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Connected Degree Equitable Domination in Graphs

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Abstract. Let G be a connected graph. A subset $S \subseteq V(G)$ is an equitable dominating set in G if for each vertex not in S there exists $u \in S$ such that $uv \in E(G)$ and $|\deg_G u - \deg_G v| \leq 1$. An equitable dominating set $S \subseteq V(G)$ is called a connected equitable dominating set of G if the subgraph $\langle S \rangle$ induced by S is connected . The minimum cardinality of such connected equitable dominating sets in G is called the *connected equitable domination number* of G and is denoted by $\gamma^{ce}(G)$.
This paper investigates the connected equitable domination in the join and corona of graphs. The connected equitable dominating sets in the join and corona of graphs are characterized and, as direct consequences, the connected equitable domination numbers of these graphs are obtained. In addition, a n exact value of some families of graphs and a realization problem are established.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Equitable dominating set, connected equitable dominating set, equitable domination number, connected equitable domination number

1. Introduction

Throughout this paper, we consider simple, finite and undirected graphs $G = (V(G), E(G))$. For a subset $S \subseteq V(G)$, the symbol $|S|$ refers to the cardinality of S. In particular, $|V(G)|$ is the *order* of G. Let G be a connected graph. If the pair $e = [u, v]$ is in $E(G)$, then e is an *edge* in G, and e is said to *join* u and v. In this case, it is customary to write $e = uv$ and say that u and v are *adjacent*, while u and e are *incident*, as v and e are. The *degree* of a vertex $v \in V(G)$, denoted by $deg_G(v)$, is the number of incident edges to v. Adjacent vertices are also called *neighbors*. For $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v)$ consisting of all vertices adjacent to v. Thus, $\deg_G(v) = |N_G(v)|$. The *maximum and minimum degree* of G are denoted by $\Delta(G)$ and $\delta(G)$, that is, $\Delta(G) = \max\{\deg_G(u) : u \in V(G)\}$ and $\delta(G) = \min\{\deg_G(u) : u \in V(G)\}$. The subgraph $\langle S \rangle$ of G induced by a subset S of $V(G)$ is the graph having vertex set S

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and whose edge set consists of those edges of G incident with two elements of G. A graph is called *connected* if every two vertices are joined by a path; otherwise, it is *disconnected*.

In graph theory, domination in graphs is one of the most extensively studied concepts in graph theory with numerous variants that model influence, control and coverage, in networked systems. We may refer to [1–5] to see some of the recent studies in domination, and to [6–8] for its various applications, such as Electrical Power Networks, Communication Network, Facility Location Problem, Land Surveying, and Routings. A subset $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. Traditional domination parameters focus on identifying minimal sets of vertices that dominate or control the entire graph structure. However, in many real-world contexts—such as sensor networks, facility location, and communication systems—there is often a need to ensure both fairness in vertex degrees and connectivity among dominating vertices. To address this, the notion of equitable domination has been introduced. We may refer to [9–13] to see some of the studies in equitable domination. A dominating set $S \subseteq V(G)$ is called a *(degree) equitable dominating set* in G if for every $v \in V(G) \setminus S$ there exists $u \in S$ such that $|deg_G(u) - deg_G(v)| \leq 1$. The *equitable domination number* of G, denoted by $\gamma^e(G)$, is the smallest cardinality among all equitable dominating set in G. The notion of equitable domination imposed a balance condition, that is, the sizes of the open neighborhoods of the dominating vertices must be as equal as possible, minimizing the difference in coverage. Some of the studies Among the many variations of domination is the connected domination which requires that the subgraph $\langle S \rangle$ induced by dominating set S is connected, reflecting situations where communication or cohesion among controlling elements is essential. We may refer to [14, 15] to see some of the studies in connected domination. Combining these two constraints "connectedness" and equitability or fairness gives rise to the concept of connected equitable domination. The notion of connected equitable domination has been introduced where dominating sets are not only degree-balanced but also induce connected subgraphs. An equitable dominating set $S \subseteq V(G)$ is called a *connected equitable dominating set* of G if the subgraph $\langle S \rangle$ induced by S is connected . The minimum cardinality among all connected equitable dominating sets in G is called the *connected equitable domination number* of G , and is denoted by $\gamma^{ce}(G)$. A subset $S \subseteq V(G)$ is called *minimal connected equitable dominating set* in G if no proper subset of S is a connected equitable dominating set in G.

The concept of connected equitable domination has many real life applications where both connectivity (connectedness) and fairness (equitability) in workload distribution or influence distribution are essential. Such theoretical and practical applications includes (1) *Distributed Computing and Load Balancing*: In distributed systems, assigning tasks to processing units can be modeled as a connected equitable dominating set which ensures that workload distribution is equitable, that is, nearly equal, and all coordinating vertices can communicate efficiently; (2) *Social Networks and Influence Modeling*: In scenarios involving opinion formation or influence spread, a connected equitable dominating set can represent a group of influential individuals who are both interconnected and exert roughly equal influence over the population; (3) *Urban Planning and Facility Location*:

When placing essential facilities such as hospitals, fire stations and etc. in a city modeled as a graph, a connected equitable dominating set ensures that all regions are covered by a connected and fairly distributed set of facilities. By combining the structural benefits of connectedness with the fairness of equitable load distribution, connected equitable domination provides a robust and efficient framework for designing and analyzing network systems; and (4) *Telecommunication and Sensor Networks*: In wireless sensor networks or ad hoc communication networks, a connected equitable dominating set can represent as a core framework where central vertices (dominating vertices) are connected and share the communication or monitoring load equitably. This ensures both robustness (connectivity) and balanced energy usage or data handling.

