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A Novel Generalization of the Inverted Nadarajah–Haghighi Distribution: Estimation Methods and Medical Applications

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Abstract. Statisticians and data analysts often use statistical probability distributions to describe and analyze their data. However, traditional distributions may not always accommodate certain data sets, making it necessary to develop new models to handle complex data structures and improve fit quality. This paper introduces a new extended lifetime model, called the New Exponential Inverted Nadarajah–Haghighi distribution (NEINH), which belongs to the new exponential-X family of distributions. This approach is designed to model complex data in a variety of applications. The article explores some of the statistical properties of this proposed distribution such as quantile function, moment, moment generating function, order statistic and others. The NEINH's parameters are estimated using Bayesian estimation method and maximum likelihood method and five other methods. The efficacy of this distribution is established by its comparative analysis with alternative distributions, using four real-world medical datasets to highlight its superior performance.

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1. Introduction

Probability distributions are fundamental for data modeling in numerous fields, such as finance, technology, Biological sciences, industry, and healthcare. Consequently, researchers have introduced new extended distributions to enhance the performance of density and hazard rate functions in applications.

Methods employed to extend distributions include compounding, parameter addition, composition, and transformation. Examples encompass [1] introduced the beta-generation approach, [2] proposed the Kumaraswamy-generated approach, and [3] developed the transformed-transformer approach, along with several other approaches.

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A novel family of lifetime distributions known as the exponential-X (NLTE-X) introduced by [4]. The basis of this distribution family is the T-X generator, where $T \sim Exp(1)$ and $W(G(x)) = -\log\left\{\frac{1-G(x)}{e^{\vartheta G(x)}}\right\}$. For the NLTE-X family, the CDF and PDF are expressed as follows:

$$F(x;\vartheta,\zeta) = 1 - \left\{\frac{1-G(x;\Omega)}{e^{\vartheta G(x;\Omega)}}\right\};\ \vartheta > 0,\ x > 0, \tag{1}$$

$$f(x;\vartheta,\zeta) = g(x)\frac{\{1+\vartheta\overline{G}(x;\Omega)\}}{e^{\vartheta G(x;\Omega)}};\ \vartheta > 0,\ x > 0, \tag{2}$$

where ϑ represents the vector of distribution parameters, and the parameter ϑ is unique to the NLTE-X family. Several distributions have been derived from the NLTE-X family, including the exponential Fréchet distribution [5], exponential inverted Topp–Leone distribution [6], exponentiated Weibull distribution [7], exponential-X power family of distributions [8], the exponential generalized inverse generalized Weibull distribution [9], and the exponential inverted Gompertz distribution [10].

Inverted distributions have attracted significant attention from researchers due to their enhanced flexibility in the structure of both the density and hazard functions compared to their non-inverted counterparts. Furthermore, inverted distributions have proven to be highly applicable in various fields, including biological research, life testing problems, chemical data analysis, and technological applications. For instance, the inverted exponential distribution [11] was applied in analyzing failure rates and repair times of computer numerical control machine tools to evaluate their reliability in industrial operations. The inverse Weibull distribution was applied to medical data [12] to model life data with decreasing failure rates and to reliability data [13] for systems with non-monotonic failure rates. The inverse Rayleigh distribution [14] was used to analyze failure times of manufactured components in industrial reliability testing. The inverted Kumaraswamy distribution [15] was applied to reliability data in industrial systems, and the inverted Lindley distribution [16] was used in medical survival data. The inverted Topp-Leone distribution [17] was applied to industrial reliability data, while the inverse Lomax distribution [18] was used in economic data for income and wealth distribution analysis. The inverse power Lomax distribution [19] and the alpha power transformed inverse Lomax distribution [20] were both applied to industrial reliability data in engineering systems.

Recently, various generalizations of inverse distributions have been presented in the literature. These include, the inverse Weibull inverse exponential distribution [21], the extended inverse Weibull distribution [22], the the Weibull inverse Rayleigh distribution [23], the Topp–Leone inverted Kumaraswamy distribution [24], the extended inverse Lindley distribution [25], and the odd Weibull inverse Topp–Leone distribution proposed by [26], the Kumaraswamy generalized inverse Lomax distribution [27].

Moreover, a novel inverted model known as the inverted Nadarajah–Haghighi (INH) distribution was introduced by [28], with a decreasing and unimodal density, along with decreasing and upside-down bathtub hazard rate shapes. Several statistical properties of the INH distribution were derived, and various methods were used to estimate the model's parameters. The suitability of the INH distribution has been demonstrated by testing it on real-life datasets. The CDF and PDF of INH are represented by

$$G_{INH}(x) = e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}};\ x > 0, \tau, \kappa > 0, \tag{3}$$

$$g_{INH}(x) = \tau\kappa x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}; x > 0, \tag{4}$$

where τ is the shape parameter and κ is the scale parameter.In addition, estimators for the parameters of the INH distribution were derived by [29] using several methods, including maximum likelihood estimation, Bayesian estimation, and the Maximum Product of Spacing approach. Moreover, the generalization of the INH distribution has been extensively investigated by various researchers, these include, the Marshall-Olkin INH distribution [30], extended odd Weibull INH distribution [31], power INH distribution [32], odd Lomax INH distribution [33], and Half Logistic INH distribution [34].

The primary aim of this article is to introduce a novel generalization of INH distribution which is derived from the NLTE-X family of distributions called the new exponential inverted Nadarajah-Haghighi (NEINH) distribution. The proposed distribution is expected to enhance the characteristics and flexibility of the density and hazard rate functions. The adaptability of the NEINH distribution is investigated to describe survival time by analysing several medical datasets. Moreover, the primary reason for employing NEINH in practice is to increase INH's flexibility by introducing new generalizations and providing a superior fit compared to other competing models.

This article is organized as follows: Section 2 introduces the NEINH model along with graphical representations. Section 3 discusses the derived properties of the NEINH. Section 4, seven estimation methods are employed to estimate the NEINH parameter, including maximum likelihood (ML), maximum product of spacing (MPS), Bayesian, ordinary least squares (OLS), weighted least squares (WLS), Cramér–von Mises (CM), and Anderson–Darling (AD). Section 5 presents extensive simulation studies to evaluate the performance of these estimators. Section 6 covers four applications in medicine. Finally, Section 7 provides concluding remarks.

2. New Exponential Inverted Nadarajah–Haghighi Distribution

The CDF and PDF of the NEINH are obtained by substituting equations (3) and (4) into (1) and (2) respectively, as follows.

$$F(x) = 1 - \frac{1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}},\ \ x > 0,\ \ \vartheta, \tau, \kappa > 0, \tag{5}$$

$$f(x) = \tau\kappa x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}} \frac{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]}{e^{\vartheta\left\{e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right\}}}. \tag{6}$$

The survival $S(x)$ function is defined as

$$S(x) = \frac{1 - e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}}. \tag{7}$$

The hazard rate function (HF), frequently employed in lifetime modelling to represent the probability of failure, is defined as

$$HF = \tau\kappa x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}} \frac{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]}{1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}. \tag{8}$$

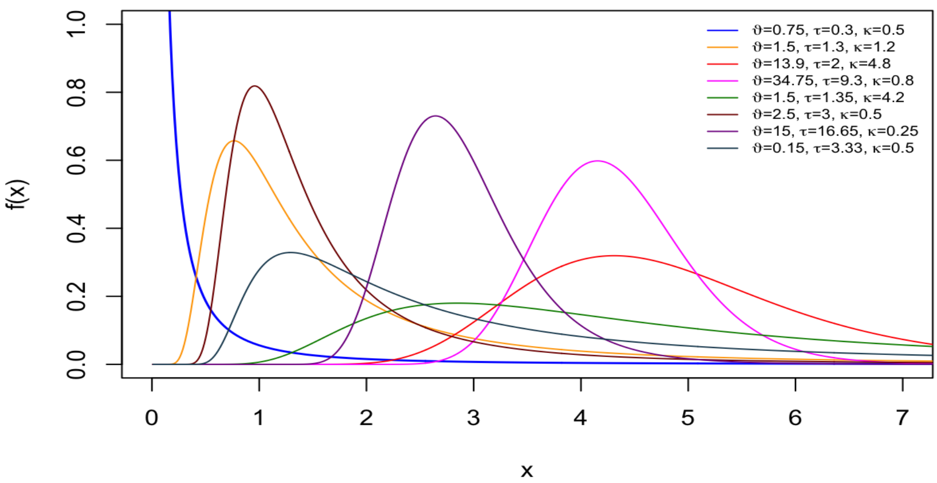


Figure 1: The NEINH density plots.

