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Asymptotic Solution of a Singularly Perturbed Integro-Differential Fractional Order Derivative Equation with Rapidly Oscillating In-Homogeneity

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Abstract. The main objective of the present article is to identify the influence of an exponentially oscillating heterogeneity and an integral operator on the structure of the asymptotic of the solution of the initial value problem for a linear singularly perturbed integro-differential equation with a fractional derivative and a rapidly oscillating heterogeneity. To construct an asymptotic solution to the problem, the algorithm of the regularization method used. The case of absence of resonance is considered, i.e. the case when the frequency of exponentially oscillating heterogeneity does not coincide with the spectrum of the limit operator of the differential part of the equation in the considered time interval. It is shown that both the rapidly oscillating heterogeneity and the kernel of the integral operator have a significant effect on the leading term of the asymptotic of the solution of the original problem.

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1. Introduction

An initial problem is considered for a singularly perturbed integro-differential equation:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon z^{(\alpha)} - A(t)z - \int_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}},$$
$$z(t_0,\varepsilon) = z^0, \quad t \in [t_0, T], \quad t_0 > 0, \tag{1.1}$$

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for a scalar unknown function $z(t,\varepsilon)$, in which $A(t),\ h_1(t),\ h_2(t),\ \beta'(t)>0,\ (\forall t\in[t_0,T])$ are known functions, $0<\alpha<1$, z^0 constant number, $\varepsilon>0$ is a small parameter. The problem is posed of constructing a regularized [1, 2] asymptotic solution to problem (1.1).

Lomov's regularization method [1, 2] was developed to construct regularized asymptotic solutions of ordinary differential equations in the case of stability of the spectrum of the limit operator. Problems devoted to the construction of regularized asymptotic solutions of Cauchy problems in the presence of weak turning points of the limit operator are considered in the works of [3–5], initialization in the work of [6]. The works of [7] considered the problems of constructing a regularized asymptotic solution to a nonlinear differential equation in a Banach space and the analytical aspects of the theory of Tikhonov systems [8]. Singularly perturbed ordinary differential equations with rapidly oscillating coefficients from the perspective of the regularization method were carried out in the work of [9]. The justification of the regularization method for linear and nonlinear integro-differential equations with a zero operator of the differential part was studied in the works of [10, 11].

Singularly perturbed integro-differential equations with rapidly oscillating coefficients and rapidly changing kernels in the case of a multiple spectrum were considered in the studies of [12–14], with rapidly oscillating coefficients and with rapidly oscillating in-homogeneities in the works of [15–21]. The Fredholm integro-differential equation with a rapidly decreasing kernel and an exponentially oscillating in-homogeneity was studied in the work of [22]. The integro-differential Cauchy problem with exponential in-homogeneity and with a spectral value that vanishes at an isolated point on a segment of an independent variable is considered in the work of [23]. The problem belongs to the class of singularly perturbed equations with an unstable spectrum and has not been considered previously in the presence of an integral operator. It is especially difficult to study it in the vicinity of zero spectral value of the in-homogeneity. In this case, it is not possible to apply the well-known procedure of the Lomov's regularization method, so the researchers chose a method for constructing the asymptotic of the solution to the original problem, based on the use of the regularized asymptotic of the fundamental solution of the corresponding homogeneous equation, the construction of which from the standpoint of the regularization method has not been considered until now.

It should be noted that singularly perturbed differential and integro-differential equations with fractional derivatives in the absence and presence of rapidly oscillating components were considered in works [24–28]. In these works, the ideas of the regularization method were generalized for equations with fractional derivatives, regularized asymptotic solutions of problems were constructed, and the influence of rapidly oscillating coefficients on the leading term of the asymptotic was studied. It should be noted that problems associated with fractional differential equations and generalized Hilfer fractional derivatives, which combine the Riemann-Liouville and Caputo fractional derivatives, are considered in [29–33].

Thus, in this work, S.A.Lomov's regularization method [1] is generalized to a singularly perturbed integro-differential equation with fractional derivatives with an exponentially oscillating right-hand side. The main goal of the study is to identify the influence of oscillating components on the structure of the asymptotic of the solution to the original problem (1.1).