In this paper, we investigate this parameter in graphs formed through two fundamental operations: the join and the corona. We provide structural characterizations of such dominating sets in these graphs and determine exact values of $\gamma^{ce}(G)$ for various graph families. A realization problem is also addressed, offering insight into how graphs can be constructed to attain prescribed values of the parameter.

For simplicity, we use the terms γ^{ce}-set, γ^{c}-set, and γ^{e} to refer to the connected equitable dominating set with cardinality $\gamma^{ce}(G)$, connected dominating set with cardinality $\gamma^{c}(G)$ and equitable dominating set with cardinality $\gamma^{e}(G)$, respectively.

The following results are due to V. Swaminathan et.al and Sivakumar et.al

Theorem 1. [15] *The connected equitable domination number of some standard graphs are*

(*i*) *For a complete graph K_n on n vertices, $\gamma^{ce}(G) = 1$.*

(*ii*) *For the paths P_n and the cycles C_n on n vertices, $\gamma^{ce}(C_n) = \gamma^{ce}(P_n) = n - 2$.*

(*iii*) *If W_n denotes the wheel on n vertices, then*

$$\gamma^{ce}(W_n) = \begin{cases} \left\lceil \frac{n+3}{3} \right\rceil, & \text{if } n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

(*iv*) *For the complete bipartite graph $K_{m,n}$, we have,*

$$\gamma^{ce}(K_{m,n}) = \begin{cases} 1, & \text{either } m = 1 \text{ or } n = 1 \\ 2, & \text{if } |m-n| \leq 1 \text{ and } m, n \geq 2 \\ m+n, & \text{if } |m-n| \geq 2, \text{ and } m, n \geq 2. \end{cases}$$

2. Results

The following result Theorem 2 is a correction of Theorem 1 (*iv*) found in [15].

Theorem 2. *For any complete bipartite $K_{m,n}$ with $m, n \geq 1$,*

$$\gamma^{ce}(K_{m,n}) = \begin{cases} 2, & \text{if } |m-n| \leq 1 \\ m+n, & \text{if } |m-n| \geq 2. \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with m vertices in one partition say A and n vertices in another partition say B. Then,

$$\deg(u) = \begin{cases} n, & if\ u \in A \\ m, & if u \in B. \end{cases}$$

If $|m-n| \leq 1$, then any set $\{x, y\}$ with $x \in A$ and $y \in B$ is a connected equitable dominating set in $K_{m,n}$. Thus, $\gamma^{ce}(K_{m,n}) = 2$. Suppose that $|m-n| \geq 2$. Let S be a minimum connected equitable dominating set in $K_{m,n}$ and suppose that $|S| < m+n$. Then, there exists $u \in V$ which is not in S. WLOG, let $u \in B$. Then, $\deg_G(u) = m$. Since S is a connected equitable dominating set in $K_{m,n}$, there exists $v \in S$ such that u is adjacent with v and $|\deg_G(u) - \deg_G(v)| \leq 1$. Clearly, $v \in A$. Therefore $\deg_G(v) = n$. Thus, $|\deg_G(u) - \deg_G(v)| = |m-n| \geq 2$, a contradiction . Therefore, $|S| = m+n$. Consequently, $\gamma^{ce}(K_{m,n}) = m+n$ if $|m-n| \geq 2$. □

Theorem 3. *For a fan graph F_n with n vertices on a path and a single vertex u connected to each vertex in P_n,*

$$\gamma^{ce}(F_n) = \begin{cases} 1, & if\ n =: 2,3 \\ 2, & if\ n = 4 \\ \lceil \frac{n}{3} \rceil + 1, & if\ n \geq 5. \end{cases}$$

Proof. Let $G = F_n$ be a fan with n vertices on a path and a single vertex u connected to each vertex in P_n. Let $V(F_n) = \{u, v_1, v_2, \ldots, v_n\}$ where u is adjacent to every vertex v_i $(1 \leq i \leq n)$ on path P_n. When $n = 2$, $\{u\}, \{v_1\}$ and $\{v_2\}$ are minimal connected equitable dominating sets in G. Thus, $\gamma^{ce}(F_n) = 1$. When $n = 3$, since $\deg_G(u) = \deg_G(v_2) = 3$ and $\deg_G(v_1) = \deg_G(v_3) = 2$, the sets $\{u\}$ and $\{v_2\}$ are the only minimal connected equitable dominating sets in G. Thus, $\gamma^{ce}(F_n) = 1$. For $n = 4$, since the $deg(v_i)$ of any vertices v_i lying on P_4 is either 2 or 3 and $\deg_G(u) = 4$, any connected dominating set in P_4 is equitable in F_4. Thus, $\gamma_c(P_4) = \lceil \frac{n}{3} \rceil = 2 = \gamma^{ce}(F_4)$. Suppose that $n \geq 5$. Then $\deg(u) = n$ and

$$\deg_G(v_i) = \begin{cases} 2, & if\ i =: 1, n \\ 3, & if\ 2 \leq i < n. \end{cases}$$

Thus, any dominating set in P_n is not equitable in F_n since $|\deg_G(u) - \deg_G(v_i)| > 2$ for all $i = 1, 2, \ldots, n$. Now, consider $S = D \cup \{u\}$ where D is a γ^e- set in P_n. Clearly, S is a connected equitable dominating set in F_n of minimum cardinality. Therefore, $\gamma^{ce}(F_n) = \gamma^e(P_n) + 1 = \lceil \frac{n}{3} \rceil + 1$. □

