x\}, f_S(c) = \{y, z\}, f_S(d) = \{e, x, y\}$$

$$g_S(a) = \{x, y\}, g_S(b) = \{e, z\}, g_S(c) = \{z\}, g_S(d) = \{e, y\}.$$

By simple calculation we get

$$(f_S \diamond g_S)(a) = (f_S \diamond g_S)(b) = \emptyset \text{ and } (f_S \diamond g_S)(c) = (f_S \diamond g_S)(d) = U.$$

Definition 4. *[7] A soft set f_S over U is called an* uni-soft semigroup *of S over U if*

$$f_S(xy) \subseteq f_S(x) \cup f_S(y)$$

for all $x, y \in S$.

Let $\theta : S \longrightarrow P(U)$ be the soft set over U defined by

$$\theta(x) = \emptyset \text{ for all } x \in S.$$

Then θ is a uni-soft semigroup of S over U and $\theta \tilde{\subseteq} f_S$ for all $f_S \in S(U)$.

Theorem 1. *Let $f_S, g_S, h_S, k_S \in S(U)$. If $f_S \tilde{\subseteq} g_S$ and $h_S \tilde{\subseteq} k_S$, then $f_S \diamond h_S \tilde{\subseteq} g_S \diamond k_S$.*

Proof. Let $a \in S$. If A_a is empty, then

$$(f_S \diamond h_S)(a) \subseteq U = (g_S \diamond k_S)(a).$$

Thus $f_S \diamond h_S \tilde{\subseteq} g_S \diamond k_S$. Assume that A_a is non-empty. Then

$$\begin{aligned}(f_S \diamond h_S)(a) &= \bigcap_{(x,y)\in A_a} \{f_S(x) \cup h_S(y)\} \\ &\subseteq \bigcap_{(x,y)\in A_a} \{g_S(x) \cup k_S(y)\} \\ &= (g_S \diamond k_S)(a).\end{aligned}$$

Thus $f_S \diamond h_S \tilde{\subseteq} g_S \diamond k_S$.

3. Main Results

In this section, we recall the concept of uni-soft left ideals, uni-soft right ideals, uni-soft quasi-ideals and uni-soft bi-ideals [6, 7]. The purpose of this section is to present several properties associated with them.

Definition 5. *A soft set f_S over U is called a* uni-soft left ideal *if*

(i) $f_S(xy) \subseteq f_S(y)$ for all $x, y \in S$;

(ii) for any $x, y \in S$, if $x \leq y$, then $f_S(x) \supseteq f_S(y)$.

Definition 6. *A soft set f_S over U is called a* uni-soft right ideal *if*

(i) $f_S(xy) \subseteq f_S(x)$ for all $x, y \in S$;

(ii) for any $x, y \in S$, if $x \leq y$, then $f_S(x) \supseteq f_S(y)$.

Definition 7. *A soft set f_S over U is called a* uni-soft quasi-ideal *if*

(i) $f_S \tilde{\subseteq} (\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta)$;

(ii) for any $x, y \in S$, if $x \leq y$, then $f_S(x) \supseteq f_S(y)$.

Definition 8. *A soft set f_S over U is called a* uni-soft bi-ideal *if*

(i) $f_S(xyz) \subseteq f_S(x) \cup f_S(z)$ for all $x, y, z \in S$;

(ii) for any $x, y \in S$, if $x \leq y$, then $f_S(x) \supseteq f_S(y)$.

Theorem 2. *Every uni-soft left (right, quasi-) ideal over U is a uni-soft semigroup.*

Proof. We prove that every uni-soft left ideal over U is a uni-soft semigroup; for other cases can be proved similarly. Let f_S be a uni-soft left ideal over U. If $x, y \in S$, then

$$f_S(xy) \subseteq f_S(y) \subseteq f_S(x) \cup f_S(y).$$

Hence f_S is a uni-soft semigroup.

In [7], the authors stated that Every uni-soft left (right) ideal over U is a uni-soft quasi-ideal over U. We give the proof of this statement by the following theorem.

