EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6128
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Group Analysis of a Class of Nonlinear Wave Equations

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Abstract. We study the nature of a (2+1)-dimensional nonlinear wave equation using the Lie symmetry analysis method. This problem is reduced to ordinary differential equations (ODEs) using non-similar subalgebras of Lie symmetries. We presented explicit solutions by solving the reduced ODEs. The conserved vectors were constructed using the Lagrange multiplier method. Using these conserved vectors, we also derived the exact solutions of the nonlinear wave equation. Consequently, 3D graphics were used to analyze and illustrate the graphical representations of the solutions.

2020 Mathematics Subject Classifications: 70G65, 35-XX, 70S10, 22E70
Key Words and Phrases: Lie symmetries, conservation laws, invariant solutions, nonlinear wave equations

1. Introduction

The study of nonlinear wave phenomena relies extensively on nonlinear partial differential equations (NLPDEs). The interplay between larger-amplitude waves spreading with lower-amplitude waves is the factor that causes the nonlinearity. Finding exact solutions is an essential problem because these equations explain the characteristics and behaviors of nonlinear phenomena. It is important to mention that various methods have been employed to address these nonlinear problems, including the ϕ^6-model expansion method [1], the exp-function method [2], the symmetry method [3], the residual power series approach [4], the homogeneous balanced method [5], the hyperbolic tangent method [6], F-expansion method [7], the unified method [8, 9], the new Jacobi elliptic functions technique [10] and the improved Sardar sub-equation approach [11].

One of the most effective ways to obtain analytical solutions for NLPDEs is to apply Lie symmetry [3]. Any solution to an NLPDE can be converted into a collection of solutions to

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6128

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the same equation via symmetry. Conservation laws play a major role in the development and analysis of various mathematical models. The most efficient technique for determining the conservation laws for NLPDEs is the multiplier method [3, 12].

It is commonly recognized that the main objective of different linear and nonlinear wave equations is to model certain wave phenomena. Ames et al. [13] used the Lie group method to study the general nonlinear form of the (1+1)-dimensional wave equation given by

$$u_{tt} - (k(u)u_x)_x = 0. \tag{1}$$

Bluman et al. [14] constructed the conserved vectors for (1), while the higher-order conservation laws and the complete categorization of Lie algebra for (1) are presented in [15]. Different types of nonlinear wave equations are presented in [16–22].

The general form of (2+1)-dimensional nonlinear wave equation [23] is given by

$$w_{tt} - (f(w)w_x)_x - (g(w)w_y)_y = 0, \tag{2}$$

where the nonlinear functions $f(w)$ and $g(w)$ determine how waves propagate. Usually, this equation simulates the propagation of waves in a nonlinear or nonhomogeneous mediums, where the response of the medium is determined by the amplitude of the wave. Two-dimensional nonlinear acoustic waves, surface water waves, and other waveforms capturing phenomena, such as wave steepening and shock production, can be modeled using (2). The formulation $f(w) = \alpha w,\ g(w) = \beta w^2$ in (2) can simulate the nonlinear dispersive waves including solitons in shallow water. By taking $f(w) = \alpha,\ g(w) = \beta w$ in (2), the beam dynamics in nonlinear elastic materials can be modeled, where the response is proportional to the deformation. In [23], the double reduction theory is applied on (2) based on the conserved vectors constructed using a partial Lagrangian.

In this study, we take the nonlinear wave equation (2) in (2+1) dimensions and use the Lie group approach to obtain its solutions. Several interesting similarity reductions were performed using similar conjugacy classes of Lie algebra, which were computed using the matrix method. Invariant solutions were obtained for these symmetry reductions. The local conserved vectors for (2) were constructed and utilized to obtain the exact solutions of (2). For certain parameter values, 3D images of the solutions in various structures were created to obtain a better analysis of the solutions.

The remainder of this paper is structured as follows: Section 2 is devoted to the classification of the Lie symmetries and the construction of the associated optimal set for (2). Section 3 addresses symmetry reduction and invariant solutions. The local conservation laws are presented in Section 4, which produce the exact solutions stated in Section 5. Section 6 presents a geometric analysis of the obtained results.

2. Lie point symmetries and optimal system

This section explains the fundamental terms required to calculate the infinitesimals of (2). We take a Lie group of infinitesimal transformations with one parameter that acts on the dependent variable w and the independent variables x, y, and t of Eq. (2)

$$\begin{aligned} \tilde{x} &\to x + \varsigma\zeta_1(x,y,t,w) + O(\varsigma^2), \\ \tilde{y} &\to y + \varsigma\zeta_2(x,y,t,w) + O(\varsigma^2), \\ \tilde{t} &\to t + \varsigma\zeta_3(x,y,t,w) + O(\varsigma^2), \\ \tilde{w} &\to w + \varsigma\Phi(x,y,t,w) + O(\varsigma^2), \end{aligned} \tag{3}$$

with $\varsigma \ll 1$ being a parameter of the group and $\zeta_1, \zeta_2, \zeta_3$ and Φ are the infinitesimal functions for the variables x, y, t and w respectively, which should be computed later. The vector field associated with the Lie group of transformations (3) takes the form

$$\mathcal{Y} = \zeta_1\frac{\partial}{\partial x} + \zeta_2\frac{\partial}{\partial y} + \zeta_3\frac{\partial}{\partial t} + \Phi\frac{\partial}{\partial w}. \tag{4}$$