Theorem 4. *For any double star graph $S_{r,s}$, where $r, s \geq 1$,*

$$\gamma^{ce}(S_{r,s}) = \begin{cases} 2, & if\ r = s = 1 \\ s+2, & if\ r = 1\ and\ s \geq 2 \\ r+2, & if\ s = 1\ and\ r \geq 2 \\ r+s+2, & if\ r, s \geq 2 \end{cases}$$

Proof. Let u and v be the two central vertices of $G = S_{r,s}$, and let U and V be the sets of all leaves adjacent to u and v, respectively, with $|U| = r$ and $|V| = s$. Then

$N_G(\{u\}) = \{v\} \cup U$ and $N_G(\{v\}) = \{u\} \cup V$ .If $r = s = 1$, then the only connected equitable dominating set in G is the set $\{u, v\}$. Thus $\gamma^{ce}(G) = 2$. Suppose that $r = 1$ and $s \geq 2$. Since every $y \in U \cup V$ is of degree 1 and $\deg_G(u) = 2$, the only connected equitable dominating set in G is the $\{u, v\} \cup V$. Thus, $\gamma^{ce}(G) = s + 2$. Similarly, if $s = 1$ and $r \geq 2$, $\gamma^{ce}(G) = r + 2$. Suppose that $r, s \geq 2$.

Let S be a connected equitable dominating set in G with $|S| = \gamma^{ce}(G)$. Suppose that $|S| < r + s + 2 = |V(G)|$ and let $x \in V \setminus S$. Then, either $x \in \{u, v\}$, $x \in U$, or $x \in V$. Suppose that x is any of the central vertices of G, say u. This means that there exists $u_j \in U$ which is an element of S, a contradiction to the assumption that S is connected. Thus, x must not be any of the central vertices. Now, either $x \in U$ or $x \in V$. Without loss of generality, let $x \in U$. Since S is a connected equitable dominating set in G, there exists $y \in S$ such that $xy \in E(G)$ and $|\deg_G(x) - \deg_G(y)| \leq 1$. Note that the only adjacent vertex of x in G is the central vertex u. Thus, $y = u$ and $|\deg_G(x) - \deg_G(y)| = |\deg_G(x) - \deg_G(u)| = |1 - (r + 1)| = r > 1$, a contradiction. Therefore, $S = V(G)$. Hence, $\gamma_e^c(G) = |S| = r + s + 2$. □

Theorem 5. *$\gamma^{ce}(G) = 1$ if and only if there exists $v \in V(G)$ such that $\deg_G(v) = |V(G)| - 1$ and $\deg_G(u) \geq |V(G)| - 2$ for all $u \neq v$ in G.*

Proof. Suppose that $\gamma^{ce}(G) = 1$. Then, there exists v in G such that $\{v\}$ is a connected equitable dominating set in G with $\deg_G(v) = |V(G)| - 1$. Now, let $u \in V(G) \setminus \{v\}$. Since $\{v\}$ is a γ^{ce}-set in G, $uv \in E(G)$ and $|\deg_G(u) - \deg_G(v)| \leq 1$. That is,

$$\begin{aligned} |\deg_G(u) - \deg_G(v)| \leq 1 &\implies |\deg_G(u) - (|V(G)| - 1) \leq 1 \\ &\implies -1 \leq \deg_G(u) - |V(G)| + 1 \leq 1 \\ &\implies |V(G)| - 2 \leq \deg_G(u) \leq |V(G)| \end{aligned}$$

Thus, $\deg_G(u) \geq |V(G)| - 2$ for all $u \neq v$ in G. Conversely suppose that there exists $v \in V(G)$ such that $\deg_G(v) = |V(G)| - 1$ and $\deg_G(u) \geq |V(G)| - 2$ for all $u \neq v$ in G. Then, $\{v\}$ is a dominating set and $|\deg_G(v) - \deg_G(u)| \leq |V(G)| - 1 - (|V(G)| - 2) \leq 1$ for all $u \neq v$ in G. Thus, $\{v\}$ is a connected equitable dominating set in G. Consequently, $\gamma^{ce}(G) = 1$. □

Remark 1. *If a subset $S \subseteq V(G)$ is a connected equitable dominating set in G and $|\deg_G(u) - \deg_G(v)| \geq 2$ for all $v \in N_G(u)$, then $u \in S$.*

Remark 2. *Every equitable dominating set is a dominating set. Thus, $\gamma(G) \leq \gamma^e(G)$.*

Remark 3. *Every connected dominating set is a dominating set. Thus, $\gamma(G) \leq \gamma^c(G)$.*

Remark 4. *Every equitable connected dominating set is a connected dominating set. Thus , $\gamma^c(G) \leq \gamma^{ce}(G)$.*

Theorem 6. *For any graph G, $\gamma(G) \leq \gamma^c(G) \leq \gamma^{ce}(G)$.*

Proof. Let S be a γ^{ce}-set in G. Then S is a connected dominating set in G. Thus, by Remark 4, $\gamma^{c}(G) \leq |S| = \gamma^{ce}(G)$. Furthermore, S is a dominating set in G. By Remark 3, the result follows. □

The next result shows that every pair of positive integers are realizable as the connected domination number and connected equitable domination number of a connected graph.

Theorem 7. *For every positive integers a and b with $1 \leq a \leq b$ there exists a connected graph G such that $\gamma^{c}(G) \leq \gamma^{ce}(G)$.*

Proof. Suppose that $a = b$. Write $V(K_{1,a-1}) = \{x, u_1, u_2, \ldots, u_{a-1}\}$ as in Figure 1. Obtain G from $K_{1,a-1}$ by adding pendant edges $v_j u_j$, $j = 1, 2, \ldots, a-1$ as shown in Figure 1.