Theorem 3. *Every uni-soft left (right) ideal over U is a uni-soft quasi-ideal over U.*

Proof. We prove that every uni-soft left ideal over U is a uni-soft quasi-ideal over U; for every uni-soft right ideal over U is a uni-soft quasi-ideal over U can be proved similarly. Let f_S be a uni-soft left ideal over U. To show that $f_S \tilde{\subseteq} (\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta)$, let $a \in S$. There are two cases to consider: $A_a = \emptyset$ and $A_a \neq \emptyset$. If A_a is empty, then

$$((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta))(a) = (\theta \diamond f_S)(a) \cup (f_S \diamond \theta)(a) = U \cup U = U \supseteq f_S(a).$$

Assume that A_a is non-empty. If $(x, y) \in A_a$, then it follows by f_S is a uni-soft left ideal over U that

$$f_S(a) \subseteq f_S(xy) \subseteq f_S(y).$$

Consequently,

$$(\theta \diamond f_S)(a) = \bigcap_{(x,y)\in A_a} \{\theta(x) \cup f_S(y)\} = \bigcap_{(x,y)\in A_a} \{\emptyset \cup f_S(y)\} = \bigcap_{(x,y)\in A_a} \{f_S(y)\} \supseteq f_S(a).$$

Hence

$$\begin{aligned} ((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta))(a) &= (\theta \diamond f_S)(a) \cup (f_S \diamond \theta)(a) \\ &\supseteq (\theta \diamond f_S)(a) \\ &\supseteq f_S(a). \end{aligned}$$

Therefore, f_S is a uni-soft quasi-ideal over U.

In the following example, we provide an example to show that the converse of the above theorem does not generally hold.

Example 2. *Let $(S, ., \leq)$ be an ordered semigroup:*

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	3	3	3	3

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(2,1)\}.$$

Then

$$f_S(x) = \begin{cases} \{0\} & , \textit{if } x = 0; \\ \{0,1\} & , \textit{if } x = 1; \\ \{0,1,2\} & , \textit{if } x = 2; \\ \{0,1,2,3\} & , \textit{if } x = 3; \end{cases}$$

is a uni-soft quasi-ideal but not a uni-soft left ideal.

Theorem 4. *Every uni-soft quasi-ideal over U is a uni-soft bi-ideal over U.*

Proof. Let f_S be a uni-soft quasi-ideal over U. Let $x, y, z \in S$. Then

$$f_S(xyz) \subseteq ((\theta \diamond f_S)\tilde{\cup}(f_S \diamond \theta))(xyz) = (\theta \diamond f_S)(xyz) \cup (f_S \diamond \theta)(xyz).$$

By $xyz \leq (xy)z$, it follows that

$$(\theta \diamond f_S)(xyz) = \bigcap_{(u,v)\in A_{xyz}} \{\theta(u) \cup f_S(v)\} = \bigcap_{(u,v)\in A_{xyz}} \{f_S(v)\} \subseteq f_S(z).$$

Similarly, by $xyz \leq x(yz)$, it follows that

$$(f_S \diamond \theta)(xyz) = \bigcap_{(u,v)\in A_{xyz}} \{f_S(u) \cup \theta(v)\} = \bigcap_{(u,v)\in A_{xyz}} \{f_S(u)\} \subseteq f_S(x).$$

Consequently,

$$f_S(xyz) \subseteq f_S(x) \cup f_S(z).$$

Therefore f_S is a uni-soft bi-ideal.

According to Theorem 3 and Theorem 4, we have the following corollary.

Corollary 1. *Every uni-soft left (right) ideal over U is a uni-soft bi-ideal over U.*

In the following example, we illustrate that the converse of Theorem 4 does not hold in general.

Example 3. *Let $(S, \cdot, \leq)$ be an ordered semigroup:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	2
2	0	0	0	0
3	0	2	0	1

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(2,0)\}.$$

Then

$$f_S(x) = \begin{cases} \{0\} & , \text{if } x = 0; \\ \{0,1\} & , \text{if } x = 1; \\ \{0,1,2\} & , \text{if } x = 2; \\ \{0,1,2,3\} & , \text{if } x = 3; \end{cases}$$

is a uni-soft bi-ideal but not a uni-soft quasi-ideal.

The following theorem provides a condition under which uni-soft quasi-ideals and uni-soft bi-ideals coincide in an ordered semigroup.