By definition of the fractional derivative [34], the fractional derivative $z^{(\alpha)}$ in terms of integer derivatives is denoted in the following form $t^{(1-\alpha)}\frac{dz}{dt}$. Accordingly, we rewrite the original fractional order equation (1.1) in the following form:

$$L_\varepsilon z(t,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{dz}{dt} - A(t)z - \int\limits_{t_0}^{t} K(t,s)z(s,\varepsilon)ds = h_1(t) + h_2(t)e^{\frac{i\beta(t)}{\varepsilon}},$$
$$z(t_0,\varepsilon) = y^0, \quad t \in [t_0,T], t_0 > 0. \tag{1.2}$$

In problem (1.2), the frequency of the rapidly oscillating in-homogeneity is $\beta'(t)$. In what follows, the function $\lambda_1(t) = A(t)$ is called the spectrum of problem (2), and function $\lambda_2(t) = -i\beta'(t)$ is the frequency of a rapidly oscillating in-homogeneity.

Problem (1.2) will be considered under the following conditions:

(i) $a(t), \beta(t), h_1(t), h_2(t) \in C[t_0,T],\ K(t,s) \in C^\infty(t_0 \le s \le t \le T)$;

(ii) $A(t) < 0\ \ \forall t \in [t_0,T]$.

We will develop an algorithm for constructing a regularized asymptotic solution [1, 2] of problem (1.2).

2. Regularization of the problem (1.2)

Denote by $\sigma = \sigma(\varepsilon)$ independent of magnitude $\sigma = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, and introduce the regularized variables:

$$\tau_1 = \frac{1}{\varepsilon}\int\limits_{t_0}^{t} \theta^{(\alpha-1)}\lambda_1(\theta)d\theta \equiv \frac{\psi_1(t)}{\varepsilon}, \qquad \tau_2 = \frac{1}{\varepsilon}\int\limits_{t_0}^{t} \lambda_2(\theta)d\theta \equiv \frac{\psi_2(t)}{\varepsilon} \tag{2.1}$$

and instead of problem (1.2), consider the problem

$$L_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)}\frac{\partial \tilde{z}}{\partial t} + \lambda_1(t)\frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)}\lambda_2(t)\frac{\partial \tilde{z}}{\partial \tau_2} -$$
$$-\lambda_1(t)\tilde{z} - \int\limits_{t_0}^{t} K(t,s)\tilde{z}\left(s, \frac{\psi(s)}{\varepsilon}, \sigma, \varepsilon\right)ds = \tag{2.2}$$
$$= h_1(t) + h_2(t)e^{\tau_2}\sigma, \tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=t_0,\tau=0} = z^0, t \in [t_0,T],$$

for the function $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$, where is indicated (according (2.1)): $\tau = (\tau_1,\tau_2),\ \psi = (\psi_1,\psi_2)$. It is clear that if $\tilde{z} = \tilde{z}(t,\tau,\sigma,\varepsilon)$ – is a solution of the problem (2.2), then the function is $\tilde{z} = \tilde{z}\left(t, \frac{\psi(t)}{\varepsilon}, \sigma, \varepsilon\right)$ an exact solution to problem (1.2), therefore, problem

(2.2) is extended with respect to problem (1.2). However, it cannot be considered fully regularized, since it does not regularize the integral

$$J\tilde{z} \equiv J\left(\tilde{z}(t,\tau,\sigma,\varepsilon)|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int_{t_0}^{t} K(t,s)\tilde{z}\left(s,\frac{\psi(s)}{\varepsilon},\sigma,\varepsilon\right)ds.$$

For its regularization, we introduce the class M_ε asymptotically invariant with respect to the operator $J\tilde{z}$ (see [1], p. 62]). Consider first the space U of vector functions $z(t,\tau,\sigma)$, representable by the sums

$$z(t,\tau,\sigma) = z_0(t,\sigma) + \sum_{i=1}^{2} z_i(t,\sigma)e^{\tau_i}, \quad z_i(t,\sigma) \in C^\infty\left([t_0,T],\mathbf{C}\right), i = \overline{0,2}. \tag{2.3}$$

In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constant $\sigma = \sigma(\varepsilon)$ and which do not affect the development of the algorithm described below, therefore, in the record of element (2.3) of this space U, we omit the dependence on $\sigma = \sigma(\varepsilon)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J.