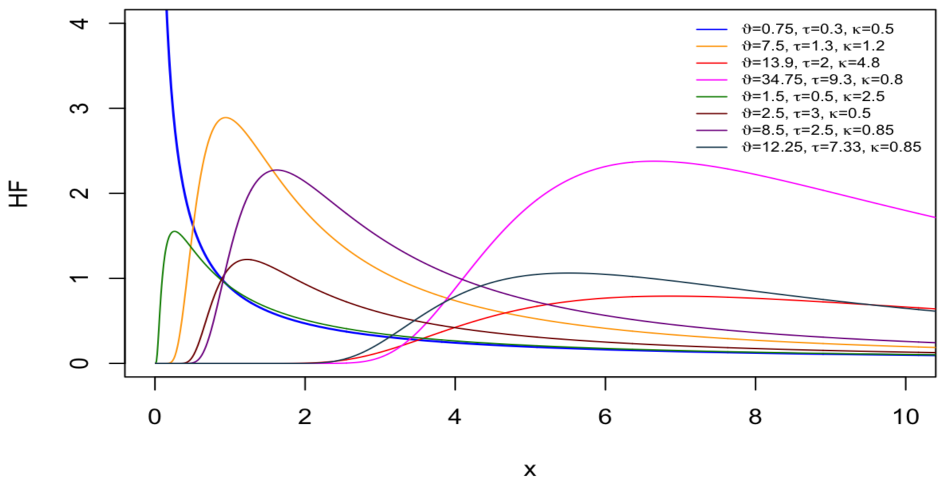


Figure 2: The NEINH HF's plots.

Several plots illustrating the PDF and HF of the NEINH model for specific parameter values are provided. The PDF in Figure (1) displays a variety of shapes, including right-skewed, left-skewed, approximately symmetrical, and decaying forms. Similarly, Figure (2) demonstrates multiple HF patterns, such as inverted, decreasing, and constant shapes. These results highlight the substantial flexibility of the NEINH model in adapting to real-world data.

2.1. Linear expression of the NEINH density function

A linear representation of NEINH's PDF has been constructed using mathematical expansions. The derivation begins with the exponential expansion with a negative exponent is given by

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}. \tag{9}$$

By applying (9), the NEINH's PDF is expressed as

$$f(x) = \sum_{\mathfrak{c}_1=0}^{\infty} \frac{(-1)^{\mathfrak{c}_1} \vartheta^{\mathfrak{c}_1}}{\mathfrak{c}_1!} \tau\kappa x^{-2} \left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{(1+\mathfrak{c}_1)\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}} \left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]\right\}. \tag{10}$$

The binomial theorem is given by

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i. \tag{11}$$

By applying (47) twice, the PDF of NEINH is simplified to the following

$$f(x) = \eta\,\tau\,\kappa\,x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}} \tag{12}$$

where

$$\eta = \sum_{\mathfrak{c}_1=0}^{\infty}\sum_{\mathfrak{c}_2=0}^{1}\sum_{\mathfrak{c}_3=0}^{\mathfrak{c}_2} \frac{(-1)^{(\mathfrak{c}_1+\mathfrak{c}_3)}\vartheta^{(\mathfrak{c}_1+\mathfrak{c}_3)}}{\mathfrak{c}_1!}\binom{1}{\mathfrak{c}_2}\binom{\mathfrak{c}_2}{\mathfrak{c}_3} e^{(1+\mathfrak{c}_1+\mathfrak{c}_3)} \tag{13}$$

3. Properties of the NEINH

In this section, several characteristic properties of the NEINH are established.

3.1. Quantile Function

Let $X \sim$ NEINH, the uth quantile function $(0 < u < 1)$ can be obtained through inverting the equation (5), and solving the non-linear equation by applying the Lambert function $W[.]$.

$$x_u = \kappa\left\{\left\{1-\log\left\{1-\frac{W\left(\vartheta e^{\vartheta}(1-u)\right)}{\vartheta}\right\}\right\}^{\frac{1}{\tau}}-1\right\}^{-1}, 0 \le u \le 1. \tag{14}$$

The median of the NEINH is calculated by substituting $u = 0.5$ into equation (14).

$$Median(x) = \kappa\left\{\left\{1-\log\left\{1-\frac{W\left(\vartheta e^{\vartheta}(0.5)\right)}{\vartheta}\right\}\right\}^{\frac{1}{\tau}}-1\right\}^{-1}. \tag{15}$$

3.2. Moment

The r^{th} moment of $X \sim$ NEINH can be derived as follows:

$$\begin{aligned}\mu_r = E\left(x^r\right) &= \int_0^{\infty} x^r f(x)dx \\ &= \eta\int_0^{\infty}\tau\,\kappa\,x^r x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}}dx.\end{aligned} \tag{16}$$

By substituting $y = (1+\frac{\kappa}{x})^{\tau}$, therefore the r^{th} moment is defined as:

$$\mu_r = \eta\sum_{\mathfrak{c}_4}^{\infty}(-1)^r\kappa^r\binom{r+\mathfrak{c}_4-1}{\mathfrak{c}_4}\frac{\Gamma(\frac{\mathfrak{c}_4}{\tau}+1)}{(1+\mathfrak{c}_1+\mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}, \quad \frac{\mathfrak{c}_4}{\tau}+1 > 0, \tag{17}$$

where η is given by (13)

3.3. Moment Generating Function

The moment-generating function (MGF) of the NEINH is expressed as

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r. \tag{18}$$

Consequently, by substituting (55) into (56), the MGF is derived as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{\mathfrak{c}_4}^{\infty} \eta\, (-1)^r \kappa^r \binom{r + \mathfrak{c}_4 - 1}{\mathfrak{c}_4} \frac{\Gamma(\frac{\mathfrak{c}_4}{\tau} + 1)}{(1 + \mathfrak{c}_1 + \mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}\,, \quad \frac{\mathfrak{c}_4}{\tau} + 1 > 0, \tag{19}$$

where η is given by (13)

3.4. Characteristic Function

The NEINH characteristic function is derived as

$$\phi_x\,(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{\mathfrak{c}_4}^{\infty} \eta\, (-1)^r \kappa^r \binom{r + \mathfrak{c}_4 - 1}{\mathfrak{c}_4} \frac{\Gamma(\frac{\mathfrak{c}_4}{\tau} + 1)}{(1 + \mathfrak{c}_1 + \mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}\,, \quad \frac{\mathfrak{c}_4}{\tau} + 1 > 0, \tag{20}$$

where η is given by (13)

3.5. Rényi entropy

The Rényi entropy is employed to measure the uncertainty associated with the random variable X. A higher Rényi entropy value indicates greater uncertainty in the data. According to [35], the Rényi entropy, $RE(\varphi)$, is defined as

$$RE(\varphi) = \frac{1}{1-\varphi} \log \left[\int_{-\infty}^{\infty} [f(x)]^{\varphi} dx \right] \tag{21}$$

By substituting $f(x)$ given in (6) into the (21) and applying some mathematical expansions $RE(\varphi)$ is presented as

$$[f(x)]^{\varphi} = \eta^* \tau^{\varphi} \kappa^{\varphi}\, (-1)^{\mathfrak{c}_1+\mathfrak{c}_3} x^{-2\varphi} \left(1 + \frac{\kappa}{x}\right)^{\varphi(\tau-1)} e^{-(\varphi+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}}, \tag{22}$$

where

$$\eta^* = \sum_{\mathfrak{c}_1=0}^{\infty} \sum_{\mathfrak{c}_2=0}^{\varphi} \sum_{\mathfrak{c}_3=0}^{\mathfrak{c}_2} \binom{\varphi}{\mathfrak{c}_2} \binom{\mathfrak{c}_2}{\mathfrak{c}_3} \frac{\varphi^{\mathfrak{c}_1}\, \vartheta^{\mathfrak{c}_1+\mathfrak{c}_2}}{\mathfrak{c}_1!} e^{(\varphi+\mathfrak{c}_1+\mathfrak{c}_3)}. \tag{23}$$

