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Interpolative Contractions for b-metric Spaces and Their Applications

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Abstract. In this research, we introduce the interpolative contractions for a pair of maps in b-metric spaces and we utilize the idea of interpolation in complete b-metric spaces to prove the associated common fixed point theorems. Our findings generalize and expand the findings of Edraoui et al. [7] and Karapınar [10] from the metric space setting to b-metric spaces. We provide instances to support our conclusions. We offer implementations to solve Fredholm-type nonlinear integral equations and functional equations that emerge in dynamic programming in order to illustrate the importance of our theoretical findings.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Common fixed points, b-metric space, integral equation, functional equation

1. Introduction

The successive approximation methods that were first developed by a number of prior mathematicians, including well-known figures like Cauchy, Liouville, Picard, Lipschitz, and others, are successfully encapsulated and reinterpreted by the Banach contraction principle. Czerwik [5] developed the idea of b-metric space, often known as metric type space, as a generalization of metric space.. Regarding the Hardy-Rogers fixed point theorem's generalization to the interpolative Hardy-Rogers type contractive mapping. Interestingly, this new kind of mapping was first developed by Karapınar, who integrated the interpolation notion into the Hardy-Rogers framework. This method probably broadens the original theorem's usefulness by generating intermediate points between known data points.

It is true that interpolation is frequently used in mathematical study to generalize different types of contractions. Researchers can broaden the application of current theorems and offer a more adaptable framework for examining fixed points in metric spaces

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by including interpolation techniques into contraction mappings. It appears that the interpolative approach has been used to generalize various contraction types and in other studies. This illustrates the interpolation approach's adaptability and efficiency in extending the notion of fixed points and offering fresh perspectives on the existence and uniqueness of solutions.

I suggest consulting the relevant paper [6, 8–16, 18] and looking into similar research in the subject to learn more about the particulars and ramifications of Karapınar's work as well as the generalization of other contraction forms utilizing the interpolative method. These resources ought to offer a more thorough comprehension of the interpolative contractive mapping of the Hardy-Rogers type and its uses in fixed point theory.

In 2018, E. Karapınar [10] introduced the notion of interpolative Kannan type contraction and established corresponding fixed point theorem in complete metric spaces.

Definition 1. [10] *Let (E, d) be a metric space. A mappings $T : E \to E$ is said to be interpolative Kannan type contraction if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Ta, Tb) \leq \lambda[d(Ta, a)]^{\alpha}[d(Tb, b)]^{1-\alpha}$$

for all $a, b \in X$ such that $Ta \neq a$.

Theorem 1. [10] *Suppose that (E, d) be a complete metric space, and T is a interpolative Kannan type contraction. Then, T has a unique common fixed point.*

Definition 2. [16] *Let (E, d) be a metric space. A mapping $T : E \to E$ is said to be interpolative Hardy-Rogers contraction if there exist a constant $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Ta, Tb) \leq k[d(a, b)^{\beta}][d(Ta, a)]^{\gamma}[d(Tb, b)]^{\alpha}\left[\frac{d(Ta, b) + d(a, Tb)}{2}\right]^{1-\alpha-\beta-\gamma}$$

for all $a, b \in X$ such that $Ta \neq a$.

Theorem 2. [16] *Suppose that (E, d) be a complete metric space, and T is a interpolative Hardy-Rogers pair. Then, T has a unique common fixed point.*

Recently, Mohamed Edraoui [7] proved the following theorem using interpolative Hardy-Rogers pair contraction in complete metric spaces.

Definition 3. [7] *Let (E, d) be a metric space. A pair of mappings $T, S : E \to E$ is said to be interpolative Hardy-Rogers pair contraction if there exist $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Ta, Sb) \leq k[d(a, b)^{\beta}][d(Ta, a)]^{\gamma}[d(Sb, b)]^{\alpha}\left[\frac{d(Ta, b) + d(a, Sb)}{2}\right]^{1-\alpha-\beta-\gamma}$$

for all $a, b \in X$ such that $Ta \neq a$ whenever $Sb \neq b$.

Theorem 3. [7] *Suppose that (E, d) be a complete metric space, and (T, S) is a interpolative Hardy-Rogers pair contraction. Then, S and T have a unique common fixed point.*

2. Main Results

In the following, we introduce interpolative contraction maps in b-metric spaces.

The definition of Hardy-Rogers-type contraction has been generalized by adding the notion of interpolation. The goal is to find new qualities and broaden the definition of a Hardy-Rogers-type contraction by interpolation.

Definition 4. *Let (E, d, s) be a b-metric space. A pair of mappings $T, S : X \to X$ is said to be interpolative Hardy-Rogers-type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$d(Ta, Sb) \leq \lambda[d(a, b)^{\beta}][d(Ta, a)]^{\gamma}[d(Sb, b)]^{\alpha} \left[\frac{d(Ta, b) + d(a, Sb)}{2s}\right]^{1-\alpha-\beta-\gamma} \tag{2.1}$$

for all $a, b \in X$ such that $Ta \neq a$ whenever $Sb \neq b$.