For the space U we take the space of functions $z\left(t,\tau,\sigma\right)$, represented by sums

$$J\tilde{z}(t,\tau,\varepsilon) \equiv \int_{t_0}^{t} K(t,s)z_0(s)ds + \int_{t_0}^{t} K(t,s)z_1(s)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds +$$

$$+\int_{t_0}^{t} K(t,s)z_2(s)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_2(\theta)d\theta}ds.$$

Integrating by parts:

$$J_1(t,\varepsilon) = \varepsilon\int_{t_0}^{t} \frac{K(t,s)z_1(s)}{s^{(\alpha-1)}\lambda_1(s)}de^{\frac{1}{\varepsilon}\int_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta} =$$

$$= \varepsilon\left[\frac{K(t,s)z_1(s)}{s^{(\alpha-1)}\lambda_1(s)}e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}\Bigg|_{s=t_0}^{s=t} -\right.$$

$$\left.-\int_{t_0}^{t}\frac{\partial}{\partial s}\left(\frac{K(t,s)z_1(s)}{s^{(\alpha-1)}\lambda_1(s)}\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds\right] =$$

$$= \varepsilon\left[\frac{K(t,t)z_1(t)}{t^{(\alpha-1)}\lambda_1(t)}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta} - \frac{K(t,t_0)z_1(t_0)}{{t_0}^{(\alpha-1)}\lambda_1(t_0)}\right] -$$

$$-\varepsilon\int\limits_{t_0}^{t}\frac{\partial}{\partial s}\left(\frac{K(t,s)z_1(s)}{s^{(\alpha-1)}\lambda_1(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}ds.$$

Continuing this process further, we will have

$$J_1(t,\varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_1^{\nu}\left(K(t,s)z_1(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\theta^{(\alpha-1)}\lambda_1(\theta)d\theta}-\left(I_1^{\nu}\left(K(t,s)z_1(s)\right)\right)_{s=t_0}\right],$$

where

$$I_1^0=\frac{1}{s^{(\alpha-1)}\lambda_1(s)}\cdot,\quad I_1^{\nu}=\frac{1}{s^{(\alpha-1)}\lambda_1(s)}\frac{\partial}{\partial s}I_1^{\nu-1},\nu\geq 1.$$

$$J_2(t,\varepsilon)=\varepsilon\int\limits_{t_0}^{t}\frac{K(t,s)z_2(s)}{\lambda_2(s)}de^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_2(\theta)d\theta}=$$

$$=\varepsilon\left[\left.\frac{K(t,s)z_2(s)}{\lambda_2(s)}e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_2(\theta)d\theta}\right|_{s=t_0}^{s=t}-\int\limits_{t_0}^{t}\frac{\partial}{\partial s}\left(\frac{K(t,s)z_2(s)}{\lambda_2(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_2(\theta)d\theta}ds\right]-$$

$$=\varepsilon\left[\frac{K(t,t)z_2(t)}{\lambda_2(t)}e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{t}\lambda_2(\theta)d\theta}-\frac{K(t,t_0)z_2(t_0)}{\lambda_2(t_0)}\right]-$$

$$-\varepsilon\int\limits_{t_0}^{t}\frac{\partial}{\partial s}\left(\frac{K(t,s)z_2(s)}{\lambda_2(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{s}\lambda_2(\theta)d\theta}ds,$$

Continuing this process further, we will have

$$J_2(t,\varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\left[\left(I_2^{\nu}\left(K(t,s)z_2(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_{t_0}^{t}\lambda_2(\theta)d\theta}-\left(I_2^{\nu}\left(K(t,s)z_2(s)\right)\right)_{s=t_0}\right],$$

where

$$I_2^0=\frac{1}{\lambda_2(s)}\cdot,\quad I_2^{\nu}=\frac{1}{\lambda_2(s)}\frac{\partial}{\partial s}I_2^{\nu-1},\nu\geq 1.$$

