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Legal Closed Hop Neighborhood Independent Sequences in Graphs

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Abstract. Let G be a graph. A sequence $Q = (x_1, x_2, ..., x_k)$ of distinct vertices of G is called a legal closed hop neighborhood independent sequence (LCHNI sequence) if it satisfies the following two conditions: $[(i)]$ $N^2_G[x_i] \setminus \bigcup_{j=1}^{i-1} N^2_G[x_j] \neq$ for each $i \in \{2, 3, ..., k\}$, and $[(ii)]$ $d_G(x_s, x_t) \neq 1$ for each $s, t \in \{1, 2, ..., k\}$, where $s \neq t$. The legal closed hop neighborhood independence number (LCHNI number) of G is the maximum length of an LCHNI sequence of G, and this is denoted by $\theta(G)$. In this paper, the authors initiate the study of a legal closed hop neighborhood independent sequence in some special graphs, shadow graphs, and the join of two graphs. In particular, the authors determine the corresponding legal closed neighborhood independence numbers of these graphs.

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Key Words and Phrases: Independent set, legal closed hop neighborhood independent sequence, legal closed hop neighborhood independence number, co-legal closed neighborhood independent sequence

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1. Introduction

An independent set in a graph is a set of vertices such that no two vertices in the set are adjacent to each other. This idea is crucial in various areas of graph theory and its applications, including network design, scheduling problems, and resource allocation. Independent sets in graphs have been studied on various types of graphs (see [1–9]).

In 2022, hop independent set in a graph and its corresponding parameter were introduced and investigated by J. Hassan et al. [10]. This defined set stated that no two distinct

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vertices in the set have a distance of two from each other. This was further investigated on different types of graphs where they obtained some interesting results. Several variants of this concept have been introduced and studied (see [11–14]).

More recently, inspired by numerous articles on independent sets of graphs, J. Hassan et al. [11], introduced another variant of independent set called legal hop independent sequence in a graph. This new variant added another property wherein the order of vertices in the set of a graph matters, and is defined as follows: A sequence $L = (w_1, \cdots, w_k)$ of distinct vertices of G is called a legal hop independent sequence if $k = 1$ or L is a hop independent and $N_G[w_i] \setminus \bigcup_{j=1}^{i-1} N_G[w_j] \neq \varnothing$ for every $i \in \{2, \cdots, k\}$. The maximum length of a legal hop independent sequence in G, denoted by $\alpha_{lh}(G)$,is called the legal hop independence number of G. The said authors have formulated characterizations and formulas of this parameter on different types of graphs.

In this paper, a new variant of independence parameter is introduced and initially investigated, and we call this legal closed hop neighborhood independent sequence (LCHNI sequence) of a graph. In this parameter, the authors put some restrictions on the usual independent set in a graph where the order of choosing vertices as well as the behavior of its closed hop neighborhoods are important. The authors believe that this newly defined parameter would open more interesting studies and applications in the future.

2. Terminologies and Notations

Let G be an undirected graph. A subset A of $V(G)$ is an *independent* set if for every pair of distinct vertices in A do not form an edge. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set in G.

Let G be an undirected graph. Let $S = (v_1, v_2, \cdots, v_k)$ be a sequence of distinct vertices of G and let $\hat{S} = \{v_1, v_2, \ldots, v_k\}$. Then S is a legal closed hop neighborhood sequence of G if $N^2_G[v_i] \setminus \cup_{j=1}^{i-1} N^2_G[v_j] \neq \varnothing$ for each $i \in \{2, \cdots, k\}$.

Let G be a graph. A sequence $Q = (x_1, x_2, ..., x_k)$ of distinct vertices of G is called a legal closed hop neighborhood independent sequence (LCHNI sequence) if it is satisfies the following two conditions:

(1.) $N^2_G[x_i] \setminus \bigcup_{j=1}^{i-1} N^2_G[x_j] \neq$ for each $i \in \{2, 3, ..., k\}$.

(2.) $d_G(x_s, x_t) \neq 1$ for each $s, t \in \{1, 2, ..., k\}$, where $s \neq t$.

The legal closed hop neighborhood independent number (LCHNI number) of G is the maximum length of an LCHNI sequence in G. The said number is denoted by $\theta(G)$. We call the corresponding set $\hat{Q}$ an LCHNI set of G. We call the corresponding set $\hat{Q}$ of Q an LCHNI set of G.