By replacing (22) in (21), and calculating the integral, the Rényi entropy of the NEINH can be derived as

$$RE(\varphi) = \frac{1}{1-\varphi} \log \left[\eta^* \sum_{\mathfrak{c}_4=0}^{\infty} \binom{-2\varphi + \mathfrak{c}_4 + 1}{\mathfrak{c}_4} \frac{(-1)^{\mathfrak{c}_1+\mathfrak{c}_3-2\varphi+2}\ \tau^{\varphi-1} \kappa^{1-\varphi}}{(\varphi + \mathfrak{c}_1 + \mathfrak{c}_3)^{\frac{1}{\tau}[\varphi(\tau-1)-(\tau-1)+i]+1}} \right. \\ \left. \times \Gamma\left(\frac{1}{\tau}[\varphi(\tau-1) - (\tau-1) + \mathfrak{c}_4] + 1 \right) \right]. \tag{24}$$

3.6. Order statistics

Let $X_{i:\mathfrak{n}}$ represent the i^{th} order statistic from a random sample $X_1, X_2, ..., X_{\mathfrak{n}}$ of the NEINH distribution. The PDF of the i^{th} order statistic, $p_{i:\mathfrak{n}}(x)$, is therefore expressed as

$$f_{i:\mathfrak{n}}(x) = \frac{\mathfrak{n}!}{(i-1)!(\mathfrak{n}-i)!} f(x) \left[F(x)\right]^{i-1} \left[1 - F(x)\right]^{\mathfrak{n}-i}. \tag{25}$$

By applying binomial expansion, the PDF of $X_{i:\mathfrak{n}}$ become

$$f_{i:\mathfrak{n}}(x) = \sum_{j=0}^{\mathfrak{n}} \frac{(-1)^j \mathfrak{n}!}{(i-1)!(\mathfrak{n}-i)!} \binom{\mathfrak{n}}{j} f(x) \left[F(x)\right]^{j+i-1}. \tag{26}$$

Substituting (5) into (26), the PDF of $X_{i:\mathfrak{n}}$

$$f_{i:\mathfrak{n}}(x) = \sum_{j=0}^{\mathfrak{n}} \frac{(-1)^j \mathfrak{n}!}{(i-1)!(\mathfrak{n}-i)!} \binom{\mathfrak{n}}{j} f(x) \left[1 - \frac{1 - e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}} \right]^{j+i-1}, \tag{27}$$

where $f(x)$ given by (6).

4. Estimation of Parameters

This section presents seven methods of estimation used to estimate the parameters of the NEINH.

4.1. ML estimation method

If $x_1, ..., x_{\mathfrak{n}}$ are NEINH RS of size $\mathfrak{n}$, the likelihood function $L(\Omega)$, and the log-likelihood function $\ell(\Omega)$ for $\Omega = (\vartheta, ă\tau, \kappa)$ are defined as follows:

$$L(\Omega) = \tau^{\mathfrak{n}}\ \kappa^{\mathfrak{n}} \prod_{i=1}^{\mathfrak{n}} x_i^{-2} \prod_{i=1}^{\mathfrak{n}} \left(1 + \frac{\kappa}{x_i}\right)^{(\tau-1)}\ e^{\left\{\sum_{i=1}^{\mathfrak{n}} \left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}\right\}} \\ \prod_{i=1}^{\mathfrak{n}} \left\{ 1 + \vartheta \left[1 - e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}} \right] \right\}\ e^{-\sum_{i=1}^{\mathfrak{n}} \vartheta e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}. \tag{28}$$

$$\begin{aligned}\ell\left(\Omega\right) =&\mathfrak{n}\log(\tau)+\mathfrak{n}\log(\kappa)-2\sum_{i=1}^{\mathfrak{n}}\log(x_i)+(\tau-1)\sum_{i=1}^{\mathfrak{n}}\log(1+\frac{\kappa}{x_i})+\sum_{i=1}^{\mathfrak{n}}\left\{1-(1+\frac{\kappa}{x_i})^{\tau}\right\}\\&+\sum_{i=1}^{\mathfrak{n}}\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]\right\}-\sum_{i=1}^{\mathfrak{n}}\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}.\end{aligned} \tag{29}$$

The following are the first derivatives of (29), with respect to $\Omega=(\vartheta,\tau,\kappa)$

$$\frac{\partial\ell}{\partial\vartheta}=\sum_{i=1}^{\mathfrak{n}}\frac{\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]}{\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]\right\}}-\sum_{i=1}^{\mathfrak{n}}e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}, \tag{30}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\tau} =&\frac{\mathfrak{n}}{\tau}+\sum_{i=1}^{\mathfrak{n}}\log\left(1+\frac{\kappa}{x_i}\right)-\sum_{i=1}^{\mathfrak{n}}\left(1+\frac{\kappa}{x_i}\right)^{\tau}\log\left(1+\frac{\kappa}{x_i}\right)\\&+\sum_{i=1}^{\mathfrak{n}}\frac{\vartheta\left(1+\frac{\kappa}{x_i}\right)^{\tau}\log\left(1+\frac{\kappa}{x_i}\right)e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}{\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]\right\}}\\&+\sum_{i=1}^{\mathfrak{n}}\vartheta\left(1+\frac{\kappa}{x_i}\right)^{\tau}\log\left(1+\frac{\kappa}{x_i}\right)e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}},\ and\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\kappa} =&\frac{\mathfrak{n}}{\kappa}+(\tau-1)\sum_{i=1}^{\mathfrak{n}}\frac{1}{x_i\left(1+\frac{\kappa}{x_i}\right)}-\tau\sum_{i=1}^{\mathfrak{n}}\left(1+\frac{\kappa}{x_i}\right)^{\tau-1}\\&+\sum_{i=1}^{\mathfrak{n}}\frac{\vartheta\tau\left(1+\frac{\kappa}{x_i}\right)^{\tau-1}e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}{x_i\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]\right\}}\\&+\sum_{i=1}^{\mathfrak{n}}\frac{\vartheta\tau}{x_i}\left(1+\frac{\kappa}{x_i}\right)^{\tau-1}e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}.\end{aligned} \tag{32}$$

Equations (30-32) can be solved through numerical methods, such as the Newton-Raphson method, which is a commonly used optimization technique.

4.2. Maximum product of spacing estimation method

The MPS method, introduced by [36], utilizes the uniform spacings of a random sample from the NEINH distribution with size $\mathfrak{n}$ to determine the MPS. This method can be applied as follows:

$$\mathcal{D}_i = F(x_{i:\mathfrak{n}}) - F(x_{i-1:\mathfrak{n}}), \quad i = 1, 2, ..., \mathfrak{n} + 1, \tag{33}$$

where $F(i)$ denotes the CDF of the observation $x_{i:\mathfrak{n}}$ from the NEINH distribution. The MPS estimates for the NEINH parameters can then be obtained by maximizing the logarithm of the geometric mean of the sample spacings, given by

$$\mathcal{M} = \frac{1}{\mathfrak{n}+1} \sum_{i=1}^{\mathfrak{n}+1} \log \mathcal{D}_i. \tag{34}$$

4.3. Bayesian estimation method

In the Bayesian approach, parameters are considered as random variables with predefined prior distributions. This method is especially valuable in survival analysis due to its capacity to incorporate prior knowledge into the research. One of the primary challenges in survival analysis is the scarcity of available data. For ϑ, τ, and κ are selected as gamma distributions. Hence,

$$\begin{aligned} \pi_1(\vartheta) &\propto \vartheta^{d_{11}-1} e^{-d_{12}\vartheta}, \;\; \vartheta > 0, \; d_{11}, d_{12} > 0, \\ \pi_2(\tau) &\propto \tau^{d_{21}-1} e^{-d_{22}\tau}, \;\; \tau > 0, \; d_{21}, b_{22} > 0, \\ \pi_3(\kappa) &\propto \kappa^{d_{31}-1} e^{-d_{32}\kappa}, \;\; \kappa > 0, \; d_{31}, d_{32} > 0. \end{aligned} \tag{35}$$

Assuming that the parameters of the proposed model are independent, The joint distribution of the priors is obtained as follows:

$$\pi(\vartheta, \tau, \kappa) \propto \vartheta^{d_{11}-1} \tau^{d_{21}-1} \kappa^{d_{31}-1} e^{-(d_{12}\vartheta + d_{22}\tau + d_{32}\kappa)}. \tag{36}$$

The posterior function of the parameters of the proposed distribution can be computed using equations (28) and (36) as follows.