The operator $\mathcal{Y}$ can be identified as the Lie point symmetry generator of (2) if it meets the invariance criterion [3]

$$\mathcal{Y}^{[2]}(w_{tt} - (f(w)w_x)_x - (g(w)w_y)_y)|_{(2)} = 0, \tag{5}$$

where $\mathcal{Y}^{[2]}$ denotes the prolongation of $\mathcal{Y}$ up to order two. We obtain a system of linear coupled PDEs for infinitesimals by setting the coefficients of the dependent variable and its differential to zero. These coupled PDEs also referred to as determining equations, are solved to produce the infinitesimals, we then substitute these infinitesimals in Eq. (4) to extract the Lie point symmetries for (2), which are stated as follows:

Case-1: $\mathbf{f(w), g(w)}$ are arbitrary [23].
In this case, the solution of (5) provides a four-dimensional Lie algebra of (2) spanned by the following vector fields

$$\mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}. \tag{6}$$

The Lie algebra for (2) is extended for the cases presented as follows:

Case-2: $\mathbf{f(w) = \alpha w,\ g(w) = \beta w^2}$.

$$\begin{aligned} &\mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w}, \\ &\qquad\qquad \mathcal{Y}_5 = y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} + 2w\frac{\partial}{\partial w}. \end{aligned} \tag{7}$$

Case-3: $\mathbf{f(w) = \alpha,\ g(w) = \beta w}$:

$$\begin{aligned} &\mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = t\frac{\partial}{\partial x} + \frac{x}{\alpha}\frac{\partial}{\partial t},\ \mathcal{Y}_5 = y\frac{\partial}{\partial y} + 2w\frac{\partial}{\partial w}, \\ &\qquad\qquad \mathcal{Y}_6 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2w\frac{\partial}{\partial w} \end{aligned} \tag{8}$$

Case-4: $\mathbf{f(w) = e^{\alpha w}},\ \mathbf{g(w) = e^{\beta w}}$:

$$\begin{aligned} \mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + \frac{\beta t}{(\beta-\alpha)}\frac{\partial}{\partial t} + \frac{2}{(\alpha-\beta)}\frac{\partial}{\partial w}, \\ \mathcal{Y}_5 = y\frac{\partial}{\partial y} + \frac{\alpha t}{(-\beta+\alpha)}\frac{\partial}{\partial t} + \frac{2}{(\beta-\alpha)}\frac{\partial}{\partial w} \end{aligned} \tag{9}$$

Case-5: $\mathbf{f(w) = w^{\alpha}},\ \mathbf{g(w) = w^{\beta}}$:

$$\begin{aligned} \mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + \frac{\beta t}{(\beta-\alpha)}\frac{\partial}{\partial t} - \frac{2w^{1+\beta}w^{1+\alpha}}{w^{\beta}(\beta-1)\,w^{1+\alpha} - w^{1+\beta}w^{\alpha}(\alpha-1)}\frac{\partial}{\partial w}, \\ \mathcal{Y}_5 = y\frac{\partial}{\partial y} + \frac{\alpha t}{(-\beta+\alpha)}\frac{\partial}{\partial t} + \frac{2w^{1+\beta}w^{1+\alpha}}{w^{\beta}(\beta-1)\,w^{1+\alpha} - w^{1+\beta}w^{\alpha}(\alpha-1)}\frac{\partial}{\partial w} \end{aligned} \tag{10}$$

Case-6: $\mathbf{f(w) = \alpha \ln w},\ \mathbf{g(w) = \beta \ln w}$:

$$\begin{aligned} \mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}, \\ \mathcal{Y}_5 = y\frac{\partial}{\partial x} - \frac{\beta x}{\alpha}\frac{\partial}{\partial y} \end{aligned} \tag{11}$$

Case-7: $\mathbf{f(w) = \alpha},\ \mathbf{g(w) = \beta}$: [22]

$$\begin{aligned} \mathcal{Y}_1 = \frac{\partial}{\partial x},\ \mathcal{Y}_2 = \frac{\partial}{\partial y},\ \mathcal{Y}_3 = \frac{\partial}{\partial t},\ \mathcal{Y}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t},\ \mathcal{Y}_5 = w\frac{\partial}{\partial w}, \\ \mathcal{Y}_6 = t\frac{\partial}{\partial y} + \frac{y}{\beta}\frac{\partial}{\partial t},\ \mathcal{Y}_7 = t\frac{\partial}{\partial x} + \frac{x}{\alpha}\frac{\partial}{\partial t},\ \mathcal{Y}_8 = y\frac{\partial}{\partial x} - \frac{\beta x}{\alpha}\frac{\partial}{\partial y}, \\ \mathcal{Y}_9 = xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + (\frac{y^2}{2\beta} + \frac{x^2}{2\alpha} + \frac{t^2}{2})\frac{\partial}{\partial t} - \frac{tw}{2}\frac{\partial}{\partial w}, \\ \mathcal{Y}_{10} = xy\frac{\partial}{\partial x} + \frac{\left(\beta\, t^2 + y^2\right)\alpha - \beta\, x^2}{2\alpha}\frac{\partial}{\partial y} + yt\frac{\partial}{\partial t} - \frac{yw}{2}\frac{\partial}{\partial w}, \\ \mathcal{Y}_{11} = \frac{\left(-\beta\, t^2 + y^2\right)\alpha - \beta\, x^2}{2\alpha}\frac{\partial}{\partial x} - \frac{\beta xy}{\alpha}\frac{\partial}{\partial y} - \frac{x\beta t}{\alpha}\frac{\partial}{\partial t} + \frac{x\beta w}{2\alpha}\frac{\partial}{\partial w} \end{aligned} \tag{12}$$