Hence, the image of the operator J on an element (2.3) of the space U can be represented as a series

$$J\tilde{z}(t,\tau,\varepsilon)=\int\limits_{t_0}^{t}K(t,s)z_0(s)ds+$$

$$+\sum_{i=1}^{2}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_i^{\nu}\left(K(t,s)z_i(s)\right)\right)_{s=t}e^{\tau_i}-\left(I_i^{\nu}\left(K(t,s)z_i(s(s)\right)\right)_{s=t_0}\right].\qquad(2.4)$$

Moreover, it is easy to show (see [35], pp. 291-294) that the series on the right in (2.4) converge for $\varepsilon \to +0$ uniformly in $t \in [0, T]$ to the corresponding integrals on the left. Let us introduce operators of order $R_m : U \to U$:

$$R_0 z(t,\tau) = \int_{t_0}^{t} K(t,s) z_0(s) ds, \qquad (2.5_0)$$

$$R_1 z(t,\tau) = \sum_{i=1}^{2} \Big[\left(I_i^0 \left(K(t,s) z_i(s)\right)\right)_{s=t} e^{\tau_i} - \left(I_i^0 \left(K(t,s) z_i(s)\right)\right)_{s=t_0} \Big], \qquad (2.5_1)$$

$$R_{\nu+1} z\,(t,\tau) = \sum_{i=1}^{2} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[\left(I_i^\nu \left(K(t,s) z_i(s)\right)\right)_{s=t} e^{\tau_i} - \right.$$
$$\left. - \left(I_i^\nu \left(K(t,s) z_i(s)\right)\right)_{s=t_0} \right], \nu \geq 1. \qquad (2.5_{\nu+1})$$

Now let $\tilde{z}(t, \tau, \varepsilon)$ be an arbitrary continuous function on $(t, \tau) \in G = [t_0, T] \times \{\tau : Re\tau_1 < 0,\ Re\tau_2 \leq 0\}$, with asymptotic expansion

$$\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z_k(t,\tau), \quad z_k(t,\tau) \in U \qquad (2.6)$$

converging as $\varepsilon \to +0$ (uniformly in $(t, \tau) \in G$). Then the image $J\tilde{z}\,(t, \tau, \varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k J z_k(t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} z_s(t,\tau)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (2.6):

$$\tilde{J}\tilde{z} \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k z_k(t,\tau)\right) = \sum_{r=0}^{\infty} \varepsilon^r \left(\sum_{k=0}^{r} R_{r-k} z_k(t,\tau)\right). \qquad (2.7)$$

Although the operator $\tilde{J}$ is formally defined, its utility is obvious, since in practice it is usual to construct the N-th approximation of the asymptotic solution of the problem (2.1), in which impose only N-th partial sums of the series (2.6), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2.1):

$$L_\varepsilon \tilde{z}(t,\tau,\sigma,\varepsilon) \equiv \varepsilon t^{(1-\alpha)} \frac{\partial \tilde{z}}{\partial t} + \lambda_1(t) \frac{\partial \tilde{z}}{\partial \tau_1} + t^{(1-\alpha)} \lambda_2(t) \frac{\partial \tilde{z}}{\partial \tau_2} - \lambda_1(t)\tilde{z} - \tilde{J}\tilde{z} =$$
$$= h_1(t) + h_2(t) e^{\tau_2}\sigma,\ \tilde{z}(t_0, 0, \sigma, \varepsilon) = z^0,\ t \in [t_0, T], \qquad (2.8)$$

where the operator $\tilde{J}$ has the form (2.7).

3. Iterative problems and their solvability in the space U

Substituting the series (2.6) into (2.8) and equating the coefficients of the same powers of ε, we obtain the following iterative problems:

$$Lz_0(t,\tau,\sigma) \equiv \lambda_1(t)\tfrac{\partial z_0}{\partial \tau_1} + t^{(1-\alpha)}\lambda_2(t)\tfrac{\partial z_0}{\partial \tau_2} - \lambda_1(t)z_0 - R_0 z_0 = \\ = h_1(t) + h_2(t)e^{\tau_2}\sigma, \ \ z_0(t_0,0) = z^0; \tag{3.1_0}$$

$$Lz_1(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_0}{\partial t} + R_1 z_0, \quad z_1(t_0,0) = 0; \tag{3.1_1}$$

$$Lz_2(t,\tau,\sigma) = -t^{(1-\alpha)}\frac{\partial z_1}{\partial t} + R_1 z_1 + R_2 z_0, \quad z_2(t_0,0) = 0; \tag{3.1_2}$$