$$\begin{aligned} \pi^*(\Omega|X) &\propto L(\vartheta, \tau, \kappa)\pi(\vartheta, \tau, \kappa) \\ &\propto \vartheta^{d_{11}-1} \tau^{\mathfrak{n}+d_{21}-1} \; \kappa^{\mathfrak{n}+d_{31}-1} \; \prod_{i=1}^{\mathfrak{n}} x_i{}^{-2} \prod_{i=1}^{\mathfrak{n}} \left(1 + \frac{\kappa}{x_i}\right)^{(\tau-1)} \\ &\times e^{-(d_{22}\tau + d_{32}\kappa)} \; e^{\left\{\sum_{i=1}^{\mathfrak{n}} \left\{1 - \left(1 + \frac{\kappa}{x_i}\right)^{\tau}\right\}\right\}} \\ &\times \prod_{i=1}^{\mathfrak{n}} \left\{1 + \vartheta \left[1 - e^{\left\{1 - \left(1 + \frac{\kappa}{x_i}\right)^{\tau}\right\}}\right]\right\} \; e^{-\vartheta\left\{d_{12} + \sum_{i=1}^{\mathfrak{n}} e^{\left\{1 - \left(1 + \frac{\kappa}{x_i}\right)^{\tau}\right\}}\right\}}. \end{aligned} \tag{37}$$

Bayesian parameter estimation for the NEINH distribution is calculated by

$$\hat{\vartheta} = \int_0^\infty \vartheta \int_0^\infty \int_0^\infty \; \pi^*(\vartheta|\mathbf{x}) \; d\tau d\kappa d\vartheta \tag{38}$$

$$\hat{\tau} = \int_0^\infty \tau \int_0^\infty \int_0^\infty \; \pi^*(\vartheta|\mathbf{x}) \; d\kappa d\vartheta d\tau \tag{39}$$

$$\hat{\kappa} = \int_0^{\infty} \kappa \int_0^{\infty} \int_0^{\infty} \pi^*(\vartheta|\mathbf{x})\ d\vartheta d\tau d\kappa \tag{40}$$

The integrals in equations (38), (39), and (40) are particularly complex. The Metropolis-Hastings (MH) algorithm is used to derive approximations for these integrals.

4.4. Ordinary and weighted least squares estimation methods

The OLS and WLS are introduced by [37]. Suppose that $x_{1:\mathfrak{n}} \leq ... \leq x_{i:\mathfrak{n}} \leq ... \leq x_{\mathfrak{n}:\mathfrak{n}}$ represent the order statistics of a RS from the NEINH. The OLS can be found by minimizing a sum of squared differences between the theoretical and the empirical CDF functions with respect to the parameters of NEINH as

$$O = \sum_{i=1}^{\mathfrak{n}} \left[1 - \frac{1 - e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}} - \mathcal{F}(i) \right]^2, \tag{41}$$

where $\mathcal{F}(i)$ represents the empirical CDF of the observation $x_{i:\mathfrak{n}}$ of the NEINH which is usually estimated by $\mathcal{F}(i) = i/(\mathfrak{n}+1)$.
The WLS estimates are obtained similarly to the OLS estimates, but with a weighted sum of squared differences as follows.

$$W = \sum_{i=1}^{\mathfrak{n}} w_i \left[1 - \frac{1 - e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}} - \mathcal{F}(i) \right]^2,\ w_i = \frac{(\mathfrak{n}+1)^2(\mathfrak{n}+2)}{i(\mathfrak{n}+1-i)}. \tag{42}$$

4.5. Cramér von Mises and Anderson Darling estimation methods

The CM and AD estimate methods are presented by [38] in the context of statistical testing. Their foundation comes from the difference between the CDF estimates and the empirical distribution function. The functions (43) and (44) minimized with respect to the parameters of NEINH to find CM and AD estimates, respectively

$$C = \frac{1}{12\mathfrak{n}} + \sum_{i=1}^{\mathfrak{n}} \left[1 - \frac{1 - exp\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}{e^{\vartheta e^{\left\{1-\left(1+\frac{\kappa}{x_i}\right)^{\tau}\right\}}}} - \frac{2i-1}{2\mathfrak{n}} \right]^2, \tag{43}$$

$$A = -\mathfrak{n} - \frac{1}{\mathfrak{n}} \sum_{i=1}^{\mathfrak{n}} (2i-1) \left[\log F(x_{i:\mathfrak{n}}) + \log \overline{F}(x_{\mathfrak{n}-i+1:\mathfrak{n}}) \right]. \tag{44}$$

5. Simulation Studies

In this section, the effectiveness of various estimation methods is assessed through numerical simulations. From the NEINH distribution, a total of $\mathcal{N} = 1000$ samples was randomly generated, using two sets of parameter values, with sample sizes of $\mathfrak{n} = 30$, 100, 200, and 500.

- $SetI : \vartheta = 1.6, \tau = 0.5, \kappa = 0.1$
- $SetII : \vartheta = 6, \tau = 0.25, \kappa = 1.7$

Monte Carlo simulation was used to obtain NEINH parameter estimates for the seven methods. All estimation results were obtained using the R programming. For each parameter, we calculated the mean estimates and the mean square error (MSE), defined as

$$MSE = \mathrm{var}(\widehat{\Omega}) + [\mathrm{Bias}(\hat{\Omega})]^2 = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathfrak{n}} (\widehat{\Omega} - \Omega_{tr})^2,$$

where, $\Omega = (\vartheta, \tau, \kappa)$ and $Bias = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathfrak{n}} \left(\hat{\Omega} - \Omega_{tr} \right)$.

The ML, MPS, Bayesian, OLS, WLS, CM, and AD parameter estimates were obtained along with their MSE as shown in Tables (1)-(2).

According to Tables (1)-(2), for most methods, the MSE decreases as the sample size increases. Additionally, parameter estimates tend to converge to true values as sample size increases. In terms of MSE, Bayesian estimation proves to be the most effective method for estimating the NEINH parameter, outperforming all other estimation techniques. Following Bayesian estimation, the ML and WLS methods also demonstrate strong performance. In contrast, the OLS, CM and AD estimators yield similar results, but tend to produce larger MSE values, particularly when the sample size is small. However, as the sample size increases to 200 or 500, the accuracy of all estimators converges, with their performance becoming approximately equivalent.

Table 1: NEINH parameters estimators and MSE for Set I.

$\mathfrak{n}$		ML	MPS	Bayesian	OLS	WLS	CM	AD
30	$\hat{\vartheta}$	1.6516	2.8819	1.5794	3.4492	1.9248	3.7545	3.9278
		(4.53 e-01)	(5.7709)	(1.95 e-02)	(2.83 e+01)	(4.94 e-01)	(3.17 e+01)	(5.89 e+01)
	$\hat{\tau}$	0.5822	0.4420	0.5203	0.5061	0.5467	0.5308	0.5382
		(3.64 e-02)	(1.81 e-02)	(4.31 e-03)	(3.78 e-02)	(3.25 e-02)	(4.43 e-02)	(5.22 e-02)
	$\hat{\kappa}$	0.1079	0.4663	0.0997	1.0927	0.1365	0.9591	1.1239
		(1.02 e-02)	(8.77 e-01)	(5.26 e-04)	(2.36 e+01)	(1.84 e-02)	(1.94 e+01)	(1.97 e+01)
100	$\hat{\vartheta}$	1.710	2.2565	1.5750	2.2912	1.7018	2.3915	2.0281
		(4.06 e-01)	(2.5933)	(1.85 e-02)	(5.3448)	(1.64 e-01)	(5.7669)	(2.9659)
	$\hat{\tau}$	0.5212	0.4648	0.5108	0.5081	0.5169	0.5135	0.5231
		(8.30 e-03)	(7.81 e-03)	(2.13 e-03)	(1.77 e-02)	(7.81 e-03)	(1.90 e-02)	(1.86 e-02)
	$\hat{\kappa}$	0.1183	0.2613	0.0999	0.3070	0.1092	0.3085	0.2108
		(8.57 e-03)	(2.60 e-01)	(4.22 e-04)	(7.20 e-01)	(3.08 e-03)	(7.21 e-01)	(2.47 e-01)
200	$\hat{\vartheta}$	1.7299	1.9743	1.5923	1.9141	1.7046	1.9585	1.7829
		(3.51 e-01)	(1.1792)	(1.66 e-02)	(1.5940)	(1.40 e-01)	(1.6763)	(9.37 e-01)
	$\hat{\tau}$	0.5083	0.4798	0.5054	0.5046	0.5049	0.5064	0.5143
		(5.53 e-03)	(5.64 e-03)	(1.29 e-03)	(9.85 e-03)	(4.16 e-03)	(9.99 e-03))	(9.75 e-03)
	$\hat{\kappa}$	0.11701	0.1652	0.1012	0.1634	0.1104	0.1646	0.1334
		(4.60 e-03)	(4.23 e-02)	(2.99 e-04)	(5.50 e-02)	(1.99 e-03)	(5.57 e-02)	(1.62 e-02)
500	$\hat{\vartheta}$	1.6565	1.6959	1.5936	1.7333	1.6507	1.7497	1.6705
		(1.39 e-01)	(2.12 e-01)	(1.51 e-02)	(4.64 e-01)	(7.42 e-02)	(4.75 e-01)	(3.39 e-01)
	$\hat{\tau}$	0.5034	0.4945	0.5031	0.5004	0.5028	0.5010	0.5051
		(2.27 e-03)	(1.26 e-03)	(4.98 e-04)	(4.65 e-03)	(1.91 e-03)	(4.67 e-03)	(3.98 e-03)
	$\hat{\kappa}$	0.1066	0.1139	0.1002	0.1226	0.1048	0.1230	0.1131
		(1.34 e-03)	(5.36 e-03)	(1.55 e-04)	(6.11 e-03)	(8.02 e-04)	(6.18 e-03)	(3.30 e-03)