2.1. Optimal System

The optimal systems for the symmetry generators of (2) are derived using the algorithm described in [3]. The adjoint action representation [15] is written as follows

$$Ad(exp(\varsigma\mathcal{Y}_i)\mathcal{Y}_j) = \mathcal{Y}_j - \varsigma[\mathcal{Y}_i, \mathcal{Y}_j] + \frac{\varsigma^2}{2!}[\mathcal{Y}_i, [\mathcal{Y}_i, \mathcal{Y}_j]] - \dots\ . \tag{13}$$

Computation of Basic Invariants

Let f be a function on Lie algebra $\mathcal{L}$ such that

$$f(Ad(exp(\varsigma\mathcal{Y})\mathcal{X})) = f(\mathcal{X}) \qquad \forall\ \mathcal{Y} \in \mathcal{L},\ \varsigma \in \mathbf{R}. \tag{14}$$

Now, if we write $f(y_1\mathcal{Y}_1 + \ldots + y_n\mathcal{Y}_n) = f(y_1 + \ldots + y_n)$, then for the basis $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_n\}$, we get [24]

$$\sum_{1\leq i,i\leq N} y_i\Theta_j([\mathcal{Y}_k, \mathcal{Y}_i])\frac{\partial f}{\partial y_i} = 0. \tag{15}$$

The basic invariants in the adjoint representations are obtained from the solution of the linear system of PDEs (15).

The adjoint transformation matrix [24] is given by

$$A = \prod_{i=1}^{n} A_i, \quad A_i = Ad(e^{\varsigma_i\mathcal{Y}_i}). \tag{16}$$

Let

$$\mathcal{Y} = \sum_{i=1}^{n} k_i\mathcal{Y}_i,\ \tilde{\mathcal{Y}} = \sum_{i=1}^{n} \tilde{k}_i\mathcal{Y}_i \in \mathcal{L}.$$

For simplicity, we write $\mathcal{Y} = (k_1\ k_2 \cdots k_n),\ \tilde{\mathcal{Y}} = (\tilde{k}_1\ \tilde{k}_2 \cdots \tilde{k}_n)$. Then, $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ are in the same conjugacy class, if we have [24]

$$\tilde{\mathcal{Y}} = \mathcal{Y}A. \tag{17}$$

The solution of (17) gives the values of ς_i in terms of k_i, which can be used to simplify $\tilde{k}_i$.

2.1.1. Optimal system for Case 1

The non-zero commutators of Lie algebra $\mathcal{L}_4$ presented in (6) are given by

$$[\mathcal{Y}_1, \mathcal{Y}_4] = \mathcal{Y}_1,\ [\mathcal{Y}_2, \mathcal{Y}_4] = \mathcal{Y}_2,\ [\mathcal{Y}_3, \mathcal{Y}_4] = \mathcal{Y}_3. \tag{18}$$

The adjoint action representations of Lie symmetries (6) are obtained using (13) and are presented in Table 1.

Table 1: Adjoint Table

$Ad(e^{\varsigma})$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$
$\mathcal{Y}_1$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \varsigma\mathcal{Y}_1$
$\mathcal{Y}_2$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \varsigma\mathcal{Y}_2$
$\mathcal{Y}_3$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \varsigma\mathcal{Y}_3$
$\mathcal{Y}_4$	$e^{\varsigma}\mathcal{Y}_1$	$e^{\varsigma}\mathcal{Y}_2$	$e^{\varsigma}\mathcal{Y}_3$	$\mathcal{Y}_4$

The adjoint transformation matrix of $\mathcal{L}_4$ is constructed using (16) and is shown below

$$A = \begin{bmatrix} e^{\varsigma_4} & 0 & 0 & 0 \\ 0 & e^{\varsigma_4} & 0 & 0 \\ 0 & 0 & e^{\varsigma_4} & 0 \\ -\varsigma_1 e^{\varsigma_4} & -\varsigma_2 e^{\varsigma_4} & -\varsigma_3 e^{\varsigma_4} & 1 \end{bmatrix}$$

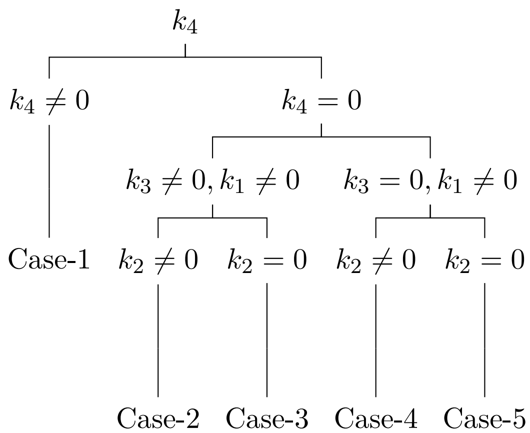
Using (17), we obtain the following system of equations, which is used to obtain ς_i:

$$\begin{aligned} \tilde{k}_1 &= k_1 e^{\varsigma_4} - k_4 \varsigma_1 e^{\varsigma_4}, \\ \tilde{k}_2 &= k_2 e^{\varsigma_4} - k_4 \varsigma_2 e^{\varsigma_4}, \\ \tilde{k}_3 &= k_3 e^{\varsigma_4} - k_4 \varsigma_3 e^{\varsigma_4}, \\ \tilde{k}_4 &= k_4. \end{aligned} \tag{19}$$