Theorem 5. *Assume* $(S, ., \leq)$ *is an ordered semigroup satisfying the following condition:*

$$\forall a \in S \ \exists x \in S \ (axa \leq a).$$

Then the uni-soft quasi-ideals over U *and the uni-soft bi-ideals over* U *coincide.*

Proof. By Theorem 4, it is sufficient to demonstrate that every uni-soft bi-ideal over U is also a uni-soft quasi-ideal over U. Let f_S be a uni-soft bi-ideal over U. To prove that f_S is a uni-soft quasi-ideal over U, consider $a \in S$. By assumption, $A_a \neq \emptyset$. Then

$$\begin{aligned}(f_S \diamond \theta)(a) &= \bigcap_{(u,v)\in A_a} \{f_S(u) \cup \theta(v)\} \\ &= \bigcap_{(u,v)\in A_a} f_S(u)\end{aligned}$$

and

$$\begin{aligned}(\theta \diamond f_S)(a) &= \bigcap_{(u,v)\in A_a} \{\theta(u) \cup f_S(v)\} \\ &= \bigcap_{(u,v)\in A_a} f_S(v).\end{aligned}$$

These imply that

$$(f_S \diamond \theta)(a) \cup (\theta \diamond f_S)(a) = \left(\bigcap_{(u,v)\in A_a} f_S(u)\right) \cup \left(\bigcap_{(u,v)\in A_a} f_S(v)\right).$$

Let $x \in S$ be such that $axa \leq a$. Suppose that there exists $y \in f_S(a)$ such that

$$y \notin (f_S \diamond \theta)(a) \cup (\theta \diamond f_S)(a)$$

Then $y \notin f_S(u_1)$ for some $(u_1, v_1) \in A_a$ and $y \notin f_S(v_2)$ for some $(u_2, v_2) \in A_a$. Observe that $u_1v_1xu_2v_2 \leq a$. By f_S is a uni-soft bi-ideal over U,

$$y \in f_S(a) \subseteq f_S(u_1v_1xu_2v_2) \subseteq f_S(u_1) \cup f_S(v_2).$$

This is a contradiction. Hence $f_S(a) \subseteq (f_S \diamond \theta)(a) \cup (\theta \diamond f_S)(a)$ and the proof is complete.

Lemma 2. *Let $f_S \in S(U)$. Then the following statements hold:*

(1) $(\theta \diamond f_S)(xy) \subseteq f_S(y)$ *for all* $x, y \in S$*;*

(2) $(\theta \diamond f_S)(xy) \subseteq (\theta \diamond f_S)(y)$ *for all* $x, y \in S$*.*

Proof. (1) Let $x, y \in S$. From $(x, y) \in A_{xy}$, it follows that $A_{xy} \neq \emptyset$. Then

$$(\theta \diamond f_S)(xy) = \bigcap_{(u,v)\in A_{xy}} \{\theta(u) \cup f_S(v)\} = \bigcap_{(u,v)\in A_{xy}} \{f_S(v)\} \subseteq f_S(y).$$

(2) Let $x, y \in S$. If A_y is empty, then

$$(\theta \diamond f_S)(xy) \subseteq U = (\theta \diamond f_S)(y).$$

Assume that A_y is non-empty. For each $(s, t) \in A_y$, we obtain that $xst \leq xy$. Thus $(xs, t) \in A_{xy}$. It follows that

$$\begin{aligned}(\theta \diamond f_S)(xy) &= \bigcap_{(u,v)\in A_{xy}} \{\theta(u) \cup f_S(v)\}\\ &= \bigcap_{(u,v)\in A_{xy}} \{f_S(v)\}\\ &\subseteq f_S(t)\\ &= \theta(s) \cup f_S(t)\end{aligned}$$

Hence

$$(\theta \diamond f_S)(xy) \subseteq \bigcap_{(u,v)\in A_{y}} \{\theta(u) \cup f_S(v)\} = (\theta \diamond f_S)(y).$$

In the same manner as the proof of Lemma 2, we can establish the following lemma:

Lemma 3. *Let f_S be a soft set over U.*

(1) $(f_S \diamond \theta)(xy) \subseteq f_S(x)$ *for all* $x, y \in S$*;*

(2) $(f_S \diamond \theta)(xy) \subseteq (f_S \diamond \theta)(x)$ *for all* $x, y \in S$*.*

Lemma 4. *Let $f_S \in S(U)$ and $x, y \in S$. If $x \leq y$, then $(\theta \diamond f_S)(y) \subseteq (\theta \diamond f_S)(x)$.*

Proof. Let $x, y \in S$ be such that $x \leq y$. Then $A_x \subseteq A_y$. If A_x is empty, then

$$(\theta \diamond f_S)(y) \subseteq U = (\theta \diamond f_S)(x).$$

Assume that A_x is non-empty. Then

$$(\theta \diamond f_S)(x) = \bigcap_{(u,v)\in A_{x}} \{\theta(u) \cup f_S(v)\} = \bigcap_{(u,v)\in A_{x}} \{f_S(v)\}.$$

Let $(s,t) \in A_x$. Then $st \leq x \leq y$ and we have

$$\begin{aligned}(\theta \diamond f_S)(y) &= \bigcap_{(u,v)\in A_y} \{\theta(u) \cup f_S(v)\} \\ &= \bigcap_{(u,v)\in A_y} \{f_S(v)\} \\ &\subseteq \bigcap_{(u,v)\in A_x} \{f_S(v)\} \\ &= (\theta \diamond f_S)(x).\end{aligned}$$

Using a technique like the one employed in proving Lemma 4, we can deduce the following lemma:

Lemma 5. *Let $f_S \in S(U)$ and $x, y \in S$. If $x \leq y$, then $(f_S \diamond \theta)(y) \subseteq (f_S \diamond \theta)(x)$.*

Lemma 6. *Let $f_S \in S(U)$ and $x, y \in S$. Then*

$$f_S \tilde{\cap} (\theta \diamond f_S)(xy) \subseteq f_S \tilde{\cap} (\theta \diamond f_S)(x)$$

Proof. Let $x, y \in S$. Since $f_S \tilde{\cap} (\theta \diamond f_S) \tilde{\subseteq} \theta \diamond f_S$,

$$f_S \tilde{\cap} (\theta \diamond f_S)(xy) \subseteq (\theta \diamond f_S)(xy)$$

By Lemma 2,

$$(\theta \diamond f_S)(xy) \subseteq f_S(x) \text{ and } (\theta \diamond f_S)(xy) \subseteq (\theta \diamond f_S)(x).$$

Then

$$(\theta \diamond f_S)(xy) \subseteq f_S(x) \cap (\theta \diamond f_S)(x) = (f_S \tilde{\cap} (\theta \diamond f_S))(x).$$

Hence

$$(f_S \tilde{\cap} (\theta \diamond f_S))(xy) \subseteq (\theta \diamond f_S)(xy) \subseteq (f_S \tilde{\cap} (\theta \diamond f_S))(x).$$

By employing a method similar to the one utilized in proving Lemma 6, we can infer the following lemma:

Lemma 7. *Let f_S be a soft set over U. Then*

$$f_S \tilde{\cap} (\theta \diamond f_S)(xy) \subseteq f_S \tilde{\cap} (\theta \diamond f_S)(y)$$

for any $x, y \in S$.

Lemma 8. *Let f_S be a soft set over U satisfying the condition*

$$x \leq y \text{ implies } f_S(y) \subseteq f_S(x) \text{ for all } x, y \in S.$$

Then the soft set $f_S \tilde{\cap} (\theta \diamond f_S)$ is a uni-soft left ideal over U.

Proof. Let $x, y \in S$. By Lemma 6, we have

$$(f_S \tilde{\cap} (\theta \diamond f_S))(xy) \subseteq (f_S \tilde{\cap} (\theta \diamond f_S))(y).$$

Now, suppose $x, y \in S$ with $x \leq y$. By Lemma 4,

$$(\theta \diamond f_S)(y) \subseteq (\theta \diamond f_S)(x).$$

By assumption we have that

$$f_S(y) \subseteq f_S(x).$$

Consequently,

$$\begin{aligned}(f_S \tilde{\cap} (\theta \diamond f_S))(y) &= f_S(y) \cap (\theta \diamond f_S)(y) \\ &\subseteq f_S(x) \cap (\theta \diamond f_S))(x) \\ &= (f_S \tilde{\cap} (\theta \diamond f_S))(x)\end{aligned}$$

Therefore, $f_S \tilde{\cap} (\theta \diamond f_S)$ is a uni-soft left ideal over U.