Table 2: NEINH parameters estimators and MSE for Set II.

$\mathfrak{n}$		**ML**	**MPS**	**Bayesian**	**OLS**	**WLS**	**CM**	**AD**
30	$\hat{\vartheta}$	5.9811	6.5403	5.9997	6.2947	6.4993	6.3630	5.7915
		(1.8997)	(4.4987)	(8.51 e-05)	(5.0136)	(2.9104)	(5.4772)	(3.1845)
	$\hat{\tau}$	0.2649	0.2500	0.2509	0.2505	0.2678	0.2603	0.2607
		(2.18 e-03)	(2.38 e-03)	(1.07 e-04)	(1.52 e-03)	(2.62 e-03)	(1.05 e-02)	(2.45 e-03)
	$\hat{\kappa}$	1.7104	2.3398	1.6995	2.4451	1.7818	2.2771	1.8991
		(1.0201)	(3.4656)	(2.61 e-04)	(4.7756)	(1.4845)	(4.0528)	(1.5404)
100	$\hat{\vartheta}$	5.9993	6.4250	5.9995	5.9542	6.2683	6.0333	5.8606
		(8.78 e-01)	(3.0018)	(7.61 e-05)	(2.2520)	(1.0812)	(2.1904)	(1.5670)
	$\hat{\tau}$	0.2544	0.2498	0.2500	0.2546	0.2550	0.2561	0.2537
		(4.14 e-04)	(6.63 e-04)	(2.97 e-05)	(9.94 e-04)	(5.61 e-04)	(9.04 e-04)	(5.94 e-04)
	$\hat{\kappa}$	1.7298	2.1397	1.6995	1.895	1.8213	1.8647	1.8253
		(4.77 e-01)	(1.8198)	(2.25 e-04)	(1.4906)	(7.14 e-01)	(1.4513)	(9.16 e-01)
200	$\hat{\vartheta}$	6.0371	6.3856	6.0001	5.9328	6.1883	5.9321	5.9153
		(1.46 e-01)	(2.0893)	(6.39 e-05)	(1.1857)	(6.02 e-01)	(1.2802)	(1.0587)
	$\hat{\tau}$	0.2513	0.2475	0.2502	0.2531	0.2531	0.2546	0.2520
		(1.91 e-04)	(3.33 e-04)	(1.31 e-05)	(3.74 e-04)	(2.88 e-04)	(4.67 e-04)	(3.32 e-04)
	$\hat{\kappa}$	1.765	2.1679	1.6996	1.7723	1.7714	1.7408	1.7957
		(3.18 e-01)	(1.8458)	(2.12 e-04)	(6.87 e-01)	(3.97 e-01)	(6.53 e-01)	(5.88 e-01)
500	$\hat{\vartheta}$	6.0117	6.3653	5.9999	5.9710	6.1079	5.9857	5.9927
		(3.05 e-01)	(1.4392)	(6.31 e-05)	(7.43 e-01)	(3.19 e-01)	(7.65 e-01)	(6.30 e-01)
	$\hat{\tau}$	0.2500	0.2474	0.2499	0.2514	0.2505	0.2517	0.2503
		(9.08 e-05)	(1.66 e-04)	(4.59 e-06)	(2.05 e-04)	(1.09 e-04)	(1.88 e-04)	(1.60 e-04)
	$\hat{\kappa}$	1.7003	2.0776	1.7004	1.7536	1.7701	1.7417	1.8009
		(1.78 e-01)	(1.2500)	(1.96 e-04)	(3.27 e-01)	(2.04 e-01)	(3.16 e-01)	(3.85 e-01)

6. Applications

This section illustrates the practical utility of the NEINH distribution by applying it to four real-world medical datasets. The datasets are given below.

To assess the adequacy of the NEINH distribution, its performance is compared against four competing distributions, with their CDFs defined as follows:

- The inverted Nadarajah Haghighi (INH) distribution given in (3)
- Power Inverted Nadarajah–Haghighi (PINH) Distribution [32]

$$F(x) = e^{\left\{1-\left(1+\frac{\kappa}{x^{\gamma}}\right)^{\tau}\right\}};\ x>0,\tau,\kappa,\gamma>0.$$

- Extended Odd Weibull Inverse Nadarajah Haghighi (EOWINH) Distribution [31]

$$F(x) = 1 - \left\{1 + c\left[\frac{e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}{1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}\right]^{d}\right\}^{\frac{-1}{c}};\ x>0,\tau,\kappa,\ c,\ d>0.$$

- Half Logistic Inverted Nadarajah Haghighi (HLINH) Distribution [34]

$$F(x) = \frac{1-\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]^{v}}{1+\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]^{v}};\ x>0,\tau,\kappa,\ v>0$$

- The odd Lomax inverted Nadarajah Haghighi (OLINH) distribution [33]

$$F(x) = 1 - b^{a}\left[b + \frac{e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}{1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}}\right]^{-a};\ x>0,\tau,\kappa,\ a,\ b>0.$$

- the Nadarajah–Haghighi (NH) distribution [39]

$$F(x) = 1 - e^{\{1-(1+\kappa x)^{\tau}\}};\ x>0,\tau,\kappa>0.$$

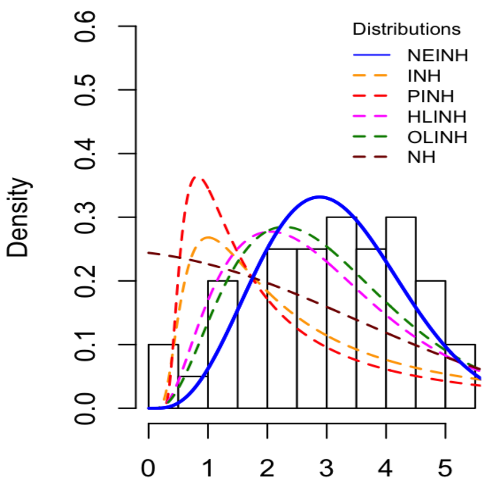
Data 1: Blood cancer

This data provides the survival duration (in years) for 40 leukemia patients treated at a Ministry of Health hospital in Saudi Arabia, as reported by [40].