The basic invariants for (6) can be calculated by finding a solution of the following system of linear PDEs, which is derived using (15).

$$\begin{aligned} & k_4 \frac{\partial \vartheta}{\partial k_1} = 0, \\ & k_4 \frac{\partial \vartheta}{\partial k_2} = 0, \\ & k_4 \frac{\partial \vartheta}{\partial k_3} = 0, \\ & k_1 \frac{\partial \vartheta}{\partial k_1} + k_2 \frac{\partial \vartheta}{\partial k_2} + k_3 \frac{\partial \vartheta}{\partial k_3} = 0. \end{aligned} \tag{20}$$

The solution of (20) gives $\vartheta(k_1, k_2, k_3, k_4) = F(k_4)$. So, the first basic invariant is k_4, which will be the first vertix of the tree.



Case-1: For $k_4 \neq 0$, the representative element is $S^1 = \mathcal{Y}_4$. By substituting $\tilde{k}_4 = 1$ in (19), we get $\varsigma_1 = \frac{k_1}{k_4}$, $\varsigma_2 = \frac{k_2}{k_4}$, $\varsigma_3 = \frac{k_3}{k_4}$.

When $k_4 = 0$, the solution of (20) provides $\vartheta(k_1, k_2, k_3, k_4) = F(\frac{k_2}{k_1}, \frac{k_3}{k_1}, k_4)$. This implies that the new invariants are $\frac{k_2}{k_1}, \frac{k_3}{k_1}$. Since $k_1 \neq 0$, so the next two vertices will be k_3 and k_2.

Case-2: For $k_4 = 0, k_3 \neq 0, k_2 \neq 0, k_1 \neq 0$, the corresponding element of the optimal system is $S^2 = \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3$.

Case-3: For $k_4 = 0, k_3 \neq 0, k_1 \neq 0, k_2 = 0$, the associated element is $S^3 = \mathcal{Y}_1 + \mathcal{Y}_3$.

Case-4: For $k_4 = 0, k_3 = 0, k_1 \neq 0, k_2 \neq 0$, the conjugacy class is of the form $S^4 = \mathcal{Y}_1 + \mathcal{Y}_2$.

Case-5: For $k_4 = 0, k_3 = 0, k_1 \neq 0, k_2 = 0$, the representative element is $S^5 = \mathcal{Y}_1$.

Hence, the one-dimensional optimal system for Lie algebra (6) is given by

$$\begin{aligned} S^1 &= \mathcal{Y}_4, \\ S^2 &= \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3, \\ S^3 &= \mathcal{Y}_1 + \mathcal{Y}_3, \\ S^4 &= \mathcal{Y}_1 + \mathcal{Y}_2, \\ S^5 &= \mathcal{Y}_1. \end{aligned} \tag{21}$$

2.1.2. Optimal system for Case 2

For Lie algebra $\mathcal{L}_5$ given in (7), we have the following non-zero commutators

$$[\mathcal{Y}_1, \mathcal{Y}_4] = \mathcal{Y}_1,\ [\mathcal{Y}_2, \mathcal{Y}_5] = \mathcal{Y}_2,\ [\mathcal{Y}_3, \mathcal{Y}_4] = 2\mathcal{Y}_3,\ [\mathcal{Y}_3, \mathcal{Y}_5] = -\mathcal{Y}_3. \tag{22}$$

The adjoint representations of (7) are given in Table 2.

Table 2: Adjoint Table

$Ad(e^{\varsigma})$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$
$\mathcal{Y}_1$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \varsigma\mathcal{Y}_1$	$\mathcal{Y}_5$
$\mathcal{Y}_2$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5 - \varsigma\mathcal{Y}_2$
$\mathcal{Y}_3$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - 2\varsigma\mathcal{Y}_3$	$\mathcal{Y}_5 + \varsigma\mathcal{Y}_3$
$\mathcal{Y}_4$	$e^{\varsigma}\mathcal{Y}_1$	$\mathcal{Y}_2$	$e^{2\varsigma}\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$
$\mathcal{Y}_5$	$\mathcal{Y}_1$	$e^{\varsigma}\mathcal{Y}_2$	$e^{-\varsigma}\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$

The adjoint transformation matrix of $\mathcal{L}_5$ is given as

$$A = \begin{bmatrix} e^{\varsigma_4} & 0 & 0 & 0 & 0 \\ 0 & e^{\varsigma_5} & 0 & 0 & 0 \\ 0 & 0 & e^{2\varsigma_4 - \varsigma_5} & 0 & 0 \\ -\varsigma_1 e^{\varsigma_4} & 0 & -2\varsigma_3 e^{2\varsigma_4 - \varsigma_5} & 1 & 0 \\ 0 & -\varsigma_2 e^{\varsigma_5} & \varsigma_3 e^{2\varsigma_4 - \varsigma_5} & 0 & 1 \end{bmatrix}$$