Let $Q_1 = (v_1, \cdots, v_n)$ and $Q_2 = (u_1, \cdots, u_m)$, $n, m \geq 1$ be two sequences of distinct vertices of G. The *concatenation* of Q_1 and Q_2, denoted by $Q_1 \oplus Q_2$, is the sequence given by

$$Q_1 \oplus Q_2 = (v_1, \cdots, v_n, u_1, \cdots, u_m).$$

Let G and H be any two graphs. The *join* $G+H$ is the graph with vertex set $V(G+H)=V(G)\cup V(H)$ and edge set $E(G+H)=E(G)\cup E(H)\cup\{uv : u\in V(G), v\in V(H)\}$. The *corona* $G\circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G\circ H$ corresponding to the vertex $v\in G$ and write $v+H^v$ for $\langle\{v\}\rangle+H^v$.

The *shadow graph* $S(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u\in V(G_1)$ to the neighbors of its corresponding vertex $u'\in V(G_2)$.

3. Results

Remark 1. *Let G be a graph. Then each of the following holds:*

(i) *A legal closed hop neighborhood sequence of G need not form an independent set of G.*

(ii) *An independent set of G need not form a legal closed hop neighborhood sequence of G.*

(iii) *A legal closed hop neighborhood independent sequence of G induces an independent set of G.*

(iv) $1\leq\theta(G)\leq\alpha(G)\leq|V(G)|$.

Theorem 1. *Let G be a graph of order n. Then $\theta(G)=|V(G)|$ if and only if $G=\overline{K}_n$.*

Proof. Suppose that $G=\overline{K}_n$. Let $V(\overline{K}_n)=\{c_1,c_2,...,c_n\}=\hat{Q}$. Then, $d_{\overline{K}_n}(c_i,c_j)=\infty\neq 1$ for each $i\neq j$ where $i,j\in\{1,2,...,n\}$. This means that $\hat{Q}$ is the maximum independent set of $\overline{K}_n$. Notice that

$$c_i\in N^2_{\overline{K_n}}[c_i]\setminus\bigcup_{j=1}^{i-1}N^2_{\bar{K}_n}[c_j]$$

for each $i\in\{2,3,...,n\}$. It follows that $Q=(c_1,c_2,...,c_n)$ is a legal closed hop neighborhood sequence of $\overline{K}_n$. Consequently, Q is the maximum legal closed hop neighborhood independent sequence of $\overline{K}_n$, and so $\theta(\overline{K}_n)=|V(\bar{K}_n)|$.

Conversely, assume that $\theta(G)=|V(G)$. Suppose further that $G\neq\overline{K}_n$. Then there exist $u,v\in V(G)$ such that $d_G(u,v)=1$. Hence, either u or v is not in any LCHNI set of G, a contradiction to the assumption that $\alpha(G)=|V(G)|$. Therefore, G must be equal to $\overline{K}_n$. □

Theorem 2. *Let n be any positive integer. Then*

$$\theta(P_n) = \begin{cases} 1, \textit{if } n=1 \\ \frac{n}{2}, \textit{if } n \textit{ is even} \\ \frac{n-1}{2}, \ \textit{if } n \geq 3 \textit{ and odd} \end{cases}$$

Proof. By Theorem 1, $\theta(P_1) = 1$. Suppose that n is even. Clearly, $\theta(P_2) = 1$. Now, for $n \geq 4$, consider $V(P_n) = \{a_1, a_2, ..., a_n\}$ and $Q = \{a_1, a_3, ..., a_{n-3}, a_n\}$. Then $\hat{Q}$ is a maximum independent set of P_n. Observe that $a_{i+2} \in N_G^2[a_i] \setminus \bigcup_{j<i} N_G^2[a_j]$ for each $i \in \{3, 5, ..., n-3\}$ and $a_n \in N_G^2[a_n] \setminus \bigcup_{m<n} N_G^2[a_m]$. It follows that Q is an LCHNI sequence of P_n. Therefore, Q is a maximum LCHNI sequence of P_n, and so $\theta(P_n) = \frac{n}{2}$ for all even numbers $n \geq 2$.