Proposition 1. *Allow (E, d, s) to be a b-metric space with two self-maps $T, S : E \to E$ and a coefficient $s \geq 1$. The pair (T, S) is assumed to be an interpolative Hardy-Rogers-type contraction. In the event that $a^{'}$ is a fixed point of S, then $a^{'}$ is a fixed point of T because of this. Additionally, in this instance, $a^{'}$ is unique.*

Theorem 4. *Suppose that (E, d, s) be a complete b-metric space, and (T, S) is an interpolative Hardy-Rogers-type contraction. Both S and T have a unique common fixed point if either T or S is b-continuous.*

Proof. *Let $a_0 \in E$ be an arbitrary point. Consider $\{a_n\}$, given as $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Sa_{2n+1}$ for each positive integer n.*
Take $a = a_{2n}$ and $b = a_{2n+1}$ in (2.1), we get

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(Ta_{2n}, Sa_{2n+1}) \\ &\leq \lambda[d(a_{2n}, a_{2n+1})^{\beta}][d(Ta_{2n}, a_{2n})]^{\gamma}[d(Sa_{2n+1}, a_{2n+1})]^{\alpha} \\ &\quad [\tfrac{1}{2s} d(Ta_{2n}, a_{2n+1}) + d(a_{2n}, Sa_{2n+1})]^{1-\alpha-\beta-\gamma} \\ &= \lambda[d(a_{2n}, a_{2n+1})^{\beta}][d(a_{2n+1}, a_{2n})]^{\gamma}[d(a_{2n+2}, a_{2n+1})]^{\alpha} \\ &\quad [\tfrac{1}{2s} d(a_{2n+1}, a_{2n+1}) + d(a_{2n}, a_{2n+2})]^{1-\alpha-\beta-\gamma} \end{aligned}$$

Then

$$\begin{aligned} [d(a_{2n+1}, a_{2n+2})]^{1-\alpha} &\leq \lambda[d(a_{2n}, a_{2n+1})]^{\beta+\gamma}[\tfrac{1}{2s} d(a_{2n}, a_{2n+2})]^{1-\alpha-\beta-\gamma} \\ &\leq \lambda[d(a_{2n}, a_{2n+1})]^{\beta+\gamma}[\tfrac{1}{2}[d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2})]]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{2.2}$$

Suppose that $d(a_{2n}, a_{2n+1}) < d(a_{2n+1}, a_{2n+2})$.
Therefore, the inequality (2.2) produces that

$$[d(a_{2n+1}, a_{2n+2})]^{1-\alpha} \leq \lambda[d(a_{2n}, a_{2n+1})]^{\beta+\gamma}[d(a_{2n+1}, a_{2n+2})]^{1-\alpha-\beta-\gamma}$$

implies that

$$[d(a_{2n+1}, a_{2n+2})]^{\beta+\gamma} \leq \lambda[d(a_{2n}, a_{2n+1})]^{\beta+\gamma}$$

which implies that

$$d(a_{2n+1}, a_{2n+2}) \leq \lambda^{\frac{1}{\beta+\gamma}} d(a_{2n}, a_{2n+1}) < d(a_{2n}, a_{2n+1}),$$

which is a contradiction. Thus, we have $d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1})$.
From (2.2), we have

$$\begin{aligned} [d(a_{2n+1}, a_{2n+2})]^{1-\alpha} &\leq \lambda[d(a_{2n}, a_{2n+1})]^{\beta+\gamma}[d(a_{2n}, a_{2n+1})]^{1-\alpha-\beta-\gamma} \\ &= \lambda[d(a_{2n}, a_{2n+1})]^{1-\alpha} \end{aligned}$$

which implies that

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &\leq \lambda^{\frac{1}{1-\alpha}} d(a_{2n}, a_{2n+1}) \\ &= \kappa d(a_{2n}, a_{2n+1}) \\ &\vdots \\ &= \kappa^{2n+1} d(a_0, a_1). \end{aligned} \tag{2.3}$$

Therefore, $d(a_{2n+1}, a_{2n+2}) \leq \kappa^{2n+1} d(a_0, a_1)$. *Now, take* $a = a_{2n}$ *and* $b = a_{2n-1}$ *in (2.1), we get*

$$\begin{aligned} d(a_{2n+1}, a_{2n}) &= d(Ta_{2n}, Sa_{2n-1}) \\ &\leq \lambda[d(a_{2n}, a_{2n-1})^{\beta}][d(Ta_{2n}, a_{2n})]^{\gamma}[d(Sa_{2n-1}, a_{2n-1})]^{\alpha} \\ &\quad [\tfrac{1}{2s} d(Ta_{2n}, a_{2n-1}) + d(a_{2n}, Sa_{2n-1})]^{1-\alpha-\beta-\gamma} \\ &= \lambda[d(a_{2n}, a_{2n-1})^{\beta}][d(a_{2n+1}, a_{2n})]^{\gamma}[d(a_{2n}, a_{2n-1})]^{\alpha} \\ &\quad [\tfrac{1}{2s} d(a_{2n+1}, a_{2n-1}) + d(a_{2n}, a_{2n})]^{1-\alpha-\beta-\gamma} \end{aligned}$$