....

$$L\,z_k(t,\tau,\sigma) = -t^{(1-\alpha)}\tfrac{\partial z_{k-1}}{\partial t} + R_k z_0 + \ldots + \\ +\ldots + R_1 z_{k-1}, \ \ z_k(t_0,0) = 0, \quad k \geq 1. \tag{3.1_k}$$

Each iterative problem (3.1_k) has the form

$$L\,z(t,\tau,\sigma) \equiv \lambda_1(t)\tfrac{\partial z}{\partial \tau_1} + t^{(1-\alpha)}\lambda_2(t)\tfrac{\partial z}{\partial \tau_2} - \lambda_1(t)z - R_0 z = \\ = H(t,\tau,\sigma), \ \ z(t_0,0) = z^*, \tag{3.2}$$

where $H(t,\tau,\sigma) = H_0(t,\sigma) + \sum\limits_{i=1}^{3} H_i(t,\sigma)e^{\tau_i}$ is the known function of space U, y_* is the known function of the complex space $\mathbf{C}$, and the operator R_0 has the form (see (2.5_0))

$$R_0 z \equiv R_0\left(z_0(t) + \sum_{j=1}^{2} z_j(t)e^{\tau_j} +\right) \triangleq \int\limits_{t_0}^{t} K(t,s)z_0(s)ds.$$

We introduce scalar (for each $t \in [t_0,T]$) product in space U:

$$\langle u,w\rangle \equiv \left\langle u_0(t) + \sum_{j=1}^{2} u_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{2} w_j(t)e^{\tau_j}\right\rangle \equiv \sum_{j=0}^{2} \left(u_j(t), w_j(t)\right)$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbf{C}:\ (u,v) = u\cdot\bar{v}$. Let us prove the following statement.

Theorem 1. *Let conditions (i), (ii) be fulfilled and the right-hand side* $H(t,\tau,\sigma) = H_0(t,\sigma) + \sum\limits_{j=1}^{2} H_j(t,\sigma)e^{\tau_j}$ *of equation (3.2) belongs to the space* U. *Then the equation (3.2) is solvable in* U, *if and only if*

$$\langle H(t,\tau), e^{\tau_1}\rangle \equiv 0 \ \ \forall t \in [t_0,T]. \tag{3.3}$$

Proof. We will determine the solution of equation (3.2) as an element (2.4) of the space U :

$$z(t,\tau,\sigma)=z_0(t,\sigma)+\sum_{j=1}^{2}z_j(t,\sigma)e^{\tau_j}. \tag{3.4}$$

Substituting (3.4) into equation (3.2), and equating here the free terms and coefficients separately for identical exponents, we obtain the following equations of equations:

$$\lambda_1(t)z_0(t,\sigma)-\int\limits_{t_0}^{t}K(t,s)z_0(s,\sigma)ds=H_0(t,\sigma), \tag{3.5}$$

$$0\cdot z_1(t,\sigma)=H_1(t,\sigma), \tag{3.5_1}$$

$$\left[t^{(1-\alpha)}\lambda_2(t)-\lambda_1(t)\right]z_2(t,\sigma)=H_2(t,\sigma). \tag{3.5_2}$$

Since the $\lambda_1(t)\neq 0$, the equation (3.5) can be written as

$$z_0(t,\sigma)=\int\limits_{t_0}^{t}\left(-\lambda_1^{-1}(t)K(t,s)\right)z_0(s,\sigma)ds-\lambda_1^{-1}(t)H_0(t,\sigma). \tag{3.5_0}$$

Due to the smoothness of the kernel $\left(-\lambda_1^{-1}(t)K(t,s)\right)$ and heterogeneity $-\lambda_1^{-1}(t)H_0(t,\sigma)$, this Volterra integral equation has a unique solution $z_0(t,\sigma)\in C^{\infty}\left([t_0,T],\mathbf{C}\right)$. The equation (3.5_2) also have unique solutions

$$z_2(t,\sigma)=\left[t^{(1-\alpha)}\lambda_2(t)-\lambda_1(t)\right]^{-1}H_2(t,\sigma)\in C^{\infty}\left([t_0,T],\mathbf{C}\right), \tag{3.6}$$

since $\lambda_2(t)$ not equal to $\lambda_1(t)$.