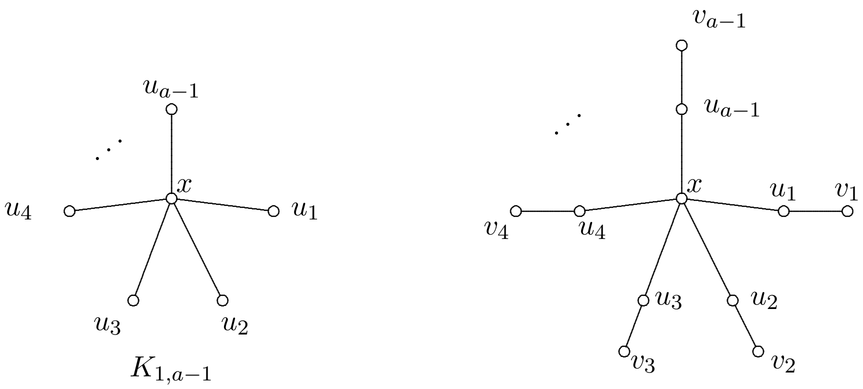

Figure 1: G obtained from $K_{1,a-1}$

Observe that the set $S = \{x, u_1, u_2, \ldots, u_{a-1}\}$ is the only minimal connected equitable dominating set, and a minimal connected dominating set of G. Thus, $\gamma^{c}(G) = |S| = a - 1 + 1 = a = b = \gamma^{ce}(G)$.

Suppose that $b = a + 1$. Obtain $G = G_1$ from $K_{1,a-1}$ by adding pendant edges $v_j u_j$, $j = 2, \ldots, a-1$, and joining the path $P_3 = [w_1, v_1, h_1]$ to vertex u_1 of $K_{1,a-1}$ as in Figure 2.

Observe that the set $\{x, u_1, u_2, \ldots, u_{a-1}\}$ is the only minimal connected dominating set of G. Thus $\gamma^{c}(G) = a-1+1 = a$. Also, note that the sets $S_1 = \{x, u_1, u_2, \ldots, u_{a-1}, v_1\}$ and $S_2 = \{x, u_1, u_2, \ldots, u_{a-1}, w_1, h_1\}$ are the only minimal connected equitable dominating sets of G. Since $S_1 = a - 1 + 1 + 1 = a + 1$ and $|S_2| = a - 1 + 1 + 2 = a + 2$, we have $\gamma^{ce}(G) = |S_1| = a - 1 + 1 + 1 = a + 1 = b = \gamma_e^{c}(G)$.

Suppose that $b = a + k$ for $k \geq 2$. Obtained $G = G_2$ from G_1 by joining $k - 1$ path $P_{3_j} = [w_j, v_j, h_j]$ to vertex u_1 of G_1 where $2 \leq j \leq k$ as in Figure 3.

Observe that the set $\{x, u_1, u_2, \ldots, u_{a-1}\}$ is the only minimal connected dominating set of G. Thus $\gamma^{c}(G) = a-1+1 = a$. Also, note that the sets $S_1 = \{x, u_1, u_2, \ldots, u_{a-1}\} \cup \left(\bigcup_{j=1}^{k} \{v_j\}\right)$ and $S_2 = \{x, u_1, u_2, \ldots, u_{a-1}\} \cup \left(\bigcup_{j=1}^{k} \{w_j, h_j\}\right)$ are the only minimal connected