We can derive the following lemma using a similar method to the proof of Lemma 8.

Lemma 9. *Let f_S be a soft set over U satisfying the condition*

$$x \leq y \text{ implies } f_S(y) \subseteq f_S(x) \text{ for all } x, y \in S.$$

Then the soft set $f_S \tilde{\cap} (f_S \diamond \theta)$ is a uni-soft right ideal over U.

Finally, the following theorem shows that uni-soft quasi-ideals can be understood simply as the unions of uni-soft right and uni-soft left ideals.

Theorem 6. *Let f_S be a soft set over U. Then f_S is a uni-soft quasi-ideal over U if and only if there exist a uni-soft left ideal g_S and a uni-soft right ideal h_S over U such that*

$$f_S = g_S \tilde{\cup} h_S.$$

Proof. Firstly, we assume that f_S is a uni-soft quasi-ideal over U. Then $f_S(y) \subseteq f(x)$ for all $x, y \in S$ with $x \leq y$. By Lemma 8 and Lemma 9, we have that $f_S \tilde{\cap} (\theta \diamond f_S)$ and $f_S \tilde{\cap} (f_S \diamond \theta)$ are a uni-soft left ideal over U and a uni-soft right ideal over U, respectively. Since f_S is a uni-soft quasi-ideal over U, it follows that

$$\begin{aligned}f_S &\tilde{\subseteq} (\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta) \\ &= ((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta)) \tilde{\cap} f_S \\ &= ((\theta \diamond f_S) \tilde{\cap} f_S) \tilde{\cup} ((f_S \diamond \theta) \tilde{\cap} f_S) \\ &\tilde{\subseteq} f_S \tilde{\cup} f_S \\ &= f_S\end{aligned}$$

Therefore, f_S is the union of a uni-soft left ideal $f_S \tilde{\cap} (\theta \diamond f_S)$ and a uni-soft right ideal $f_S \tilde{\cap} (f_S \diamond \theta)$.

Conversely, assume that there exist a uni-soft left ideal g_S and a uni-soft right ideal h_S over U such that

$$f_S = g_S \tilde{\cup} h_S.$$

Let $a \in S$. Then

$$f_S(a) \subseteq ((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta))(a).$$

To see this, if A_a is empty, then

$$f_S(a) \subseteq U = U \cup U = (\theta \diamond f_S)(a) \cup (f_S \diamond \theta)(a) = ((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta))(a).$$

Assume that A_a is non-empty. Then

$$(\theta \diamond f_S)(a) = \bigcap_{(u,v) \in A_a} \{\theta(u) \cup f_S(v)\} = \bigcap_{(u,v) \in A_a} \{f_S(v)\}.$$

If $(x, y) \in A_a$, then $xy \leq a$; hence

$$g_S(a) \subseteq g_S(xy) \subseteq g_S(y).$$

Thus

$$g_S(a) \subseteq \bigcap_{(u,v) \in A_a} \{f_S(v)\} \subseteq (\theta \diamond f_S)(a).$$

Therefore $f_S(a) \subseteq (\theta \diamond f_S)(a) \tilde{\cup} (f_S \diamond \theta)(a) = ((\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta))(a)$. This implies that

$$f_S \tilde{\subseteq} (\theta \diamond f_S) \tilde{\cup} (f_S \diamond \theta).$$

Let $x, y \in S$ be such that $x \leq y$. Then

$$\begin{aligned} f_S(y) &= (g_S \tilde{\cup} h_S)(y) \\ &= g_S(y) \cup h_S(y) \\ &\subseteq g_S(x) \cup h_S(x) \\ &= (g_S \tilde{\cup} h_S)(x) \\ &= f_S(x). \end{aligned}$$

Hence f_S is a uni-soft quasi-ideal over U.

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