Next, suppose that n is an odd positive integer. For $n = 3$, let $V(P_3) = \{b_1, b_2, b_3\}$. Since $N_{P_3}^2[b_1] = N_{P_3}^2[b_3]$, it follows that $\theta(P_3) = 1$. Now, assume that $n \geq 5$ and odd. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$, and consider $T = \{v_1, v_3, ..., v_{n-2}\}$. Then $\hat{T}$ is an independent set of P_n and $v_{i+2} \in N_G^2[v_i] \setminus \bigcup_{j<i} N_G^2[v_j]$ for all $i \in \{3, 5, ..., n-2\}$. Thus, T is an LCHNI sequence of P_n. Since $N_G^2[v_n] \subseteq N_G^2[v_{n-2}]$, it follows that T is a maximum LCHNI sequence of P_n. Consequently, $\theta(P_n) = \frac{n-1}{2}$ for all odd positive integers $n \geq 3$. □

Theorem 3. *Let $n \geq 3$ be any positive integer. Then*

$$\theta(C_n) = \begin{cases} \alpha(C_n), \textit{if } n \textit{ is odd} \\ \alpha(C_n) - 1, \textit{if } n \textit{ is even} \end{cases}$$

Proof. Suppose that n is odd. Clearly, for $n = 3$, $\theta(C_n) = 1$. Now for $n \geq 5$, let $V(C_n) = \{c_1, c_2, ..., c_n\}$ and consider $S = (c_1, c_3, ..., c_{n-2})$. Then $\hat{S}$ is a maximum independent set of C_n. Notice that $c_{i+2} \in N_G^2[c_i] \setminus \bigcup_{j<i} N_G^2[c_j]$ for each $i \in \{3, 5, ..., n-2\}$. This means that S is a legal closed hop neighborhood sequence of C_n. Thus, S is a maximum LCHNI sequence of C_n, and so $\theta(C_n) = \alpha(C_n)$ for all odd numbers $n \geq 3$.

Next, suppose that n is even. For $n = 4$, let $V(C_4) = \{a_1, a_2, a_3, a_4\}$. Then $N_{C_4}^2[a_1] = N_{C_4}^2[a_3]$ and $N_{C_4}^2[a_2] = N_{C_4}^2[a_4]$, where $d_{c_4}(a_1, a_3) = 2 = d_{c_4}(a_2, a_4)$. It follows that $\theta(C_4) = 1$.

For $n = 6$, let $V(C_6) = \{b_1, b_2, ..., b_6\}$.Notice that $N_{c_6}^2[b_1] = N_{c_6}^2[b_3] = N_{c_6}^2[b_4]$ and $N_{c_6}^2[b_2] = N_{c_6}^2[b_4] = N_{c_6}^2[b_6]$, where $d_{c_6}(b_1, b_3) = d_{c_6}(b_3, b_5) = d_{c_6}(b_1, b_5) = 2$ and $d_{c_6}(b_2, b_4) = d_{c_6}(b_4, b_6) = d_{c_6}(b_2, b_6) = 2$. This means that only one element in $\{b_1, b_3, b_5\}$ could be in any legal closed hop neighborhood independent sequence of C_6. Similarly, for the set $\{b_2, b_4, b_6\}$. Since $d_{C_6}(b_1, b_1) = 1 = d_{C_6}(b_1, b_6)$, $d_{C_6}(b_1, b_4) = 3$, and $N_{c_6}^2[b_4] \setminus N_{c_6}^2[b_1] = \{b_2, b_4, b_6\}$, it follows that $T = (b_1, b_4)$ is a maximum LCHNI sequence of C_6, and so $\theta(C_6) = 2$.

Now for $n \geq 8$, let $V(C_n) = \{x_1, x_2, ..., x_n\}$ and consider $W = \{x_1, x_3, ..., x_{n-5}, x_{n-2}\}$. Then $\hat{W}$ is an independent set of C_n with $|\hat{W}| = \alpha(C_n) - 1$. Notice that $x_{i+2} \in$

$N^2_{C_n}[x_i] \setminus \bigcup_{j<i} N^2_{C_n}[x_j]$ for all $i \in \{3,5,...,n-5\}$ and $x_n, x_{n-2}, x_{n-4} \in N^2_{C_n}[x_{n-2}] \setminus \bigcup_{j\in\{1,3,...,n-5\}} N^2_{C_n}[x_j]$. Thus, W is a LCHNI sequence of C_n. Since

$$N^2_{C_n}[x_{n-3}] \subseteq N^2_{C_n}[x_{n-5}] \bigcup N^2_{C_n}[x_i]$$

and

$$N^2_{C_n}[x_{n-1}] \subseteq N^2_{C_n}[x_{n-5}] \bigcup N^2_{C_n}[x_i],$$

it follows that W is a maximum LCHNI sequence of C_n. Since $M = \{x_1, x_3, \ldots, x_{n-1}\}$ is a maximum independent set of C_n for all positive even integers $n \geq 4$. It follows that $\theta(C_n) = \alpha(C_n) - 1$ for all positive even numbers $n \geq 4$. □

The following concept will be used to study the behavior of LCHNI sequences in the join of any two graphs.