Then

$$\begin{aligned} [d(a_{2n+1}, a_{2n})]^{1-\gamma} &\leq \lambda[d(a_{2n}, a_{2n-1})]^{\beta+\alpha}[\tfrac{1}{2s} d(a_{2n+1}, a_{2n-1})]^{1-\alpha-\beta-\gamma} \\ &\leq \lambda[d(a_{2n}, a_{2n-1})]^{\beta+\alpha}[\tfrac{1}{2}(d(a_{2n-1}, a_{2n}) + d(a_{2n}, a_{2n+1}))]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{2.4}$$

Suppose that $d(a_{2n-1}, a_{2n}) < d(a_{2n}, a_{2n+1})$.
Thus, the inequality (2.4) produces that

$$[d(a_{2n+1}, a_{2n})]^{1-\gamma} \leq \lambda[d(a_{2n}, a_{2n-1})]^{\beta+\alpha}[d(a_{2n+1}, a_{2n})]^{1-\alpha-\beta-\gamma}$$

implies that

$$[d(a_{2n}, a_{2n+1})]^{\beta+\alpha} \leq \lambda[d(a_{2n-1}, a_{2n})]^{\beta+\alpha}$$

which implies that

$$d(a_{2n}, a_{2n+1}) \leq \lambda^{\frac{1}{\beta+\alpha}} d(a_{2n-1}, a_{2n}) < d(a_{2n-1}, a_{2n}),$$

which is a contradiction. Thus, we have $d(a_{2n}, a_{2n+1}) \leq d(a_{2n-1}, a_{2n})$.
From (2.4), we have

$$\begin{aligned} [d(a_{2n}, a_{2n+1})]^{1-\gamma} &\leq \lambda[d(a_{2n-1}, a_{2n})]^{\beta+\alpha}[d(a_{2n-1}, a_{2n})]^{1-\alpha-\beta-\gamma} \\ &= \lambda[d(a_{2n-1}, a_{2n})]^{1-\gamma} \end{aligned}$$

which implies that

$$\begin{aligned} d(a_{2n}, a_{2n+1}) &\leq \lambda^{\frac{1}{1-\gamma}} d(a_{2n-1}, a_{2n}) \\ &= \iota d(a_{2n-1}, a_{2n}) \\ &\vdots \\ &= \iota^{2n} d(a_0, a_1). \end{aligned} \tag{2.5}$$

Therefore, $d(a_{2n}, a_{2n+1}) \leq \iota^{2n} d(a_0, a_1)$.
It follows from (2.3) and (2.5), we deduce that

$$d(a_n, a_{n+1}) \leq k^n d(a_0, a_1) \text{ for all } n \in \mathbb{N}$$

where $k = \min\{\kappa, \iota\} < 1$.
For $m > 0$. *By employing the b-triangular inequality, we arrive at*

$$\begin{aligned} d(a_n, a_{n+m}) &\leq s d(a_n, a_{n+1}) + s^2 d(a_{n+1}, a_{n+2}) + \ldots + s^m d(a_{n+m-1}, a_{n+m}) \\ &\leq s k^n d(a_0, a_1) + s^2 k^{n+1} d(a_0, a_1) + \ldots + s^m k^{n+m-1} d(a_0, a_1) \\ &= s k^n [1 + sk + s^2 k^2 + \ldots + s^{m-1} k^{m-1}] d(a_0, a_1) \\ &\leq s k^n [1 + sk + (sk)^2 + \ldots + (sk)^{m-1} + \ldots] d(a_0, a_1) \\ &= \frac{s k^n}{1 - sk} d(a_0, a_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{a_n\}$ *is a b-Cauchy sequence in* (E, d, s) *and by completeness, there exists* $a^{'}$ *such that* $\lim_{n\to\infty} a_n = a^{'}$.
Hence, $a^{'} = \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} Ta_{2n}$, *and* $a^{'} = \lim_{n\to\infty} a_{2n+2} = \lim_{n\to\infty} Sa_{2n+1}$
so that $a^{'} = \lim_{n\to\infty} Ta_{2n} = \lim_{n\to\infty} Sa_{2n+1}$.
We assume that T *is b-continuous.*
Since $a_{2n} \to a^{'}$ *as* $n \to \infty$, *we have* $Ta_{2n} \to Ta^{'}$ *as* $n \to \infty$.
Now, $0 \leq d(a^{'}, Ta^{'}) \leq s[d(a^{'}, Ta_{2n}) + d(Ta_{2n}, Ta^{'})] \to 0$ *as* $n \to \infty$.
$a^{'}$ *is a fixed point of* T *as a result.*
According to Proposition 1, $a^{'}$ *is a unique common fixed point of* T *and* S.

An example supporting Theorem 4 is shown below.