The equation (3.5_1) is solvable in space $C^{\infty}\left([t_0,T],\mathbf{C}\right)$ if and only if $(H_1(t,\tau),e^{\tau_1})\equiv$ $\equiv 0 \ \ \forall t\in[t_0,T]$ hold. It is not difficult to see that these identities coincide with identities (3.3). Thus, condition (3.3) is necessary and sufficient for the solvability of equations (3.2) in the space U. The Theorem 1 is proved.

Remark 1. If identity (3.3) holds, then under conditions (i), (ii), equation (3.2) has the following solution in the space U :

$$z(t,\tau,\sigma)=z_0(t,\sigma)+\alpha_1(t,\sigma)e^{\tau_1}+\left[t^{(1-\alpha)}\lambda_2(t)-\lambda_1(t)\right]^{-1}H_2(t,\sigma)e^{\tau_2}, \tag{3.7}$$

where $\alpha_1(t,\sigma)\in C^{\infty}\left([t_0,T],\mathbf{C}\right)$ is arbitrary function, $z_0(t,\sigma)$ is the solution of an integral equation (3.5_0).

4. The unique solvability of the general iterative problem in the space U. Residual term theorem

As can be seen from (3.7), the solution to equation (3.2) is determined ambiguously. However, if it is subject to additional conditions:

$$\begin{aligned} & z(t_0,0)=z_*, \\ & \left\langle -t^{(1-\alpha)}\tfrac{\partial z}{\partial t}+R_1 z+Q(t,\tau,\sigma),e^{\tau_1}\right\rangle \equiv 0 \quad \forall t\in[t_0,T], \end{aligned} \tag{4.1}$$

where $Q(t,\tau,\sigma)=Q_0(t,\sigma)+\sum\limits_{j=1}^{2}Q_j(t,\sigma)e^{\tau_j}$ is the known function of the space U, z_* is a constant vector of the complex space $\mathbf{C}$, then problem (3.2) will be uniquely solvable in the space U. More precisely, the following result holds.

Theorem 2. *Let conditions (i), (ii) be satisfied, the right-hand side $H(t,\tau,\sigma)$ of the equation (3.2) belongs to the space U and satisfies the orthogonality condition (3.3). Then equation (3.2) under additional conditions (4.1) is uniquely solvable in U.*

Proof. Under condition (3.3), the equation (3.2) has a solution (3.7) in the space U, where the function $\alpha_1(t,\sigma)\in C^\infty([t_0,T],\mathbf{C})$, are still arbitrary. Subordinate (3.7) to the first condition (4.1), i.e. $z(t_0,0)=z_*$. We obtain

$$\alpha_1(t_0,\sigma)=z^*, \tag{4.2}$$

where is denoted:

$$z^*=z_*+\lambda_1^{-1}(t_0)H_0(t_0,\sigma)-\alpha_1(t_0,\sigma)-\left[{t_0}^{(1-\alpha)}\lambda_2(t_0)-\lambda_1(t_0)\right]^{-1}H_2(t_0,\sigma).$$

Let us now subject solution (3.7) to the second condition (4.1). The right side of this equation:

$$\begin{aligned} -t^{(1-\alpha)}\frac{\partial z_0}{\partial t}+R_1 z_0+Q\left(t,\tau,\sigma\right) = & -\frac{d}{dt}\left(t^{(1-\alpha)}z_0(t,\sigma)\right)-\frac{d}{dt}\left(t^{(1-\alpha)}\alpha_1(t,\sigma\right)e^{\tau_1}- \\ & -\frac{d}{dt}\left(t^{(1-\alpha)}z_0(t,\sigma)\left[t^{(1-\alpha)}\lambda_2(t)-\lambda_1(t)\right]^{-1}H_2(t,\sigma)\right)e^{\tau_2}+ \\ & +\left[\frac{K(t,t)\alpha_1(t,\sigma)}{t^{(1-\alpha)}\lambda_1(t)}e^{\tau_1}-\frac{K(t,t_0)\alpha_1(t_0,\sigma)}{{t_0}^{(1-\alpha)}\lambda_1(t_0)}\right]+ \\ & +\frac{K(t,t_0)z_2(t)}{\lambda_2(t)}e^{\tau_2}-\frac{K(t,t)z_2(t_0,\sigma)}{\lambda_2(t_0)}+Q(t,\tau,\sigma). \end{aligned} \tag{4.3}$$