Definition 1. *Let G be a graph. Then a sequence P of distinct vertices of G is called a co-legal closed neighborhood independent sequence (CLCNI sequence) of G if P is a legal closed neighborhood sequence in $\bar{G}$ and P is an independent set of G.*

The co-legal closed neighborhood independence number (CLCNI number) of G is the maximum length of a CLCNI sequence of G, and is denoted by $\alpha_{cl}(G)$.

The following theorem will be used to prove the characterization of an LCHNI sequence on the join of two graphs.

Theorem 4. *[15] Let G and H be any two graphs. A sequence S of distinct vertices of $G+H$ is a legal closed hop neighborhood sequence if and only if one of the following holds:*

(i) S is a co-legal closed neighborhood sequence in G (legal closed neighborhood sequence in $\overline{G}$).

(ii) S is a co-legal closed neighborhood sequence in H (legal closed neighborhood sequence in $\overline{H}$).

(iii) S is a concatenation $S_G \oplus S_H$, where S_G and S_H are co-legal closed neighborhood sequences in G and H, respectively.

Theorem 5. *Let G and H be any two graphs. Then a sequence F of distinct vertices of $G+H$ is an LCHNI sequence of $G+H$ if and only if F satisfies any of the following conditions:*

(i) F is a CLCNI sequence of G.

(ii) F is a CLCNI sequence of H.

Proof. Suppose that F is an LCHNI sequence of $G+H$. Then $\hat{F}$ is an independent set of $G+H$. Thus, either $\hat{F}$ is an independent set of G or $\hat{F}$ is an independent set of H. By Theorem 4, F is either a CLCNI sequence of G or H. Hence, F is either a CLCNI sequence of G or H.

Conversely, suppose that (i) holds. Then by Theorem 4, F is a legal closed hop neighborhood sequence of $G+H$. Since $\hat{F}$ is an independent set of G, it follows that F is a LCHNI sequence of $G+H$. Similarly, the assertion holds when (ii) is true. □

Theorem 6. *Let G and H be any two graphs. Then $\theta(G+H) = max\{\alpha_{cl}(G), \alpha_{cl}(H)\}$*

Proof. We may assume that $\alpha_{cl}(G) \geq \alpha_{cl}(H)$. First, let P be a maximum LCHNI sequence of $G+H$. Then by Theorem 3, P is a CLCNI sequence of G. Hence, $\theta(G+H) = |\hat{P}| \leq \alpha_{cl}(G)$.

Next, suppose that F is a maximum CLCNI sequence of G. Then F is an LCHNI sequence of $G+H$ by Theorem 3. It follows that $\theta(G+H) \geq |\hat{F}| = \alpha_{cl}(G)$. This establishes the desired equality. □

Corollary 1. *Let n and m be any two positive natural numbers. Then each of the following holds:*

(i) $\theta(F_n) = \alpha_{cl}(P_n)$

(ii) $\theta(W_n) = \alpha_{cl}(C_n)$

(iii) $\theta(S_n) = \alpha_{cl}(\bar{K}_n) = 1$

(iv) $\theta(P_n + P_m) = max\{\alpha_{cl}(P_n), \alpha_{cl}(P_m)\}$

(v) $\theta(C_n + C_m) = max\{\alpha_{cl}(C_n), \alpha_{cl}(C_m)\}$

(vi) $\theta(K_n + K_m) = max\{\alpha_{cl}(K_n), \alpha_{cl}(K_m)\} = 1$

Now, let's study the behavior of an LCHNI sequence in the shadow graphs. We shall note the following notations:

Let G_1 and G_2 be two copies of a graph G in the definition of the shadow graph $S(G)$. If $Q_{G_1} \subseteq V(G_1)$ and $Q_{G_2} \subseteq V(G_2)$, then the sets Q'_{G_1} and Q'_{G_2} are defined as follows:

$$Q'_{G_1} = \{x' \in V(G_2) : x \in Q_{G_1}\} \text{ and } Q'_{G_2} = \{b \in V(G_1) : b' \in Q_{G_2}\}.$$