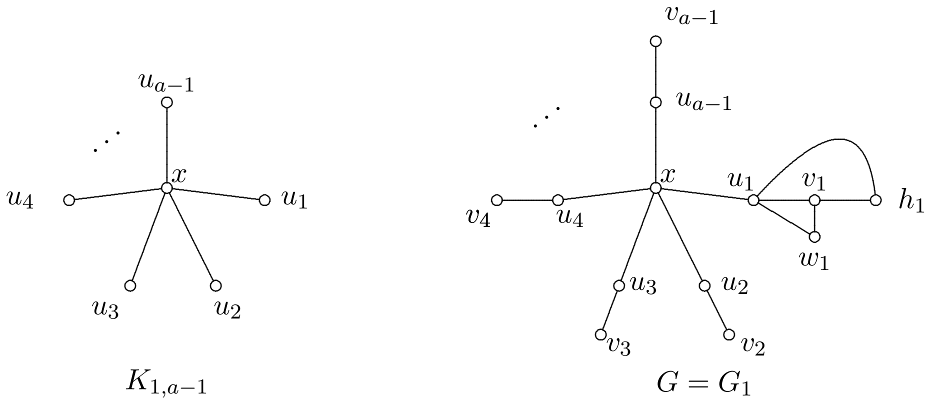

Figure 2: G_1 obtained from $K_{1,a-1}$

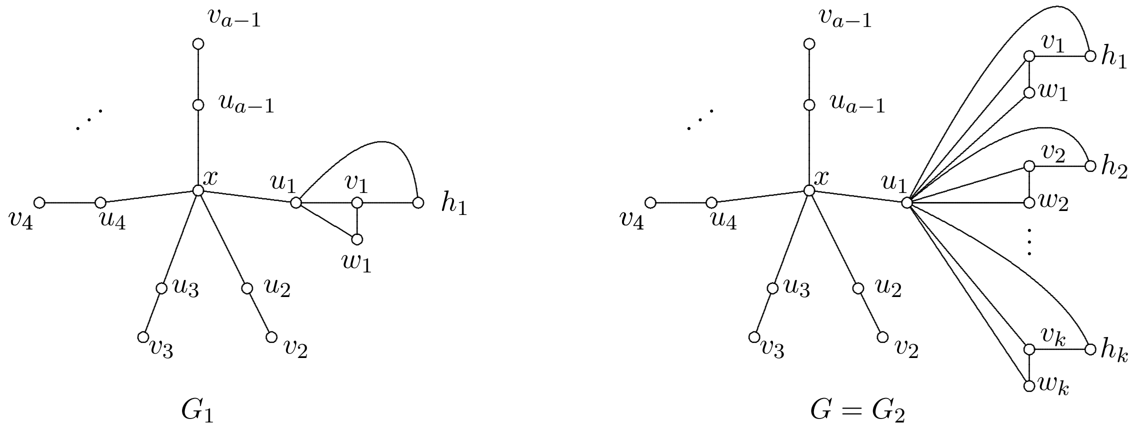

Figure 3: G_2 obtained from G_1

equitable dominating sets of G. Now, $|S_1| = a-1+1+k = a+k$ and $|S_2| = a-1+2k = a+2k-1$. Since $k > 1$, $|S_2| = a+2k-1 > a+k = |S_1|$. Thus, $\gamma^{ce}(G) = |S_1| = a+k = b > \gamma^c(G) = a$. $\square$

Theorem 8. *For any regular graph G, $\gamma^{ce}(G) = \gamma^c(G)$.*

Proof. Let $S \subseteq V(G)$ be a γ^c-set of G and $u \in V(G) \setminus S$. Since S is a dominating set in G, there exists $v \in S$ such that $uv \in E(G)$. Since G is a regular graph, say n regular, $\deg_G(x) = n$ for all $x \in V(G)$. Thus, $|\deg_G(u) - \deg_G(v)| = 0 < 1$ implying that S is an equitable set in G. Thus S is a connected equitable dominating set in G. Hence, $\gamma^{ce}(G) \le |S| = \gamma^c(G)$. From Remark 4, $\gamma^{ce}(G) = \gamma^c(G)$. $\square$

3. The Join of Graphs

The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 9. *Let G be a graph such that $\Delta(G) \leq |V(G) - 3|$. Then $S \subseteq V(K_1 + G)$ is a connected equitable dominating set of $K_1 + G$ if and only if $S = V(K_1) \cup S_G$ where S_G is an equitable dominating set of G.*

Proof. Suppose S is a connected equitable dominating set of $K_1 + G$. Since $\Delta(G) \leq |V(G) - 3|$ and $\deg_{K_1+G}(v) = |V(G)|$, where $v \in V(K_1)$, it follows that $v \in S$. Let $S_G = V(G) \cap S$. Since $\{v\}$ is not an equitable dominating set of $K_1 + G$, $S_G \neq \varnothing$. Let $u \in V(G) \setminus S_G$. Then $\exists w \in S$ such that $|\deg_{K_1+G}(w) - \deg_{K_1+G}(u)| \leq 1$. Since $\deg_{K_1+G}(u) = 1 + \deg_G(u) \leq 1 + |V(G)| - 3 = |V(G)| - 2$ and $\deg_{K_1+G}(v) = |V(G)|$, $w \neq v$. Thus, $w \in S_G$ and $|\deg_G(w) - \deg_G(u)| \leq 1$. This implies that S_G is an equitable dominating set of G. The converse is clear. □

Corollary 1. *Let G be a graph such that $\Delta(G) \leq |V(G)| - 3$. Then*

$$\gamma^{ce}(K_1 + G) = 1 + \gamma^e(G).$$

Theorem 10. *Let G be a graph such that $\Delta(G) \geq n - 2$, where $n = |V(G)|$. Then S is an equitable connected dominating set of $K_1 + G$ if and only if it satisfies one of the following statements:*

(*i.*) *$S \subseteq V(G)$ and is a connected equitable dominating set of G.*

(*ii.*) *$S = V(K_1) \cup S_G$ such that $S_G \subseteq V(G)$ equitably dominates $V(G) \setminus (D_G \cup S_G)$ where $D_G = \{z \in V(G) \setminus S_G : \deg_G(z) \geq n - 2\}$.*

Proof. Suppose S is a connected equitable dominating set of $K_1 + G$. Let $v \in V(K_1)$ and set $S_G = V(G) \cap S$. Suppose first that $v \notin S$. Then $S \subseteq V(G)$. Since S is a connected equitable dominating set of $K_1 + G$, S is a connected equitable dominating set of G. Next, suppose that $v \in S$. Let $z \in V(G) \setminus D_G$. Then $\deg_G(z) \leq n-3$, that is, $\deg_{K_1+G}(z) \leq n-2$. Since $\deg_{K_1+G}(v) = n$ and S is a connected equitable dominating set of K_1+G, there exists $y \in S_G$ such that $|\deg_G(y) - \deg_G(z)| \leq 1$. This implies that every element of $V(G) \setminus D_G$ is equitably dominated by S_G. For the converse, suppose that (i) holds. Then S contains a vertex $u \in V(G)$ with $\deg_G(u) \geq n-2$. Hence, $|\deg_{K_1+G}(v) - \deg_{K_1+G}(u)| \leq 1$. Further, since S is a connected equitable dominating set of G, S is a connected equitable dominating set of $K_1 + G$. Next, suppose that (ii) holds. Let $z \in D_G$. Then v equitably dominates z. Moreover, since S_G equitably dominates $V(G) \setminus D_G$, S is a connected equitable dominating set of $K_1 + G$. □

Corollary 2. *Let G be a graph such that $\Delta(G) \geq n - 2$, where $n = |V(G)|$. Then*

$$\gamma^{ce}(K_1 + G) = \min\{\gamma^{ce}(G), 1 + \gamma^e(\langle V(G) \setminus D^*_G \rangle)\}$$

where $D^*_G = \{z \in V(G) : \deg_G(z) \geq n - 2\}$.