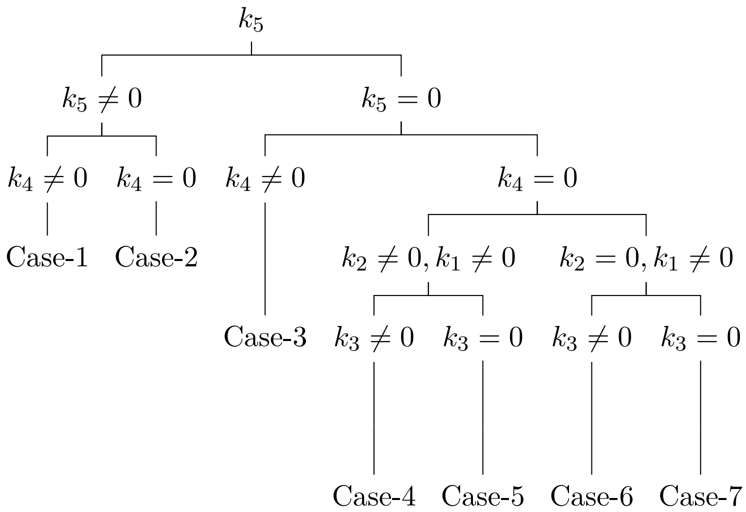
Eq. (17) yields the following system of equations

$$\begin{aligned} \tilde{k}_1 &= k_1 e^{\varsigma_4} - k_4 \varsigma_1 e^{\varsigma_4}, \\ \tilde{k}_2 &= k_2 e^{\varsigma_5} - k_5 \varsigma_2 e^{\varsigma_5}, \\ \tilde{k}_3 &= k_3 e^{2\varsigma_4 - \varsigma_5} - 2k_4 \varsigma_3 e^{2\varsigma_4 - \varsigma_5} + k_5 \varsigma_3 e^{2\varsigma_4 - \varsigma_5}, \\ \tilde{k}_4 &= k_4, \\ \tilde{k}_5 &= k_5. \end{aligned} \tag{23}$$

By using the formula (15), we obtain the following system of linear PDEs

$$\begin{aligned} &k_4 \frac{\partial \vartheta}{\partial k_1} = 0, \\ &k_5 \frac{\partial \vartheta}{\partial k_2} = 0, \\ &k_5 \frac{\partial \vartheta}{\partial k_3} - 2k_4 \frac{\partial \vartheta}{\partial k_3} = 0, \\ &k_1 \frac{\partial \vartheta}{\partial k_1} + 2k_3 \frac{\partial \vartheta}{\partial k_3} = 0, \\ &k_2 \frac{\partial \vartheta}{\partial k_2} + k_3 \frac{\partial \vartheta}{\partial k_3} = 0. \end{aligned} \tag{24}$$

By solving equation (24), we get $\Phi(k_1, k_2, k_3, k_4, k_5) = F(k_4, k_5)$. So, the first two vertices of the tree are k_4 and k_5.



Case-1: For $k_5 \neq 0, k_4 \neq 0$, we have $M^1 = \mathcal{Y}_4 + \mathcal{Y}_5$. By substituting $\tilde{k}_4 = \tilde{k}_5 = 1$ in (23), we obtain $\varsigma_1 = \frac{k_1}{k_4}$, $\varsigma_2 = \frac{k_2}{k_5}$, $\varsigma_3 = \frac{k_3}{2k_4 - k_5}$.
Case-2: For $k_5 \neq 0, k_4 = 0$, we have $M^2 = \mathcal{Y}_5 + a\mathcal{Y}_1,\ a \in \mathbb{R}$. By substituting $\tilde{k}_1 = \tilde{k}_5 = 1$ in (23), we obtain $\varsigma_2 = \frac{k_2}{k_5}$, $\varsigma_3 = \frac{k_3}{2k_4 - k_5}$.
Case-3: For $k_5 = 0, k_4 \neq 0$, we have $M^3 = \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R}$. By substituting $\tilde{k}_2 = \tilde{k}_4 = 1$ in (23), we obtain $\varsigma_1 = \frac{k_1}{k_4}$, $\varsigma_3 = \frac{k_3}{2k_4 - k_5}$.
When $k_4 = k_5 = 0$, the solution of (24) gives $\vartheta(k_1, k_2, k_3, k_4, k_5) = F(\frac{k_2 k_3}{k_1^2}, k_4, k_5)$. This means that the new invariant is $\frac{k_2 k_3}{k_1^2}$. Since $k_1 \neq 0$, so the next two vertices will be k_2 and k_3.
Case-4: For $k_4 = k_5 = 0, k_3 \neq 0, k_2 \neq 0, k_1 \neq 0$, we have $M^4 = \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3$.
Case-5: For $k_4 = k_5 = 0, k_3 = 0, k_2 \neq 0, k_1 \neq 0$, we have $M^5 = \mathcal{Y}_1 + \mathcal{Y}_2$.
Case-6: For $k_4 = k_5 = 0, k_3 \neq 0, k_2 = 0, k_1 \neq 0$, we have $M^6 = \mathcal{Y}_1 + \mathcal{Y}_3$.