Example 1. *Let* $E = [0, 1]$. *We define* $d : E \times E \to \mathbb{R}^+$ *by*

$$d(a, b) = \begin{cases} 0, & \text{if } a = b, \\ \frac{11}{15}, & \text{if } a, b \in [0, \frac{2}{3}], \\ \frac{23}{25} + \frac{a+b}{26}, & \text{if } a, b \in (\frac{2}{3}, 1], \\ \frac{121}{250}, & \text{otherwise.} \end{cases}$$

When (E, d, s) *has the coefficient* $s = \frac{51}{49}$, *it is evident that it is a complete b-metric space. Here we observe that when* $a = \frac{9}{10}, b = 1$ *and* $c \in (0, \frac{2}{3}]$, *we have*
$d(a, b) = \frac{23}{25} + \frac{a+b}{26} = \frac{1291}{1300} \neq \frac{121}{125} = \frac{121}{250} + \frac{121}{250} = d(a, c) + d(c, b)$ *so that* d *is not a metric.*
We specify $T, S : E \to E$ *by*

$$T(a) = \begin{cases} a, & \text{if } a \in [0, \frac{2}{3}), \\ \frac{4}{3} - a, & \text{if } a \in [\frac{2}{3}, 1] \end{cases} \quad \text{and} \quad S(a) = \begin{cases} \frac{a^2+3}{4}, & \text{if } a \in [0, \frac{2}{3}), \\ 1 - \frac{a}{2}, & \text{if } a \in [\frac{2}{3}, 1]. \end{cases}$$

Clearly, T is b-continuous.
We take $\lambda = \frac{99}{100}, \alpha = \beta = \gamma = \frac{1}{20}$. Then clearly $\alpha + \beta + \gamma < 1$.
Keeping generality intact, we suppose that $a \geq b$.

Case(i): $a, b \in [0, \frac{2}{3})$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{121}{250} &\leq \tfrac{99}{100} \left[\tfrac{11}{15}\right]^{\frac{1}{20}} \left[\tfrac{11}{15}\right]^{\frac{1}{20}} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{223685}{382500}\right]^{\frac{17}{20}} \\ &= \lambda \left[d(a,b)\right]^{\beta} \left[d(a,Ta)\right]^{\gamma} \left[d(b,Sb)\right]^{\alpha} \left[\tfrac{d(b,Ta)+d(a,Sb)}{2s}\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

Case(ii): $a, b \in (\frac{2}{3}, 1]$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{11}{15} &\leq \tfrac{99}{100} \left[\tfrac{23}{25} + \tfrac{a+b}{26}\right]^{\frac{1}{20}} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{223685}{382500}\right]^{\frac{17}{20}} \\ &= \lambda \left[d(a,b)\right]^{\beta} \left[d(a,Ta)\right]^{\gamma} \left[d(b,Sb)\right]^{\alpha} \left[\tfrac{d(b,Ta)+d(a,Sb)}{2s}\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

Case(iii): $a \in (\frac{2}{3}, 1], b \in [0, \frac{2}{3})$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{121}{250} &\leq \tfrac{99}{100} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{121}{250}\right]^{\frac{1}{20}} \left[\tfrac{3038}{3825}\right]^{\frac{17}{20}} \\ &= \lambda \left[d(a,b)\right]^{\beta} \left[d(a,Ta)\right]^{\gamma} \left[d(b,Sb)\right]^{\alpha} \left[\tfrac{d(b,Ta)+d(a,Sb)}{2s}\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

From all above cases we conclude that (T, S) is a pair of interpolative Hardy-Rogers contraction maps.
As a result, T and S satisfy every hypothesis of Theorem 4, and $\frac{2}{3}$ is the only joint fixed point.

Remark 1. *Theorem 4 and Example 1 extend and generalize Theorem 3 to b-metric spaces.*

Theorem 5. *Suppose that (E, d, s) be a complete b-metric space, and the maps T, S satisfy the following condition: there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Ta, Sb) \leq \lambda [d(Ta, a)]^{\alpha} [d(Sb, b)]^{1-\alpha}$$

$\forall\, a, b \in X$ such that $Ta \neq a$ whenever $Sa \neq a$. Both S and T have a unique common fixed point if either T or S is b-continuous.

Proof. Since the proof resembles Theorem 4, we left it out.

Example 2. *Let $E = \mathbb{R}^+$. We define $d : E \times E \to \mathbb{R}^+$ by*

$$d(a,b) = \begin{cases} 0, & \text{if } a = b, \\ 4, & \text{if } a, b \in [0,1], \\ 5 + \frac{1}{a+b}, & \text{if } a, b \in (1, \infty), \\ \frac{27}{10}, & \text{otherwise.} \end{cases}$$

When (E, d, s) has the coefficient $s = \frac{489}{480}$, it is evident that it is a complete b-metric space. Here we observe that when $a = \frac{11}{10}, b = \frac{12}{10}$ and $c \in (0, 1]$, we have $d(a, b) = 5 + \frac{1}{a+b} = \frac{125}{23} \neq \frac{27}{5} = \frac{27}{10} + \frac{27}{10} = d(a, c) + d(c, b)$ *so that d is not a metric. We define $T, S : E \to E$ by*

$$T(a) = \begin{cases} \log(1 + a), & \text{if } a \in [0, 1), \\ \frac{2}{a^2+1}, & \text{if } a \in [1, \infty) \end{cases} \quad \text{and} \quad S(a) = \begin{cases} \exp^{a}, & \text{if } a \in [0, 1), \\ \frac{1+a}{2}, & \text{if } a \in [1, \infty). \end{cases}$$

Clearly, T is b-continuous.
We take $\lambda = \frac{99}{100}, \alpha = \frac{4}{5}$. Assuming $a \geq b$, we maintain generality.