Now multiplying (4.3) scalarly by e^{τ_1}, we obtain the system of ordinary differential equations

$$-t^{(1-\alpha)}\frac{d\alpha_1(t,\sigma)}{dt}+\left[\frac{K(t,t)}{t^{(1-\alpha)}\lambda_1(t)}-(1-\alpha)t^{(-\alpha)}+\right]\alpha_1(t,\sigma)=0 \;\;\Leftrightarrow$$

$$\Leftrightarrow \quad t^{(1-\alpha)}\frac{d\alpha_1(t,\sigma)}{dt} = \left[\frac{K(t,t)}{t^{(1-\alpha)}\lambda_1(t)} - (1-\alpha)t^{(-\alpha)}\right]\alpha_1(t,\sigma).$$

Adding the initial condition (4.2) to it, we can uniquely find the functions $\alpha_1(t,\sigma)$, and, hence, construct a solution (3.7) of the problem (3.2) in the space U in a unique way. The Theorem 2 is proved.

Applying Theorems 1 and 2 to iterative problems (3.1_k), we find uniquely their solutions in the space U and construct series (2.6). Just as in [1, 35, 36] we prove the following statement.

Theorem 3. *Let conditions (i), (ii) be satisfied for the equation (1.2). Then, for* $\varepsilon \in (0, \varepsilon_0](\varepsilon_0 > 0$ *is sufficiently small), the equation (1.2) has a unique solution* $y(t,\varepsilon) \in \in C^1([t_0,T],\mathbf{C})$; *in this case, the estimate*

$$||z(t,\varepsilon) - z_{\varepsilon N}(t)||_{C[0,T]} \leq c_N\varepsilon^{N+1}, N = 0,1,2,\ldots,$$

holds. Here $y_{\varepsilon N}(t)$ *is the restriction (at* $\tau = \frac{\psi(t)}{\varepsilon}$*) of the N-th partial sum of the series (2.6) (with coefficients* $z_k(t,\tau) \in U$*, satisfying the iterative problems* (3.1_k)*), and the constant* $c_N > 0$ *does not depend on* ε *at* $\varepsilon \in (0, \varepsilon_0]$.

5. Constructing a solution to the first iterative problem

Using Theorem 1, we will try to find a solution to the first iteration problem (3.1_k). Since the right-hand side $h_1(t) + h_2(t)e^{\tau_{n+2}}\sigma$ of the equation (3.1_0), satisfy condition (3.3), this system has (according to (3.7)) a solution in the space U in the form

$$z_0(t,\tau,\sigma) = z_0^{(0)}(t,\sigma) + \alpha_1^{(0)}(t,\sigma)e^{\tau_1} + z_2^{(0)}(t,\sigma)e^{\tau_2}\sigma, \tag{5.1}$$

where $\alpha_1^{(0)}(t,\sigma) \in C^\infty([t_0,T],\mathbf{C})$ is arbitrary function, $z_2^{(0)}(t,\sigma) = \left[t^{(1-\alpha)}\lambda_2(t) - \lambda_1(t)\right]^{-1}h_2(t)$, $z_0^{(0)}(t,\sigma)$ is the solution of the integral equation

$$z_0^{(0)}(t,\sigma) = \int\limits_{t_0}^{t}\left(-\lambda_1^{-1}(t)K(t,s)\right)z_0(s,\sigma)ds - \lambda_1^{-1}(t)h_1(t).$$

Submitting (5.1) to the initial condition $z_0(t_0,0) = z^0$, we will have

$$z_0^{(0)}(t_0,\sigma) + \alpha_1^{(0)}(t_0,\sigma) + z_2^{(0)}(t_0,\sigma)\sigma = z^0, \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \alpha_1^{(0)}(t_0,\sigma) = z^0 + \lambda_1^{-1}(t_0)h_1(t_0) - \left[{t_0}^{(1-\alpha)}\lambda_2(t_0) - \lambda_1(t_0)\right]^{-1}h_2(t_0)\sigma. \tag{5.2}$$