Case-7: For $k_4 = k_5 = 0, k_2 = k_3 = 0, k_1 \neq 0$, we have $M^7 = \mathcal{Y}_1$.
So, the one-dimensional optimal system of subalgebras of Lie algebra (7) is presented as

$$\begin{aligned} M^1 &= \mathcal{Y}_4 + \mathcal{Y}_5, \\ M^2 &= \mathcal{Y}_5 + a\mathcal{Y}_1,\ a \in \mathbb{R}, \\ M^3 &= \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R}, \\ M^4 &= \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3, \\ M^5 &= \mathcal{Y}_1 + \mathcal{Y}_2, \\ M^6 &= \mathcal{Y}_1 + \mathcal{Y}_3, \\ M^7 &= \mathcal{Y}_1. \end{aligned} \tag{25}$$

2.1.3. Optimal system for Case 3

For Lie algebra $\mathcal{L}_6$ presented in (8), the non-zero commutators are provided by

$$[\mathcal{Y}_1, \mathcal{Y}_4] = \frac{\mathcal{Y}_3}{\alpha},\ [\mathcal{Y}_1, \mathcal{Y}_6] = \mathcal{Y}_1,\ [\mathcal{Y}_2, \mathcal{Y}_5] = \mathcal{Y}_2,\ [\mathcal{Y}_3, \mathcal{Y}_4] = \mathcal{Y}_1,\ [\mathcal{Y}_3, \mathcal{Y}_6] = \mathcal{Y}_3. \tag{26}$$

In Table 3, the adjoint actions of (8) are presented.

Table 3: Adjoint Table

$Ad(e^{\varsigma})$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$	$\mathcal{Y}_6$
$\mathcal{Y}_1$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \frac{\varsigma}{\alpha}\mathcal{Y}_3$	$\mathcal{Y}_5$	$\mathcal{Y}_6 - \varsigma\mathcal{Y}_1$
$\mathcal{Y}_2$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5 - \varsigma\mathcal{Y}_2$	$\mathcal{Y}_6$
$\mathcal{Y}_3$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4 - \varsigma\mathcal{Y}_1$	$\mathcal{Y}_5$	$\mathcal{Y}_6 - \varsigma\mathcal{Y}_3$
$\mathcal{Y}_4$	$\frac{(e^{\frac{\varsigma}{\sqrt{\alpha}}} - e^{\frac{-\varsigma}{\sqrt{\alpha}}})}{2\sqrt{\alpha}}\mathcal{Y}_3 + \frac{(e^{\frac{\varsigma}{\sqrt{\alpha}}} + e^{\frac{-\varsigma}{\sqrt{\alpha}}})}{2}\mathcal{Y}_1$	$\mathcal{Y}_2$	$\frac{(e^{\frac{\varsigma}{\sqrt{\alpha}}} + e^{\frac{-\varsigma}{\sqrt{\alpha}}})}{2}\mathcal{Y}_3 - \frac{(e^{\frac{-\varsigma}{\sqrt{\alpha}}} - e^{\frac{\varsigma}{\sqrt{\alpha}}})\sqrt{\alpha}}{2}\mathcal{Y}_1$	$\mathcal{Y}_4$	$\mathcal{Y}_5$	$\mathcal{Y}_6$
$\mathcal{Y}_5$	$\mathcal{Y}_1$	$e^{\varsigma}\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$	$\mathcal{Y}_6$
$\mathcal{Y}_6$	$e^{\varsigma}\mathcal{Y}_1$	$\mathcal{Y}_2$	$e^{\varsigma}\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$	$\mathcal{Y}_6$

For Lie algebra (8), the adjoint transformation matrix is presented as follows

$$A = \begin{bmatrix} \frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & \frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & 0 & 0 \\ 0 & e^{\varsigma_5} & 0 & 0 & 0 & 0 \\ \frac{-\sqrt{\alpha}(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & \frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & 0 & 0 \\ \frac{-\varsigma_3(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + \frac{\varsigma_1(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}} & 0 & \frac{-\varsigma_3(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}} - \frac{\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\alpha} & 1 & 0 & 0 \\ 0 & -\varsigma_2 e^{\varsigma_5} & 0 & 0 & 1 & 0 \\ \frac{-\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + \frac{\varsigma_3\sqrt{\alpha}(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & \frac{-\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} - e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}} - \frac{\varsigma_3\sqrt{\alpha}(e^{\frac{\varsigma_4}{\sqrt{\alpha}}} + e^{-\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} & 0 & 0 & 1 \end{bmatrix}$$