Theorem 11. *$\gamma^{ce}(G+K_1)=1$ if and only if $\Delta(G+K_1)\leq \deg_G(v)+2$ for all v in G.*

Proof. Let G a connected graph of order n. Let $v_1, v_2, \ldots, v_n$ ne vertices in G. Suppose that $\gamma^{ce}(G+K_1)=1$. Then there exists a vertex u in $G+K_1$ which is an equitable dominating in $G+K_1$. Consider the following cases:

Case $1: u\in V(K_1)$. Then

$$\begin{aligned} 1\geq |\deg_{G+K_1}(u)-\deg_{G+K_1}(v_i)| &= |n-(\deg_G(v_i)+1)| \\ &= |(n-1)-\deg_G(v_i)|. \end{aligned}$$

Thus, $\deg_G(v_i)\geq n-2$ for all $i=1,2,\ldots,n$. Hence, $\Delta(G+K_1)=\deg_{G+K_1}(u)=n\leq \deg_G(v_i)+2$ for all v_i in G.

Case $2: If\ u\in V(G).$, then $|\deg_{G+K_1}(u)-\deg_{G+K_1}(v)|\leq 1$ for all $v\neq u$ in $G+K_1$. This means that $\deg_{G+K_1}(u)\leq \deg_{G+K_1}(v)+1$. Thus,

$$\begin{aligned} \Delta(G+K_1) &= \deg_{G+K_1}(u) \\ &\leq \deg_{G+K_1}(v)+1\ ,\ v\in V(G+K_1)\setminus\{u\} \\ &= \deg_G(v)+1+1\ ,\ v\neq u \text{ in } G \\ &= \deg_G(v)+2. \end{aligned}$$

For the converse, suppose that $\Delta(G+K_1)\leq \deg_G(v)+2$ for all v in G. Let u in $G+K_1$ be of maximum degree. If $x\in V(K_1)$, then $\deg_{G+K_1}(u)=\Delta(G+K_1)\deg_{G+K_1}(v)+1$ for all v in G. That is, $\deg_{G+K_1}(v)\geq \deg_{G+K_1}(u)-1$ for all v in G. This implies that G is a $(t,t+1)$ bi-regular graph. Take $\{u\}$. Observe that u is an equitable dominating in $G+K_1$. Thus, $\gamma^{ce}(G+K_1)=1$. Similarly, if $u\in V(G)$, $\deg_{G+K_1}(v)\geq \deg_{G+K_1}(u)-1$ for all v in G. Taking $\{u\}$ which is a connected equitable dominating set in $G+K_1$. Therefore, $\gamma^{ce}(G+K_1)=1$. □

Theorem 12. *Let G be a complete graph of order n and H be any graph of order m with $n-m=k>0$. Then*

$$\gamma^{ce}(G+H)=1+\gamma^{e}(\langle V(H\setminus D_H\rangle)$$

where $V(H)\setminus D_H=\{v\in V(H):\deg_H(v)<\Delta(G)-(k+1)=n-k-2\}$ and $D_H=\{z\in V(H):\deg_H(z)\geq\Delta(G)-(k+1)=n-k-2\}$.

Proof. Let $u\in V(G)$ and $S\subseteq V(H)\setminus D_H^*$ be a γ^e-set of $\langle V(H\setminus D_H\rangle$ where $D_H=\{z\in V(H):\deg_H(z)\geq\Delta(G)-(k+1)=n-k-2\}$. Put $C=\{u\}\cup S$. We will show that C is a γ^{ce}-set of $G+H$. Let $v\in V(G+H)\setminus C$. If $v\in V(G)$, then $v\in N_{G+H}(u)$ and

$$\begin{aligned} |\deg_{G+H}(u)-\deg_{G+H}(v)| &= |\deg_G(u)+|V(H)|-(\deg_G(v)+|V(H)|)| \\ &= |\deg_G(u)-\deg_G(v)| \\ &= (n-1)-(n-1) \\ &= 0 \\ &< 1. \end{aligned}$$

Suppose that $v \in V(H)$. If $\deg_H(v) \geq n-k-2$, the v is equitably dominated by u, that is, $v \in N_{G+H}(u)$ and

$$\begin{aligned}
|\deg_{G+H}(u) - \deg_{G+H}(v)| &= |\deg_G(u) + |V(H)| - (\deg_H(v) + |V(G)|)| \\
&= |\deg_G(u) - \deg_H(v) - (|V(G)| - |V(H)|)| \\
&= |\deg_G(u) - \deg_H(v) - (n-m)| \\
&= |\deg_G(u) - \deg_H(v) - k| \\
&< |\deg_G(u) - (n-k-2) - k| \\
&= |n-1-n+k+2-k| \\
&= 1.
\end{aligned}$$

Suppose that $\deg_H(v) < n-k-2$. Since S is an equitable dominating set in $\langle V(H \setminus D_H \rangle$ there exists $y \in S$ such that $v \in N_{G+H}(y)$ and

$$\begin{aligned}
|\deg_{G+H}(y) - \deg_{G+H}(v)| &= |\deg_H(y) + |V(G)| - (\deg_H(v) + |V(G)|)| \\
&= |\deg_H(y) - \deg_H(v)| \\
&< 1.
\end{aligned}$$