Case(i): $a, b \in [0, 1)$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{27}{10} &\leq \tfrac{99}{100} \, [4]^{\frac{4}{5}} \left[\tfrac{27}{10}\right]^{\frac{1}{5}} \\ &= \lambda \, [d(a, Ta)]^{\alpha} \, [d(b, Sb)]^{1-\alpha} \end{aligned}$$

Case(ii): $a, b \in (1, \infty)$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{27}{10} &\leq \tfrac{99}{100} \, [4]^{\frac{4}{5}} \left[\tfrac{27}{10}\right]^{\frac{1}{5}} \\ &= \lambda \, [d(a, Ta)]^{\alpha} \, [d(b, Sb)]^{1-\alpha} \end{aligned}$$

Case(iii): $a \in (1, \infty), b \in [0, 1)$.

$$\begin{aligned} d(Ta, Sb) = \tfrac{27}{10} &\leq \tfrac{99}{100} \, [4]^{\frac{4}{5}} \left[\tfrac{27}{10}\right]^{\frac{1}{5}} \\ &= \lambda \, [d(a, Ta)]^{\alpha} \, [d(b, Sb)]^{1-\alpha} \end{aligned}$$

From all above cases we conclude that (T, S) is an interpolative Kannan-type contraction maps.
Thus, 1 is the only common fixed point, and T and S satisfy every hypothesis of Theorem 5.

Corollary 1. *Suppose that (E, d, s) be a complete b-metric space, and the map T satisfies the following condition: there exist a constant $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha+\beta+\gamma < 1$, such that*

$$d(Ta, Tb) \leq \lambda [d(a, b)^{\beta}][d(Ta, a)]^{\gamma}[d(Tb, b)]^{\alpha} \left[\frac{d(Ta, b) + d(a, Tb)}{2s}\right]^{1-\alpha-\beta-\gamma}$$

for all $a, b \in X$ such that $Ta \neq a$. Then T has a unique fixed point in E.

Remark 2. *Corollary 1 extend and generalize Theorem 2 to b-metric spaces.*

Corollary 2. *Suppose that (E, d, s) be a complete b-metric space, and the map T satisfies the following condition: there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Ta, Tb) \leq \lambda [d(Ta, a)]^{\gamma} [d(Tb, b)]^{1-\alpha}$$

for all $a, b \in X$ such that $Ta \neq a$. Then T has a unique fixed point in E.

Remark 3. *Corollary 2 extend and generalize Theorem 1 to b-metric spaces.*

3. Nonlinear integral equations: An Approach

The primary objective of this section is to determine the solution to an integral problem. If $[a, b]$ is a closed and bounded integral in $\mathbb{R}$, then $\Omega = C[a, b]$ is a set of real valued continuous functions on $[a, b].d : \Omega \times \Omega \to \mathbb{R}^+$ is what we define. For every $\xi, \eta \in \Omega$, $d(\xi, \eta) = \max\limits_{t\in[a,b]} |\xi(t) - \eta(t)|^p$, where $p > 1$ is a real number. Therefore (Ω, d) is a complete b-metric space with $s = 2^{p-1}$. Several authors have studied the unique solution of a system of nonlinear integral equations [1–3, 17]. We demonstrate the existence of a single common solution for a system of two nonlinear integral equations of Fredholm type, which is defined by

$$\begin{cases} \xi(t) = f(t) + \mu \int\limits_a^b \mathcal{D}_1(t, r, \xi(r))dr, \\ \zeta(t) = f(t) + \mu \int\limits_a^b \mathcal{D}_2(t, r, \zeta(r))dr \end{cases} \tag{3.1}$$

where $\xi \in C[a, b], \mu \in \mathbb{R}, t, r \in [a, b], \mathcal{D}_1, \mathcal{D}_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous functions.