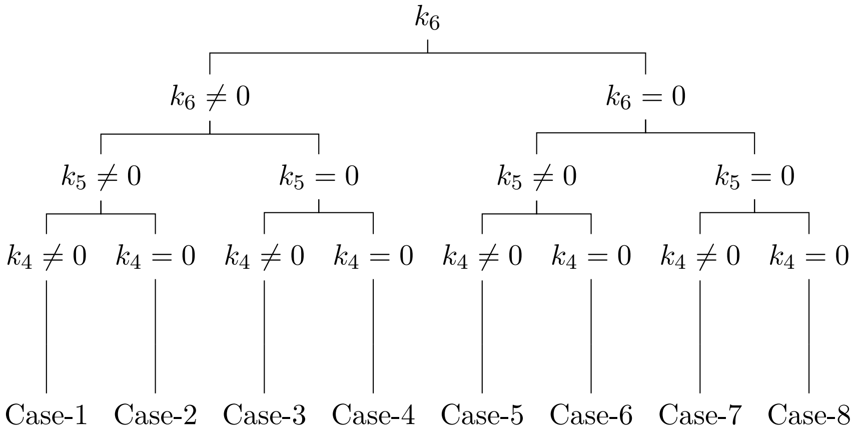
By using adjoint transformation matrix in (17), we obtain the following system of equations

$$
\begin{aligned}
\tilde{k}_1 &= k_1\frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} - k_3\frac{\sqrt{\alpha}(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + k_4\Big(\frac{-\varsigma_3(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + \frac{\varsigma_1(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}}\Big) \\
&\quad + k_6\Big(\frac{-\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + \frac{\varsigma_3\sqrt{\alpha}(e^{\frac{-\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2}\Big), \\
\tilde{k}_2 &= k_2e^{\varsigma_5} - k_5\varsigma_2e^{\varsigma_5}, \\
\tilde{k}_3 &= k_1\frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + k_3\frac{(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2} + k_4\Big(\frac{-\varsigma_3(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}} - \frac{\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\alpha}\Big) \\
&\quad + k_6\Big(\frac{-\varsigma_1(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}-e^{\frac{-\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2\sqrt{\alpha}} - \frac{\varsigma_3\sqrt{\alpha}(e^{\frac{\varsigma_4}{\sqrt{\alpha}}}+e^{-\frac{\varsigma_4}{\sqrt{\alpha}}})e^{\varsigma_6}}{2}\Big), \\
\tilde{k}_4 &= k_4, \\
\tilde{k}_5 &= k_5, \\
\tilde{k}_6 &= k_6.
\end{aligned}
\tag{27}
$$

For the computation of basic invariants, we have the following system of linear PDEs derived from (15).

$$
\begin{aligned}
&k_6\frac{\partial\vartheta}{\partial k_1} + \frac{k_4}{\alpha}\frac{\partial\vartheta}{\partial k_3} = 0, \\
&k_5\frac{\partial\vartheta}{\partial k_2} = 0, \\
&k_4\frac{\partial\vartheta}{\partial k_1} + k_6\frac{\partial\vartheta}{\partial k_3} = 0, \\
&k_3\frac{\partial\vartheta}{\partial k_1} + \frac{k_1}{\alpha}\frac{\partial\vartheta}{\partial k_3} = 0, \\
&k_2\frac{\partial\vartheta}{\partial k_2} = 0, \\
&k_1\frac{\partial\vartheta}{\partial k_1} + k_3\frac{\partial\vartheta}{\partial k_3} = 0.
\end{aligned}
\tag{28}
$$

This implies $\Phi(k_1, k_2, k_3, k_4, k_5, k_6) = F(k_4, k_5, k_6)$. So, the invariants k_4, k_5 and k_6 are the vertices of the tree.