Theorem 7. *Let G be a non-trivial connected graph. Then Q is an LCHNI sequence in $S(G)$ if and only if one of the following conditions holds:*

(i) *Q is an LCHNI sequence in G_1.*

(ii) *Q is an LCHNI sequence in G_2.*

(iii) *$Q = Q_{G_1} \oplus Q_{G_2}$ such that $\hat{Q}_{G_1} \cup \hat{Q}'_{G_2}$ and $\hat{Q}'_{G_1} \cup \hat{Q}_{G_2}$ are LCHNI sets in G_1 and G_2, respectively.*

Proof. Let Q be an LCHNI sequence in $S(G)$. Let $\hat{Q}_{G_1} = \hat{Q} \cap V(G_1)$ and $\hat{Q}_{G_2} = \hat{Q} \cap V(G_2)$. If $\hat{Q}_{G_2} = \varnothing$, then $Q = Q_{G_1}$ is an LCHNI sequence in G_1. Moreover, if $\hat{Q}_{G_1} = \varnothing$, then $Q = Q_{G_2}$ is an LCHNI sequence in G_2. Therefore, either (i) or (ii) holds.

Now assume that $\hat{Q}_{G_1} \neq \varnothing$ and $\hat{Q}_{G_2} \neq \varnothing$. Let $\hat{Q}_{G_1} \cup \hat{Q}'_{G_2} = \{v_1, v_2, \ldots, v_k\}$, and consider $v_i, v_j \in \hat{Q}_{G_1} \cup \hat{Q}'_{G_2}$ and consider the following cases:

Case 1: If $v_i, v_j \in \hat{Q}_{G_1} = \hat{Q} \cap V(G)$, then $d_{G_1}(v_i, v_j) \neq 1$ since $\hat{Q}$ is an independent set in $S(G)$ with $\hat{Q}_{G_1} \subseteq \hat{Q}$.

Case 2: If $v_i, v_j \in \hat{Q'}_{G_2}$, then $v'_i, v'_j \in \hat{Q}_{G_2}$. Since $\hat{Q}_{G_2} \subseteq \hat{Q}$ and $\hat{Q}$ is an independent, it follows that $d_{G_2}(v_i, v_j) \neq 1$, and we are done.

Case 3: Assume that $v_i \in \hat{Q}_{G_1}$ and $v_j \in \hat{Q'}_{G_2}$. Then $v'_j \in \hat{Q}_{G_2} \subseteq \hat{Q}$. Thus, $d_{S(G)}(v_i, v_j) \neq 1$ implying that $d_{G_1}(v_i, v_j) \neq 1$. Similarly, the same result follows when $v_j \in \hat{Q}_{G_1}$ and $v_i \in \hat{Q'}_{G_2}$.

Now, let $\hat{Q}_{G_1} = \{v_s : s \in S \subseteq \{1, 2, \ldots, k\}\}$ and $\hat{Q'}_{G_2} = \{v_t : t \in \{1, 2, \ldots, k\} \setminus S\}$. Then $v'_t \in \hat{Q}_{G_2}$ for all $t \in \{1, 2, \ldots, v_k\} \setminus S$. Since $N^2_{S(G)}[v'_t] = N^2_{S(G)}[v_t]$ and Q is a legal closed neighborhood sequence of $S(G)$, it follows that $N^2_G[v_i] \setminus \bigcup_{j=1}^{i-1} N^2_G[v_j] \neq \varnothing$ for all $i \in \{2, 3, \ldots, k\}$. Thus, $\hat{Q}_{G_1} \cup \hat{Q}'_{G_2}$ is an LCHNI set of G_1. Similarly, $\hat{Q}'_{G_1} \cup \hat{Q}_{G_2}$ is an LCHNI set of G_2.

The converse is clear. □

The following result follows from Theorem 7.

Corollary 2. *Let G be a non-trivial connected graph. Then $\theta(S(G)) = \theta(G)$.*

4. Conclusion

The concept of legal closed hop neighborhood independent sequence has been introduced and initially investigated in this study. Characterizations and formulas for the legal closed hop neighborhood independence have been obtained on some special graphs, shadow graphs, and on the join of any two graphs. Further exploration of this parameter on other graphs under some operations, as well as its relationships with other parameters may lead to deep insights and may show connections with other graph properties. Interested researchers may provide a real-world application of the legal closed hop neighborhood independent, and may investigate the complexity of this concept.

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