Thus, C is a connected equitable dominating set in $G+H$. Hence, $\gamma^{ce}(G+H) \leq |C|$. Suppose that $\gamma^{ce}(G+H) < |C|$. Then there exists a γ^{ce}-set in $G+H$ say A with $|A| < |C|$. Suppose that $A = C \setminus \{x\}$ with $x \in \{u\} \cup S$ with S a γ^e-set of $\langle V(H \setminus D_H^* \rangle$. If $x = u$, then for all $z \in V(G+H)$ whose degree is less than $n+k-2$ is not equitably dominated by A, a contradiction. If $x \in S$, then there exists an element $w \in \langle V(H \setminus D_H \rangle$ which nor equitably dominated by A, a contradiction. Thus, $\gamma^{ce}(G+H) = |C|$. Therefore, $\gamma^{ce}(G+H) = 1 + \gamma^e(\langle V(H \setminus D_H \rangle)$. □

Theorem 13. *Let G be a complete graph of order n and H be any graph of order m with $m - n = k \geq 0$. Then*

$$\gamma^{ce}(G+H) = 1 + \gamma^e(\langle V(H \setminus D_H^* \rangle)$$

where $V(H) \setminus D_H^* = \{v \in V(H) : \deg_H(v) < \Delta(G) + (k-1) = n+k-2\}$ and $D_H^* = \{z \in V(H) : \deg_H(z) \geq \Delta(G) + (k-1) = n+k-2\}$

Proof. Let $u \in V(G)$ and $S \subseteq V(H) \setminus D_H^*$ be a γ^e-set of $\langle V(H \setminus D_H^* \rangle$ where $D_H^* = \{z \in V(H) : \deg_H(z) \geq \Delta(G) + (k-1) = n-k-2\}$. Put $C = \{u\} \cup S$. We will show that C is a γ^{ce}-set of $G+H$. Let $v \in V(G+H) \setminus C$. If $v \in V(G)$, then $v \in N_{G+H}(u)$ and

$$\begin{aligned}
|\deg_{G+H}(u) - \deg_{G+H}(v)| &= |\deg_G(u) + |V(H)| - (\deg_G(v) + |V(H)|)| \\
&= |\deg_G(u) - \deg_G(v)| \\
&= (n-1) - (n-1) \\
&= 0 \\
&< 1.
\end{aligned}$$

Suppose that $v \in V(H)$. If $\deg_H(v) \geq n+k-2$, the v is equitably dominated by u, that is, $v \in N_{G+H}(u)$ and

$$\begin{aligned}
|\deg_{G+H}(u) - \deg_{G+H}(v)| &= |\deg_G(u) + |V(H)| - (\deg_H(v) + |V(G)|)| \\
&= |\deg_G(u) - \deg_H(v) + (|V(H)| - |V(G)|)| \\
&= |\deg_G(u) - \deg_H(v) + (n-m)| \\
&= |\deg_G(u) - \deg_H(v) + k| \\
&< |\deg_G(u) - (n+k-2) + k| \\
&= |n-1-n-k+2+k| \\
&= 1.
\end{aligned}$$

Suppose that $\deg_H(v) < n+k-2$. Since S is an equitable dominating set in $\langle V(H \setminus D_H^*\rangle$ there exists $y \in S$ such that $v \in N_{G+H}(y)$ and

$$\begin{aligned}
|\deg_{G+H}(y) - \deg_{G+H}(v)| &= |\deg_H(y) + |V(G)| - (\deg_H(v) + |V(G)|)| \\
&= |\deg_H(y) - \deg_H(v)| \\
&< 1.
\end{aligned}$$

Thus, C is a connected equitable dominating set in $G+H$. Hence, $\gamma^{ce}(G+H) \leq |C|$. Suppose that $\gamma^{ce}(G+H) < |C|$. Then there exists a γ^{ce}-set in $G+H$ say A with $|A| < |C|$. Suppose that $A = C \setminus \{x\}$ with $x \in \{u\} \cup S$ with S a γ^e-set of $\langle V(H \setminus D_H^*\rangle$. If $x = u$, then for all $z \in V(G+H)$ whose degree is less than $n+k-2$ is not equitably dominated by A, a contradiction. If $x \in S$, then there exists an element $w \in \langle V(H \setminus D_H^*\rangle$ which nor equitably dominated by A, a contradiction. Thus, $\gamma^{ce}(G+H) = |C|$. Therefore, $\gamma^{ce}(G+H) = 1 + \gamma^e(\langle V(H \setminus D_H^*\rangle)$. □

4. Corona of Graphs

Let G and H be graphs of order m and n, respectively. The corona $G \circ H$ of G and H is the graph obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. For every$v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$.

Theorem 14. *Let G be a connected non-trivial graph and H be any graph of order n. The $C \subseteq V(G \circ H)$ is a connected equitable dominating set of $G \circ H$ if and only if $C = V(G) \cup (\bigcup_{v \in V(G)} B_v)$, where for each $v \in V(G)$,*

(i.) B_v is an equitable dominating set of H^v whenever $\Delta(H) \leq n-2$ or $\deg_G(v) \geq 2$ and

(ii.) B_v dominates equitably $V(H^v) \setminus S_v$, where $S_v = B_v \cup \{x \in V(H^v) : \deg_H(x) = n-1\}$ whenever $\deg_G(v) = 1$ and $\Delta(H) = n-1$.