Consider two mappings $\mathcal{F}_1, \mathcal{F}_2 : \Omega \to \Omega$ that are specified by

$$\begin{cases} \mathcal{F}_1(\xi(t)) = f(t) + \mu \int\limits_a^b \mathcal{D}_1(t, r, \xi(r))dr, \\ \mathcal{F}_2(\xi(t)) = f(t) + \mu \int\limits_a^b \mathcal{D}_2(t, r, \xi(r))dr \end{cases} \tag{3.2}$$

Make the following assumptions:

(i) there exists a continuous function $\gamma : [a, b] \times [a, b] \to \mathbb{R}^+$, such that
$\max\limits_{r\in[a,b]} \int\limits_a^b \gamma(t, r)dr \leq 1$;

(ii) there exists a constant $K \in (0, 1)$ such that for all $t, r \in [a, b], \xi, \zeta \in \mathbb{R}$, and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, the following condition is satisfied:
$|\mathcal{D}_1(t, r, \xi_1(r)) - \mathcal{D}_2(t, r, \xi_2(r)|^p \leq \frac{K}{(b-a)^{p-1}2^{6p-6}}\gamma(t, r)\Delta(\xi_1, \xi_2)$,
where

$$\begin{aligned} \Delta(\xi_1, \xi_2) &= [|\xi_1(r) - \xi_2(r)|^p]^\beta \, [|\xi_1(r) - \mathcal{F}_1\xi_1(r)|^p]^\gamma \, [|\xi_1(r) - \mathcal{F}_1\xi_1(r)|^p]^\alpha \\ &\left[\frac{|\xi_1(r) - \mathcal{F}_2\xi_2(r)|^p + |\xi_2(r) - \mathcal{F}_1\xi_1(r)|^p}{2^p}\right]^{1-\alpha-\beta-\gamma} \end{aligned}$$

(iii) $|\mu| \leq 1$.

Theorem 6. *The requirements (i) − (iii) hold if* (3.2) *is used to define* $\mathcal{F}_1, \mathcal{F}_2 : \Omega \to \Omega$. *Next, there is a unique common solution in* Ω *for the system of nonlinear integral equations* (3.1).

Proof. *Let $\xi, \eta \in \Omega$ and let $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$ using Hölder's inequality and from the conditions $(i) - (iii)$, for all t, we have*

$$
\begin{aligned}
& d(\mathcal{F}_1\xi_1, \mathcal{F}_2\xi_2) = \max_{t\in[a,b]} |\mathcal{F}_1\xi_1(t) - \mathcal{F}_2\xi_2(t)|^p \\
& = |\mu|^p \max_{t\in[a,b]} \left| \int_a^b \mathcal{D}_1(t,r,\xi_1(r) - \int_a^b \mathcal{D}_2(t,r,\xi_2(r)dr \right|^p \\
& = |\mu|^p \max_{t\in[a,b]} \left| \int_a^b (\mathcal{D}_1(t,r,\xi_1(r) - \mathcal{D}_2(t,r,\xi_2(r))dr \right|^p \\
& \leq \left[|\mu|^p \max_{t\in[a,b]} \left(\int_a^b 1^p dr \right)^{\frac{1}{q}} \left(\int_a^b |(\mathcal{D}_1(t,r,\xi_1(r) - \mathcal{D}_2(t,r,\xi_2(r))|^p \, dr \right)^{\frac{1}{p}} \right]^p \\
& \leq (b-a)^{\frac{p}{q}} \max_{t\in[a,b]} \left(\int_a^b |(\mathcal{D}_1(t,r,\xi_1(r) - \mathcal{D}_2(t,r,\xi_2(r))|^p \, dr \right) \\
& = (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b |(\mathcal{D}_1(t,r,\xi_1(r) - \mathcal{D}_2(t,r,\xi_2(r))|^p \, dr \right) \\
& \leq (b-a)^{p-1} \max_{t\in[a,b]} \int_a^b \frac{K}{(b-a)^{p-1}2^{6p-6}} \gamma(t,r)\Delta(\xi_1,\xi_2)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
d(\mathcal{F}_1\xi_1, \mathcal{F}_2\xi_2) \ & \leq \tfrac{K}{s^6} \left[|\xi_1(r) - \xi_2(r)|^p\right]^\beta \left[|\xi_1(r) - \mathcal{F}_1\xi_1(r)|^p\right]^\gamma \left[|\xi_1(r) - \mathcal{F}_1\xi_1(r)|^p\right]^\alpha \\
& \quad \left[\tfrac{|\xi_1(r) - \mathcal{F}_2\xi_2(r)|^p + |\xi_2(r) - \mathcal{F}_1\xi_1(r)|^p}{2^p}\right]^{1-\alpha-\beta-\gamma} \\
& = \lambda\Delta(\xi_1,\xi_2)
\end{aligned}
$$

where $\lambda = \frac{K}{s^2} \in (0,1)$.

As a result, $\mathcal{F}_1, \mathcal{F}_2$ have a unique common solution of the system of nonlinear integral equations specified in (3.1), since all the requirements of Theorem 4 are satisfied.

4. Application to dynamic programming

The decision space is $\mathcal{D} \subseteq \mathcal{X}_1$, and the state space is $\mathcal{S} \subseteq \mathcal{X}_2$. $\mathcal{X}_1$ and $\mathcal{X}_2$ are assumed to be two Banach spaces in this section. All bounded real valued functions on $\mathcal{S}$ have a Banach space called $\Omega(\mathcal{S})$, whose b-metric is defined as follows: $d(\xi, \zeta) = \sup_{t\in\mathcal{S}} \mid \xi(t) - \zeta(t) \mid^p$, $\forall$ $\xi, \zeta \in \Omega(\mathcal{S})$ with coefficient $s = 2^{p-1}$ and the norm is defined as $\|\mathcal{F}\| = \sup\{\mid \mathcal{F}(t) \mid : t \in \mathcal{S}\}$, where $\mathcal{F} \in \Omega(\mathcal{S})$.