For a complete calculation of the function $\alpha_1^{(0)}(t,\sigma)$, we pass to the next iterative problem (3.1_1). Substituting the solution (5.1) of the equation (3.1_0) into it, we obtain the following system of equations:

$$Ly_1(t,\tau) = -\frac{d}{dt}\left(t^{(1-\alpha)}z_0^{(0)}(t,\sigma)\right) - \frac{d}{dt}\left(t^{(1-\alpha)}\alpha_1^{(0)}(t,\sigma)\right)e^{\tau_1} -$$

$$-\frac{d}{dt}\left(t^{(1-\alpha)}z_2^{(0)}(t,\sigma)\sigma e^{\tau_2}\right)+R_1z_0=-(1-\alpha)t^{(-\alpha)}z_0^{(0)}(t,\sigma)-$$

$$-t^{(1-\alpha)}\dot{z}_0^{(0)}(t,\sigma)-\left((1-\alpha)t^{(-\alpha)}\alpha_1^{(0)}(t,\sigma)+t^{(1-\alpha)}\dot{\alpha}_1^{(0)}(t,\sigma)\right)e^{\tau_1}-$$

$$-\left((1-\alpha)t^{(-\alpha)}z_2^{(0)}(t,\sigma)-t^{(1-\alpha)}\dot{z}_2^{(0)}(t,\sigma)\right)\sigma e^{\tau_2}+$$

$$+\left[\frac{K(t,t)\alpha_1(t,\sigma)}{t^{(1-\alpha)}\lambda_1(t)}e^{\tau_1}-\frac{K(t,t_0)\alpha_1(t_0,\sigma)}{t_0{}^{(1-\alpha)}\lambda_1(t_0)}\right]+$$

$$+\frac{K(t,t_0)z_2(t)}{\lambda_2(t)}e^{\tau_2}-\frac{K(t,t)z_2(t_0,\sigma)}{\lambda_2(t_0)}.$$

Performing here scalar multiplication, we obtain the following system of ordinary differential equations

$$-t^{(1-\alpha)}\frac{d\alpha_1^{(0)}(t,\sigma)}{dt}+\left[\frac{K(t,t)}{t^{(1-\alpha)}\lambda_1(t)}-(1-\alpha)t^{(-\alpha)}\right]\alpha_1^{(0)}(t,\sigma)=0.$$

Adding the initial condition (5.2) to this equation, we find $\alpha_k^{(0)}(t)$:

$$\alpha_1^{(0)}(t,\sigma)=\alpha_1^{(0)}(t_0,\sigma)e^{\int\limits_{t_0}^{t}\left[\frac{K(\theta,\theta)-(1-\alpha)\theta^{(1-2\alpha)}}{\theta^{2(1-\alpha)}A(\theta)}\right]d\theta},$$

and hence the solution (5.1) of the problem (3.1_0) will be found uniquely in the space U. In this case, the leading term of the asymptotic has the following form:

$$\begin{gathered}z_{\varepsilon 0}(t)=z_0^{(0)}(t)+z_2^{(0)}(t)e^{+\frac{i\beta(t)}{\varepsilon}}+\big(z^0+A^{-1}(t_0)h_1(t_0)-\\-\left[t_0{}^{(1-\alpha)}(-i\beta'(t_0))-A(t_0)\right]^{-1}h_2(t_0)\sigma\big)\,e^{\int\limits_{t_0}^{t}\left[\frac{K(\theta,\theta)-(1-\alpha)\theta^{(1-2\alpha)}}{\theta^{2(1-\alpha)}A(\theta)}\right]d\theta+\frac{1}{\varepsilon}\int\limits_{t_0}^{t}A(\theta)d\theta}.\end{gathered}\tag{5.3}$$

6. Conclussion

From expression (5.3) for it is evident that the application of S.A.Lomov's regularization method to the construction of the leading term of the asymptotic of the solution to problem (1.2) is significantly influenced by both the rapidly oscillating in-homogeneity and the kernel of the integral operator.

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