Proof. Suppose that C is a connected equitable dominating set of $G \circ H$. Let $A = C \cap V(G)$ and let $B_v = C \cap V(H^v)$ for each $v \in V(G)$. Since $\langle C \rangle$ is connected subgraph of $G \circ H$, $\langle A \rangle$ is a connected subgraph of G. If there exists $v \in V(G) \setminus A$, then $B_v \neq \varnothing$ since C is a dominating set of $G \circ H$. This implies that $\langle C \rangle$ is not connected, contrary to our assumption that C is a connected dominating set of $G \circ H$. Thus, $A = V(G)$. Next, let $v \in V(G)$ and let $q \in V(H^v) \setminus B_v$. Then there exists $w \in C \cap N_{G \circ H}(q)$ such that $|\deg_{G \circ H}(w) - \deg_{G \circ H}(q)| \leq 1$. If $\Delta(H) \leq n-2$, then $\deg_{G \circ H}(v) \geq n+1$ and $\deg_{G \circ H}(q) \leq n-1$. If $\deg_G(v) \geq 2$, then $\deg_{G \circ H}(v) \geq n+2$ and $\deg_{G \circ H}(q) \leq n$. Hence, $w \neq v$ whenever $\Delta(H) \leq n-2$ or $\deg_G(v) \geq 2$. It follows that $w \in B^v \cap N_{H^v}(q)$ and $|\deg_{H^v}(w) - \deg_{H^v}(q)| \leq 1$.This shows that B_v is an equitable dominating set of H^v whenever $\Delta(H) \leq n-2$ or $\deg_G(v) \geq 2$. Suppose that $\deg_G(v) = 1$ and $\Delta(H) = n-1$. Suppose further that $\deg_{H^v}(q) \neq n-1$. Then $\deg_{G \circ H}(v) = n+1$ and $\deg_{G \circ H}(q) \leq n-1$. This implies that $w \neq v$. Hence, $w \in B_v$ and $|\deg_{H^v}(w) - \deg_{H^v}(q)| \leq 1$. Consequently, B_v dominates equitably $V(H^v) \setminus S_v$. The converse is easy. □

The next result is a consequence of the Theorem above.

Corollary 3. *Let G be a connected graph of order $m \geq 2$ and H be any graph of order n. If $\delta(G) \geq 2$ or $\Delta(H) \leq n-2$, then*

$$\gamma^{ce}(G \circ H) = [1 + \gamma^e(H)]m.$$

Proof. Let C be a γ^{ce}-set of $G \circ H$. By Theorem 14, $C = V(G) \cup (\bigcup_{v \in V(G)} B_v)$, where B_v is a connected equitable dominating set of H^v for each $v \in V(G)$. Since C is a γ^{ce}-set , B_v is a γ^e-set of H^v for each $v \in V(G)$. Thus, $\gamma^{ce}(G \circ H) = |C| = m + m\gamma^e(H) = m[1 + \gamma^e(H)]$. □

Corollary 4. *For any graph G of order n,*

$$\gamma^{ce}(G \circ K_p) = n + |S|$$

where $S = \{u \in V(G) : \deg_G(u) \geq 2\}$.

Proof. Follows from Theorem 14 and Corollary 3. □

Corollary 5. *For any graph G of order n, and a r-regular graph H with $r \neq |V(H)| - 1$,*

$$\gamma^{ce}(G \circ H) = n + n\gamma(H).$$

Proof. Let $v \in V(G)$ and $C_v \subseteq V(H^v)$ be a γ-set in H^v. Then C_v is an equitable dominating set in H^v since H is r-regular. Furthermore, since $r \neq |V(H)| - 1$, the set $\bigcup_{u \in V(G)} \{v\} \cup C_v$ is a connected equitable dominating set in $G \circ H$ of minimum cardinality. Thus, $\gamma^{ce}(G \circ H) = n + n|C_v| = n + n\gamma(H)$.

□

5. Conclusion

In this paper, we have investigated the concept of connected equitable domination in graphs focusing on some families of graphs, and on graphs arising from the join and corona of graphs. Specifically, the connected equitable dominating sets in the join and corona of graphs are characterized. We also obtained the exact value of $\gamma^{ce}(G)$ for some families of graphs; and graphs formed in the join and corona of graphs. Additionally, a realization problem is established.

The significance of this study lie in its extension of domination theory to connected equitable domination through combining both connectivity and equity constraints, reflecting more realistic models of control, influence, and resource distribution in networked systems; and focus on more complex structure of graphs: graphs obtained in the join and corona of graphs. Unlike classical domination parameters that focus solely on coverage or connectivity, the connected equitable domination framework ensures balanced distribution of domination responsibilities among vertices while maintaining network cohesion.This study provides a foundation for efficient network design in practical domains such as sensor placement, facility location, communication networks, and distributed computing, where both connectivity and fairness are critical. The exact values and characterizations obtained here can also serve as benchmarks for algorithm development, particularly in designing heuristics and approximation algorithms for complex networks. Nonetheless, the work is confined to unweighted, undirected, and simple graphs, which may limit its direct applicability to more complex or dynamic networks. The analysis focuses specifically on the join and corona operations, leaving room to explore the concept on some other binary operations such as lexicographic product or Cartesian product of graphs, on weighted graphs, directed graphs, or dynamic environments, potentially yielding deeper insights into equitable network optimization under evolving constraints. Furthermore, while the results are mathematically rigorous, the study does not address algorithmic complexity or include empirical validation, which could be valuable extensions in future research.

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