0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025	2.036	2.162
2.211	2.370	2.532	2.693	2.805	2.910	2.912	3.192	3.263	3.348
3.348	3.427	3.499	3.534	3.767	3.751	3.858	3.986	4.049	4.244
4.323	4.381	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381

Table 3: Analysis of MLs and GoFs for data 1

Distributions	NEINH	INH	PINH	HLINH	OLINH	NH
Estimates	$\hat{\vartheta} = 4.12e{+}04$					
	$\hat{\tau} = 0.2418$	$\hat{\tau} = 0.9907$	$\hat{\tau} = 18.8841$	$\hat{\tau} = 0.5688$	$\hat{\tau} = 0.3483$	$\hat{\tau} = 5.4006$
	$\hat{\kappa} = 8.85e{+}04$	$\hat{\kappa} = 2.0420$	$\hat{\kappa} = 0.0537$	$\hat{\kappa} = 18.0788$	$\hat{\kappa} = 541.9925$	$\hat{\kappa} =0.0451$
			$\hat{\gamma} = 0.7218$	$\hat{v} = 9.0194$	$\hat{a} =189.8362$	
					$\hat{b} =1.4510$	
$-\ell$	-74.0331	-91.4842	-97.3297	-74.6817	-74.2407	-77.9047
CAIC	154.7330	187.2928	201.3261	158.5064	157.6244	160.1338
AIC	154.0663	186.9685 0	200.6595	157.3636	156.4815	159.8095
BIC	159.1329	190.3462	205.7261	164.1191	163.2370	163.1872
HQIC	155.8982	188.1898	202.4914	159.8062	158.9241	161.0308
K-S	0.1178	0.3129	0.2942	0.1674	0.1648	0.2786
p-value	0.6345	0.0007	0.0019	0.2119	0.2268	0.0040



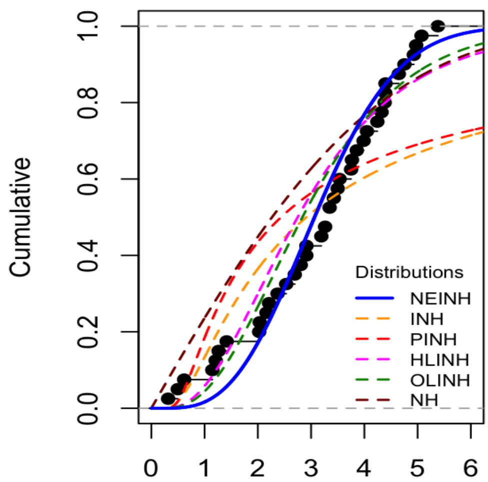


Figure 3: The NEINH distribution is evaluated against other distributions using data 1. (Right): CDFs of the distributions. (Left): observed versus expected frequencies for each distribution.

Data 2: Head and Neck cancer

The survival data of 44 patients diagnosed with head and neck cancer are examined.

The data were originally reported by [41] and subsequently analyzed by [42].

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
63.47	68.46	78.26	74.47	81.43	84	92	94	110	112
119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519
633	725	817	1776						

Table 4: Analysis of MLs and GoFs for data 2

Distributions	NEINH	INH	PINH	HLINH	OLINH	NH
Estimates	$\hat{\vartheta} = 5.0284$					
	$\hat{\tau} = 0.3737$	$\hat{\tau} = 0.8516$	$\hat{\tau} = 0.4465$	$\hat{\tau} = 0.7329$	$\hat{\tau} = 155.4218$	$\hat{\tau} = 0.6877$
	$\hat{\kappa} = 2599.9991$	$\hat{\kappa} = 99.7859$	$\hat{\kappa} = 2103.9957$	$\hat{\kappa} = 129.6323$	$\hat{\kappa} = 0.1205$	$\hat{\kappa} =0.0086$
			$\hat{\gamma} = 1.4338$	$\hat{\upsilon} = 1.6545$	$\hat{a} =2.4432$	
					$\hat{b} =16.7024$	
$-\ell$	-277.4739	279.3727	-277.7396	-277.5251	-277.5157	-280.8588
CAIC	561.5477	563.0380	562.0793	561.6501	564.0570	566.0103
AIC	560.9477	562.7453	561.4793	561.0501	563.0314	565.7177
BIC	566.3003	566.3137	566.8319	566.4027	570.1681	569.2860
HQIC	562.9327	564.0686	563.4643	563.0351	565.6780	567.0410
K-S	0.0552	0.0758	0.0646	0.0645	0.0627	0.1051
p-value	0.9982	0.9450	0.9870	0.9873	0.9907	0.6765

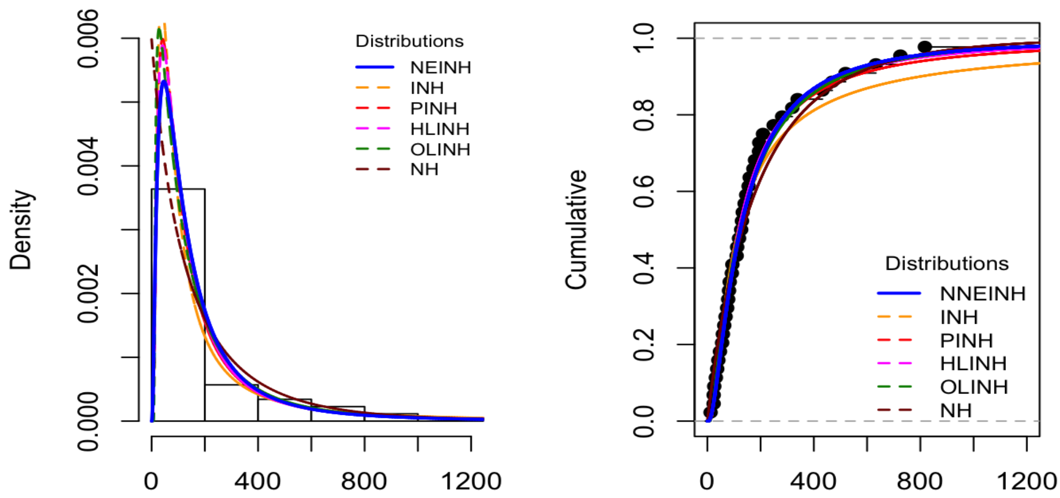


Figure 4: The NEINH distribution is evaluated against other distributions using data 2. (Right): CDFs of the distributions. (Left): observed versus expected frequencies for each distribution.

Data 3: Acute bone cancer

The survival times, measured in days, of 73 patients diagnosed with acute bone cancer are provided in, [43]:

0.09	0.76	1.81	1.10	3.72	0.72	2.49	1.00	0.53	0.66
31.61	0.60	0.20	1.61	1.88	0.70	1.36	0.43	3.16	1.57
4.93	11.07	1.63	1.39	4.54	3.12	86.01	1.92	0.92	4.04
1.16	2.26	0.20	0.94	1.82	3.99	1.46	2.75	1.38	2.76
1.86	2.68	1.76	0.67	1.29	1.56	2.83	0.71	1.48	2.41
0.66	0.65	2.36	1.29	13.75	0.67	3.70	0.76	3.63	0.68
2.65	0.95	2.30	2.57	0.61	3.93	1.56	1.29	9.94	1.67
1.42	4.18	1.37							

Table 5: Analysis of MLs and GoFs for data 3

Distributions	NEINH	INH	PINH	HLINH	OLINH	NH
Estimates	$\hat{\vartheta} = 3.9488$					
	$\hat{\tau} = 0.4440$	$\hat{\tau} = 0.7596$	$\hat{\tau} = 0.3254$	$\hat{\tau} = 0.6073$	$\hat{\tau} = 0.4302$	$\hat{\tau} = 0.5223$
	$\hat{\kappa} = 16.7693$	$\hat{\kappa} = 1.6226$	$\hat{\kappa} = 9.2579$	$\hat{\kappa} = 2.9132$	$\hat{\kappa} = 24.8730$	$\hat{\kappa} =1.04362$
			$\hat{\gamma} = 1.8333$	$\hat{v} = 2.0633$	$\hat{a} = 1.7242$	
					$\hat{b} =0.2033$	
$-\ell$	-141.0170	-146.8324	-142.1592	-142.2841	-141.2973	-154.0040
CAIC	288.3818	297.8363	290.6663	290.9160	291.1828	312.1795
AIC	288.0340	297.6649	290.3184	290.5682	290.5946	312.0080
BIC	294.9053	302.2458	297.1898	297.4396	299.7564	316.5889
HQIC	290.7723	299.4904	293.0568	293.3065	294.2457	313.8336
K-S	0.0880	0.1505	0.0916	0.0926	0.0926	0.1825
p-value	0.6225	0.0730	0.5718	0.5571	0.5575	0.0153