$\Omega(\mathcal{S}, d)$ is obviously a complete b-metric space. According to Bellman et al. [4], the functional equation in dynamic programming has the following basic form:
$f(\xi) = \underset{\zeta\in\tilde{\mathcal{D}}}{H}(\xi, \zeta, f(T(\xi, \zeta))), \xi \in \mathcal{S}$, where T indicates the process transformation, $f(\xi)$ indicates the optimal return function with the initial state ξ, and opt stands for sup or

inf. The state and decision vectors are denoted by ξ and ζ, respectively.
We look at the functional equation system

$$\begin{cases} f_1(\nu_s) = \underset{\nu_d \in \tilde{\mathcal{D}}}{\text{opt}} (\eta_1(\nu_s, \nu_d) + \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))))\forall \nu_s \in \mathcal{S}, \\ f_2(\nu_s) = \underset{\nu_d \in \tilde{\mathcal{D}}}{\text{opt}} (\eta_2(\nu_s, \nu_d) + \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))))\forall \nu_s \in \mathcal{S} \end{cases} \tag{4.1}$$

where the state vector is ν_s, the decision vector is ν_d, the process transformations are represented by ρ_1, ρ_2, and the optimal return functions with initial state ν_s are indicated by $f_1(\nu_s), f_2(\nu_s)$.
Let $\mathcal{F}_1, \mathcal{F}_2 : \Omega(\mathcal{S}) \to \Omega(\mathcal{S})$ be two mappings defined by;

$$\begin{cases} \mathcal{F}_1 f_1(\nu_s) = \underset{\nu_d \in \tilde{\mathcal{D}}}{\text{opt}} (\eta_1(\nu_s, \nu_d) + \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))))\forall \nu_s \in \mathcal{S}, \\ \mathcal{F}_2 f_2(\nu_s) = \underset{\nu_d \in \tilde{\mathcal{D}}}{\text{opt}} (\eta_2(\nu_s, \nu_d) + \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))))\forall \nu_s \in \mathcal{S} \end{cases} \tag{4.2}$$

Assume the following:

($\mathcal{D}_a$) for all $(\nu_s, \nu_d, f_1, f_2) \in \mathcal{S} \times \mathcal{D} \times \Omega(\mathcal{S}) \times \Omega(\mathcal{S}) \times \Omega(\mathcal{S}) \times \Omega(\mathcal{S})$ and there exist $0 < h < 1$ and $0 < \alpha < 1$, such that;

$$\begin{aligned} &\mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid \\ &\leq [\tfrac{h}{2^{4p-4}} M(f_1, f_2)]^{\frac{1}{p}} \end{aligned}$$

where

$$M(f_1, f_2) = [|f_1 - \mathcal{F}_1 f_1|^p]^\alpha \, [|f_2 - \mathcal{F}_2 f_2|^p]^{1-\alpha}$$

($\mathcal{D}_b$) ρ_i, ξ_i are bounded $i = 1, 2$.

Theorem 7. *Assume $\mathcal{F}_1, \mathcal{F}_2 : \Omega(\mathcal{S}) \to \Omega(\mathcal{S})$ be defined by (4.2) for which the conditions $\mathcal{D}_a$ and $\mathcal{D}_b$ are satisfied. Then, there is a unique bounded common solution in $\Omega(S)$ for the system of functional equations provided by (4.1).*

Proof. *Let $\nu_s \in \mathcal{S}, f_1, f_2 \in \Omega(\mathcal{S})$ and $\epsilon > 0$.*
As ρ_i, ξ_i are bounded for $i = 1, 2$ $\exists$ $L \geq 0 \ni$

$$\sup\{||\rho_1(\nu_s, \nu_d)||, ||\rho_2(\nu_s, \nu_d)||, ||\xi_2(\nu_s, \nu_d, t)|| : (\nu_s, \nu_d, t) \in \mathcal{S} \times \mathcal{D} \times \mathbb{R}\} \leq L. \tag{4.3}$$

From the inequalities (4.2) and (4.3), we conclude that $\mathcal{F}_1, \mathcal{F}_2$ are self mappings of $\Omega(\mathcal{S})$
First assume that $\underset{\nu_s \in \tilde{\mathcal{D}}}{opt} = \underset{\nu_d \in \mathcal{D}}{\inf}$.
The inequality (4.2) allows us to determine $\nu_d \in \mathcal{D}$ and $(\nu_s, f, g) \in \mathcal{S} \times \Omega(\mathcal{S}) \times \Omega(\mathcal{S})$ such that

$$\mathcal{F}_1 f_1(\nu_s) > \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) + \eta_1(\nu_s, \nu_d) - \epsilon \tag{4.4}$$

$$\mathcal{F}_1 f_2(\nu_s) > \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) + \eta_2(\nu_s, \nu_d) - \epsilon \tag{4.5}$$

$$\mathcal{F}_1 f_1(\nu_s) \leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) + \eta_1(\nu_s, \nu_d) \tag{4.6}$$

$$\mathcal{F}_1 f_2(\nu_s) \leq \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) + \eta_2(\nu_s, \nu_d) \tag{4.7}$$