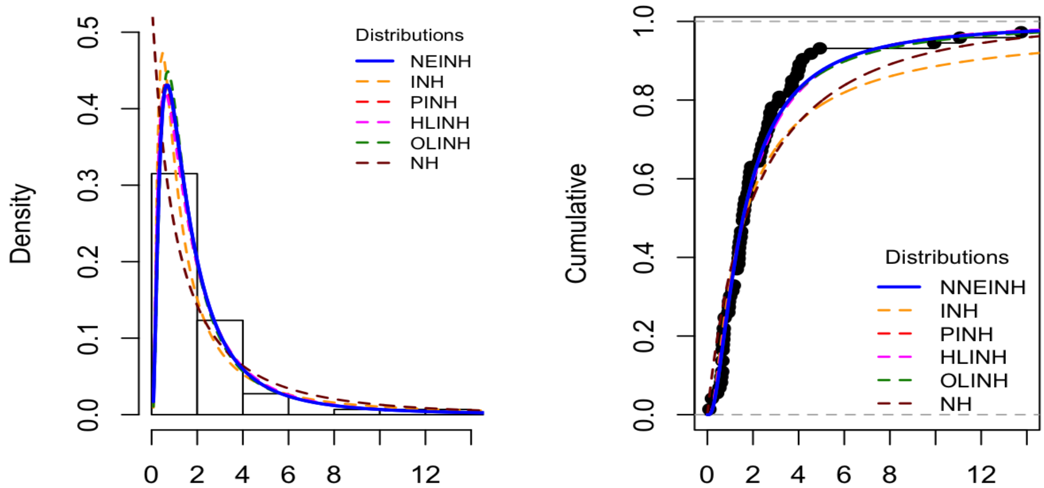


Figure 5: The NEINH distribution is evaluated against other distributions using data 3. (Right): CDFs of the distributions. (Left): observed versus expected frequencies for each distribution.

Data 4: Pain Relief Times

This dataset contains the pain relief times, recorded in minutes, for 20 patients who received an analgesic. Originally reported by [44], the data has also been reanalyzed by [45] and [46].

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

Table 6: Analysis of MLs and GoFs for data 4

Distributions	NEINH	INH	PINH	HLINH	OLINH	NH
Estimates	$\hat{\vartheta} = 6.7462$					
	$\hat{\tau} = 21.9181$	$\hat{\tau} = 88.9201$	$\hat{\tau}$ =0.3344	$\hat{\tau} = 36.5257$	$\hat{\tau} = 0.4381$	$\hat{\tau} = 55.0732$
	$\hat{\kappa} = 0.1001$	$\hat{\kappa} = 0.0148$	$\hat{\kappa} = 162.9943$	$\hat{\kappa} = 0.0521$	$\hat{\kappa} = 231.2687$	$\hat{\kappa}$ =0.0069
			$\hat{\gamma} = 6.4866$	$\hat{v} = 6.6494$	$\hat{a} = 4.910$	
					$\hat{b}$ =0.0038	
$-\ell$	-15.5056	-26.7583	-16.1470	-15.9439	-17.6737	-27.8239
CAIC	38.5113	58.2224	39.79404	39.3879	46.0140	60.3537
AIC	37.0113	57.5166	38.2940	37.8879	43.3474	59.6478
BIC	39.9985	59.5080	41.2812	40.8750	47.3303	61.6393
HQIC	37.5944	57.9053	38.8771	38.4710	44.1249	60.0365
K-S	0.0983	0.3866	0.1362	0.1292	0.1462	0.4054
p-value	0.9903	0.0050	0.8519	0.8918	0.7860	0.0027

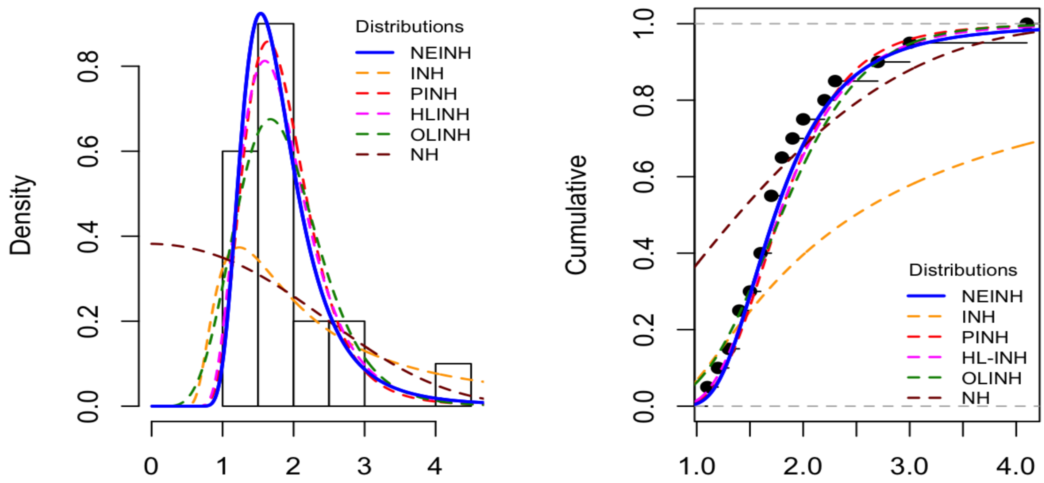


Figure 6: The NEINH distribution is evaluated against other distributions using the data 4. (Right): CDFs of the distributions. (Left): observed versus expected frequencies for each distribution.

The parameters for each distribution were estimated using ML, and their respective log-likelihoods were obtained. To evaluate the NEINH distribution's performance, several goodness-of-fit (GoF) tests were applied, such as the corrected Akaike information criterion (CAIC), Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and the Kolmogorov-Smirnov (K-S) test. The K-S test also provided the corresponding *p-value*. The optimal model is identified by the lowest values for these criteria and the highest *p-value*.

MLEs of the parameters, along with the log-likelihood values and GoF statistics for each model, are shown in Tables (3-6). These results suggest that the NEINH distribution consistently provides the lowest values for CAIC, AIC, BIC, HQIC, and K-S statistics while delivering the most favorable *p-values* when compared to the other models. Furthermore, Figures (3-6) illustrate that the NEINH distribution closely fits the observed data. Thus, compared to other models, NEINH is the best-fitting distribution for the datasets under analysis.

7. Concluding Remarks

The Exponential Inverted Nadarajah–Haghighi (NEINH) distribution, derived from the NLTE-X family, is introduced in this article to offer enhanced flexibility in analyzing real-world data. Its adaptable hazard rate function allows it to model a variety of hazard behaviours, making it applicable in fields like medicine, biology, and engineering. To estimate its parameters, seven methods ML, MPS, Bayesian, OLS, WLS, CM, and AD were employed. A comprehensive simulation study revealed that Bayesian performed the best, as indicated by the lowest MSEs. Additionally, the NEINH distribution was tested on four medical datasets, where it provided a better fit compared to competing models. The NEINH distribution is expected to play a significant role in advancing research in lifetime and survival analysis.

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Appendix

A. Proof (1)

We need to find the linear representation of NEINH's PDF which is presented in (6).

The derivation begins with the exponential expansion with a negative exponent is given in (9)

By applying (9) into term $e^{\vartheta\left\{e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right\}}$ in (6), the NEINH's PDF is expressed as

$$f(x) = \sum_{\mathfrak{c}_1=0}^{\infty} \frac{(-1)^{\mathfrak{c}_1}\vartheta^{\mathfrak{c}_1}}{\mathfrak{c}_1!}\tau\kappa x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{(1+\mathfrak{c}_1)\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]\right\}. \tag{45}$$

By applying (11) into term $\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]\right\}$ in (45), we got the following.

$$\left\{1+\vartheta\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]\right\} = \sum_{\mathfrak{c}_2=0}^{1}\binom{1}{\mathfrak{c}_2}\vartheta^{\mathfrak{c}_2}\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]^{\mathfrak{c}_2} \tag{46}$$

Again, the following binomial theorem is used.

$$(1-x)^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^i x^i. \tag{47}$$

By applying (47) into term $\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]$ in (46), we get the following.

$$\left[1-e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]^{\mathfrak{c}_2} = \sum_{\mathfrak{c}_3=0}^{c_2}\binom{c_2}{\mathfrak{c}_3}(-1)^{c_3}\left[e^{\left\{1-\left(1+\frac{\kappa}{x}\right)^{\tau}\right\}}\right]^{\mathfrak{c}_3} \tag{48}$$

After the simplification in (46) and (48), the PDF of NEINH is written as the following.

$$f(x) = \sum_{\mathfrak{c}_1=0}^{\infty}\sum_{\mathfrak{c}_2=0}^{1}\sum_{\mathfrak{c}_3=0}^{\mathfrak{c}_2}\frac{(-1)^{(\mathfrak{c}_1+\mathfrak{c}_3)}\vartheta^{(\mathfrak{c}_1+\mathfrak{c}_3)}\tau\kappa}{\mathfrak{c}_1!}\binom{1}{\mathfrak{c}_2}\binom{\mathfrak{c}_2}{\mathfrak{c}_3}e^{(1+\mathfrak{c}_1+\mathfrak{c}_3)}x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1}e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}}, \tag{49}$$