By using the inequalities (4.4) and (4.7), we get that

$$\begin{cases} \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) > \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) - \epsilon \\ \qquad \geq -\{\mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ \qquad + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid + \epsilon\} \end{cases} \tag{4.8}$$

Also, from (4.5) and (4.6), we have

$$\begin{cases} \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon \\ \qquad \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ \qquad + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid + \epsilon \end{cases} \tag{4.9}$$

By using (4.8) and (4.9), we get that

$$\begin{aligned} \mid \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \mid &< \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ &\quad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon \\ &\leq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ &\quad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon \end{aligned}$$

Now, we support that $\underset{\nu_d \in \tilde{\mathcal{D}}}{opt} = \sup_{\nu_d \in \mathcal{D}}$.
From (4.2), we can determine $\nu_d \in \mathcal{D}$ *and* $(\nu_s, f, g) \in \mathcal{S} \times \Omega(\mathcal{S}) \times \Omega(\mathcal{S})$ *such that*

$$\mathcal{F}_1 f_1(\nu_s) < \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) + \eta_1(\nu_s, \nu_d) + \epsilon \tag{4.10}$$

$$\mathcal{F}_1 f_2(\nu_s) < \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) + \eta_2(\nu_s, \nu_d) + \epsilon \tag{4.11}$$

$$\mathcal{F}_1 f_1(\nu_s) < \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) + \eta_1(\nu_s, \nu_d) \tag{4.12}$$

$$\mathcal{F}_1 f_2(\nu_s) < \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) + \eta_2(\nu_s, \nu_d) \tag{4.13}$$

Using the inequalities (4.10) and (4.13), we have

$$\begin{cases} \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) < \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon \\ \qquad \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid + \epsilon \end{cases} \tag{4.14}$$

Also, from the inequalities (4.11) and (4.12), we get that

$$\begin{cases} \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \geq \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) - \epsilon \\ \qquad \geq -\{\mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon\} \end{cases} \tag{4.15}$$

From (4.14) and (4.15), we have

$$\begin{cases} \mid \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \mid < \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \\ \qquad + \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) - \epsilon \\ \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) + \epsilon \end{cases} \tag{4.16}$$

On taking $\epsilon \to 0$ *in (4.16), we obtain that*

$$\begin{aligned} \mid \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \mid \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid \end{aligned}$$

From the condition $(\mathcal{D}_b)$, *we have*

$$\begin{aligned} &\mid \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \mid \leq \mid \xi_1(\nu_s, \nu_d, f_1(\rho_1(\nu_s, \nu_d))) - \xi_2(\nu_s, \nu_d, f_2(\rho_2(\nu_s, \nu_d))) \mid \\ &\qquad\qquad + \mid \eta_1(\nu_s, \nu_d) - \eta_2(\nu_s, \nu_d) \mid \\ &\leq [\tfrac{h}{2^{4p-4}} M(f_1, f_2)]^{\frac{1}{p}} \\ &\leq \left[\sup_{\nu_s \in \mathcal{S}} (\tfrac{h}{2^{4p-4}} \left[|f_1 - \mathcal{F}_1 f_1|^p\right]^\alpha \left[|f_2 - \mathcal{F}_2 f_2|^p\right]^{1-\alpha} \right]^{\frac{1}{p}} \end{aligned}$$

which implies that

$$\sup_{\nu_s \in \mathcal{S}} \mid \mathcal{F}_1 f_1(\nu_s) - \mathcal{F}_1 f_2(\nu_s) \mid^p \quad \leq \tfrac{h}{2^{4p-4}} \sup_{\nu_s \in \mathcal{S}} \left[|f_1 - \mathcal{F}_1 f_1|^p\right]^\alpha \left[|f_2 - \mathcal{F}_2 f_2|^p\right]^{1-\alpha}.$$

Now, for all $f_1, f_2 \in \Omega(\mathcal{S})$, *we have*

$$d(\mathcal{F}_1 f_1, \mathcal{F}_1 f_2) \leq \tfrac{h}{2^{4p-4}} \left[d(f_1, \mathcal{F}_1 f_1)\right]^\alpha \left[d(f_2, \mathcal{F}_2 f_2)\right]^{1-\alpha}.$$

Consequently, $\mathcal{F}_1, \mathcal{F}_2$ *have a unique bounded common solution to the system of functional equations (4.1) since all the requirements of Theorem 5 are met.*

5. Conclusion and future work

In this paper, we studied fixed point results for interpolative contraction mappings in b-metric spaces. Using similar approaches, it can be studied new fixed point results on metric and some generalized metric spaces. The investigation of certain circumstances to exclude the identity map of E from Theorem 4 and Theorem 5 and related results is a worthwhile problem for future effort.

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