We can put the constant in one symbol (η). , the PDF of NEINH is simplified as follows.

$$f(x) = \eta\tau\kappa x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}},$$

where

$$\eta = \sum_{\mathfrak{c}_1=0}^{\infty}\sum_{\mathfrak{c}_2=0}^{1}\sum_{\mathfrak{c}_3=0}^{\mathfrak{c}_2} \frac{(-1)^{(\mathfrak{c}_1+\mathfrak{c}_3)}\vartheta^{(\mathfrak{c}_1+\mathfrak{c}_3)}}{\mathfrak{c}_1!}\binom{1}{\mathfrak{c}_2}\binom{\mathfrak{c}_2}{\mathfrak{c}_3} e^{(1+\mathfrak{c}_1+\mathfrak{c}_3)}$$

B. Proof (2)

The r^{th} moment of $X \sim$ NEINH can be derived as follows:

$$\mu_r = E\left(x^r\right) = \int_0^{\infty} x^r f(x)dx \tag{50}$$

$$= \eta\int_0^{\infty} \tau\kappa x^r x^{-2}\left(1+\frac{\kappa}{x}\right)^{\tau-1} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)\left(1+\frac{\kappa}{x}\right)^{\tau}} dx. \tag{51}$$

Substitute x in (51) by $y = \left(1+\frac{\kappa}{x}\right)^{\tau}$ as follows.

$$y = \left(1+\tfrac{\kappa}{x}\right)^{\tau} \qquad dy = \tau\left(1+\tfrac{\kappa}{x}\right)^{\tau-1}\kappa x^{-2}dx$$

$$y^{1/\tau} = 1+\tfrac{\kappa}{x} \qquad dx = \tfrac{dyx^2}{\tau\kappa}\left(1+\tfrac{\kappa}{x}\right)^{\tau-1}$$

$$x = -\kappa\left(1-y^{1/\tau}\right)^{-1}$$

Therfore,

$$\mu_r = \eta\int_0^{\infty}(-1)^r\kappa^r\left(1-y^{1/\tau}\right)^{-r} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)y}dy. \tag{52}$$

By using the binomial theorem (53) into term $\left(1-y^{1/\tau}\right)^{-r}$ in (52) as follows.

$$(1-x)^{-n} = \sum_{i=0}^{\infty}\binom{n+1-i}{i}x^i. \tag{53}$$

$$(1-y^{1/\tau})^{-r} = \sum_{c_4=0}^{\infty}\binom{r+1-c_4}{i}y^{\frac{c_4}{\tau}}. \tag{54}$$

Therfore,

$$\mu_r = \eta(-1)^r\kappa^r\sum_{c_4=0}^{\infty}\binom{r+1-c_4}{i}\frac{1}{(1+\mathfrak{c}_1+\mathfrak{c}_3)^{\frac{c_4}{\tau}+1}}\int_0^{\infty}\left[(1+\mathfrak{c}_1+\mathfrak{c}_3)y\right]^{\frac{c_4}{\tau}+1-1} e^{-(1+\mathfrak{c}_1+\mathfrak{c}_3)y}(1+\mathfrak{c}_1+\mathfrak{c}_3)dy.$$

Therefore, by using the gamma function, the μ_r is written as follows.

$$\mu_r = \eta \sum_{\mathfrak{c}_4}^{\infty} (-1)^r \kappa^r \binom{r+\mathfrak{c}_4-1}{\mathfrak{c}_4} \frac{\Gamma(\frac{\mathfrak{c}_4}{\tau}+1)}{(1+\mathfrak{c}_1+\mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}, \quad \frac{\mathfrak{c}_4}{\tau}+1>0, \tag{55}$$

where η is given by (13).

The moment-generating function (MGF) and the characteristic function of the NEINH are derived by substituting the μ_r in (56) and (57) by (55).

$$M_X(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r. \tag{56}$$

$$\phi_x(t) = E(e^{itx}) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mu_r. \tag{57}$$

Therefore, the moment-generating function, $M_X(t)$ and $\phi_x(t)$ of the NEINH are written as follows.

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{\mathfrak{c}_4}^{\infty} \eta(-1)^r \kappa^r \binom{r+\mathfrak{c}_4-1}{\mathfrak{c}_4} \frac{\Gamma(\frac{\mathfrak{c}_4}{\tau}+1)}{(1+\mathfrak{c}_1+\mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}, \quad \frac{\mathfrak{c}_4}{\tau}+1>0,$$

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{\mathfrak{c}_4}^{\infty} \eta(-1)^r \kappa^r \binom{r+\mathfrak{c}_4-1}{\mathfrak{c}_4} \frac{\Gamma(\frac{\mathfrak{c}_4}{\tau}+1)}{(1+\mathfrak{c}_1+\mathfrak{c}_3)^{\frac{\mathfrak{c}_4}{\tau}+1}}, \quad \frac{\mathfrak{c}_4}{\tau}+1>0,$$

where η is given by (13)

C. R Code

```
# NEINH
# PDF
dNEINH=function(x,pars ){
  th=pars[1]   ;a=pars[2]   ;b=pars[3]
  p1=(1+(b/x))^a
  p11=(1+(b/x))^(a-1)
  p2=exp(1-p1)
  p3=1-p2
  p4=exp(th*p2)
  p5=1+(th*p3)
```

```
  pdf=a*b*x^(-2)*p11*p2*(1/p4)*p5
  return(pdf)
}
# CDF
pNEINH=function(q,th,a,b ){
  p1=(1+(b/q))^a
  p2=exp(1-p1)
  p5=exp(th*p2)
  cdf=1-((1-p2)/p5)
  return(cdf)
}
# Quantile
qNEINH=function(p,pars ){
  th=pars[1]   ;a=pars[2]   ;b=pars[3]
  u=runif(p,0,1)
  z=th*exp(th)*(1-u)
  y=log(1-(lambertW0(z)/th))
  f=(1-y)^(1/a)
  k=(f-1)^(-1)
  x=b*k
  return(x)}
# hazard rate function h(x)
hNEINH=function(x,th,a,b){#qEIW
  h=dNEINH(x,th,a,b)/(1-pNEINH(x,th, a,b))
  return(h)
}
# log likelihood function
lh_NEINH=function(par,x){
  L=sum(log(dNEINH(x,par[1], par[2],par[3])))
  return(-L)}
############################################################################
#In this program Bayesian methods are used to estimation the parameters #
############################################################################
## Define the Prior distributions:

prior_MO=function(pars)
{
  th=pars[1]   ;a=pars[2]   ;b=pars[3]
  prior_th=dgamma(th,5,0.20,log=TRUE)#5
  prior_a=dgamma(a,5,0.20,log=TRUE)#5
  prior_b=dgamma(b,5,0.20,log=TRUE)#5
  return(prior_th + prior_a +prior_b)
}
```

```
## Dfined the functions
################################
##1: The likelihood function of  for Bayesian method
Bayes.likl=function(pars,data)
{
  x=data
  prior= prior_MO(pars)
  log_likelihood=sum(log(dNEINH(x,pars)))
  return(log_likelihood + prior)
}
#################################################################
#################################################################
Estimate0=array(c(0),dim=c(3,3,4),dimnames=list(c("theta","alpha","beta"),
                                                 c("Estimate","Bias", "MSE")
                                                 ,c("n=30","n=100","n=200","n=
n=30  ##  sample size
m=1000##  number of iteration
conv=c();mess=c()
thB0=c();aB0=c();bB0=c()
################################################################
param=c(1.6,  0.5,  0.1)
set.seed(123456789)
sample=qNEINH(n*m,param)
x= matrix(sample, nrow = n, ncol=m)
for(i  in  1:m){
  x2=x[,i]
  mcmc_r=Metro_Hastings(li_func=Bayes.likl,pars=c(1.6,  0.5,  0.1),
                        par_names=c("theta"  ,"alpha"  ,"beta" ),data=x2)
  thB0[i]=mean(mcmc_r$trace[,1])
  aB0[i]=mean(mcmc_r$trace[,2])
  bB0[i]=mean(mcmc_r$trace[,3])
  print(i)}
```