Case-1: For $k_6 \neq 0, k_5 \neq 0, k_4 \neq 0$, we have $N^1 = \mathcal{Y}_4 + \mathcal{Y}_5 + \mathcal{Y}_6$. By substituting $\tilde{k}_4 = \tilde{k}_5 = \tilde{k}_6 = 1$ in (27), we obtain $\varsigma_1 = \frac{\sqrt{\alpha}k_1k_6+k_3}{(\frac{k_4}{\alpha}+\frac{\sqrt{\alpha}}{2}{k_6}^2)},\ \varsigma_2 = \frac{k_2}{k_5},\ \varsigma_3 = \frac{k_1}{k_4} - \frac{k_6}{2k_4}\left(\frac{\sqrt{\alpha}k_1k_6+k_3}{(\frac{k_4}{\alpha}+\frac{\sqrt{\alpha}}{2}{k_6}^2)}\right),\ \varsigma_4 = 0$.
Case-2: For $k_6 \neq 0, k_5 \neq 0, k_4 = 0$, we have $N^2 = \mathcal{Y}_5 + \mathcal{Y}_6$. By substituting $\tilde{k}_5 = \tilde{k}_6 = 1$ in (27), we obtain $\varsigma_1 = \frac{2k_1}{k_6},\ \varsigma_2 = \frac{k_2}{k_5},\ \varsigma_3 = \frac{k_3}{\sqrt{\alpha}k_6},\ \varsigma_4 = 0$.
Case-3: For $k_6 \neq 0, k_5 = 0, k_4 \neq 0$, we have $N^3 = \mathcal{Y}_6 + \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R}$. By substituting $\tilde{k}_2 = \tilde{k}_4 = \tilde{k}_6 = 1$ in (27), we obtain $\varsigma_1 = \frac{\sqrt{\alpha}k_1k_6+k_3}{(\frac{k_4}{\alpha}+\frac{\sqrt{\alpha}}{2}{k_6}^2)},\ \varsigma_3 = \frac{k_1}{k_4} - \frac{k_6}{2k_4}\left(\frac{\sqrt{\alpha}k_1k_6+k_3}{(\frac{k_4}{\alpha}+\frac{\sqrt{\alpha}}{2}{k_6}^2)}\right),\ \varsigma_4 = 0$.
Case-4: For $k_6 \neq 0, k_5 = 0, k_4 = 0$, we have $N^4 = \mathcal{Y}_6 + a\mathcal{Y}_2,\ a \in \mathbb{R}$. By substituting $\tilde{k}_2 = \tilde{k}_6 = 1$ in (27), we obtain $\varsigma_1 = \frac{2k_1}{k_6},\ \varsigma_3 = \frac{k_3}{\sqrt{\alpha}k_6},\ \varsigma_4 = 0$.
Case-5: For $k_6 = 0, k_5 \neq 0, k_4 \neq 0$, we have $N^5 = \mathcal{Y}_4 + \mathcal{Y}_5$. By substituting $\tilde{k}_4 = \tilde{k}_5 = 1$ in (27), we obtain $\varsigma_1 = \frac{\alpha k_3}{k_4},\ \varsigma_2 = \frac{k_2}{k_5},\ \varsigma_3 = \frac{k_1}{k_4},\ \varsigma_4 = 0$.
Case-6: $k_6 = 0, k_5 \neq 0, k_4 = 0$. For $\varsigma_2 = \frac{k_2}{k_5},\ \varsigma_4 = \left(\ln\left(\frac{k_1-k_3}{k_1+k_3}\right)\right)^{\frac{\sqrt{\alpha}}{2}}$, we get $N^6 = \mathcal{Y}_5 + a\mathcal{Y}_1,\ a \in \mathbb{R}$. If we choose $\varsigma_2 = \frac{k_2}{k_5},\ \varsigma_4 = \left(\ln\left(\frac{\sqrt{\alpha}k_3-k_1}{\sqrt{\alpha}k_3+k_1}\right)\right)^{\frac{\sqrt{\alpha}}{2}}$, we get $N^7 = \mathcal{Y}_5 + a\mathcal{Y}_3,\ a \in \mathbb{R}$.
Case-7: For $k_6 = 0, k_5 = 0, k_4 \neq 0$, we have $N^8 = \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R}$. By substituting $\tilde{k}_2 = \tilde{k}_4 = 1$ in (27), we obtain $\varsigma_1 = \frac{\alpha k_3}{k_4},\ \varsigma_3 = \frac{k_1}{k_4},\ \varsigma_4 = 0$.
Case-8: $k_6 = 0, k_5 = 0, k_4 = 0$. For $\varsigma_4 = \left(\ln\left(\frac{k_1-k_3}{k_1+k_3}\right)\right)^{\frac{\sqrt{\alpha}}{2}}$, we get $N^9 = \mathcal{Y}_2 + a\mathcal{Y}_1,\ a \in \mathbb{R}$. If we choose $\varsigma_4 = \left(\ln\left(\frac{\sqrt{\alpha}k_3-k_1}{\sqrt{\alpha}k_3+k_1}\right)\right)^{\frac{\sqrt{\alpha}}{2}}$, we get $N^{10} = \mathcal{Y}_2 + a\mathcal{Y}_3,\ a \in \mathbb{R}$.

So, the one-dimensional optimal system of subalgebras of Lie algebra (8) is presented as

$$\begin{aligned}
N^1 &= \mathcal{Y}_4 + \mathcal{Y}_5 + \mathcal{Y}_6,\\
N^2 &= \mathcal{Y}_5 + \mathcal{Y}_6,\\
N^3 &= \mathcal{Y}_6 + \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R},\\
N^4 &= \mathcal{Y}_6 + a\mathcal{Y}_2,\ a \in \mathbb{R},\\
N^5 &= \mathcal{Y}_4 + \mathcal{Y}_5,\\
N^6 &= \mathcal{Y}_5 + a\mathcal{Y}_1,\ a \in \mathbb{R},\\
N^7 &= \mathcal{Y}_5 + a\mathcal{Y}_3,\ a \in \mathbb{R},\\
N^8 &= \mathcal{Y}_4 + a\mathcal{Y}_2,\ a \in \mathbb{R},\\
N^9 &= \mathcal{Y}_2 + a\mathcal{Y}_1,\ a \in \mathbb{R},\\
N^{10} &= \mathcal{Y}_2 + a\mathcal{Y}_3,\ a \in \mathbb{R}.
\end{aligned} \tag{29}$$

3. Group invariant solutions

3.1. Similarity reductions for Case 1

Case-a: Consider $\mathcal{Y}_1 = \frac{\partial}{\partial x}$.
The solution of the characteristic equation for $\mathcal{Y}_1$ provides the symmetry invariants

$$\begin{aligned}
&p = y,\\
&q = t,\\
&w(x, y, t) = \mathcal{H}(p, q).
\end{aligned} \tag{30}$$

By inserting (30) in (2), we get

$$\mathcal{H}_{qq} - g'{\mathcal{H}_p}^2 - g\mathcal{H}_{pp} = 0. \tag{31}$$

Infinitesimals for (31) are as follows

$$\begin{aligned}
&\zeta_p = c_1 p + c_2,\\
&\zeta_q = c_1 q + c_3,\\
&\Phi_{\mathcal{H}} = 0.
\end{aligned} \tag{32}$$

Case-a1: In (32), set $c_3 = 1$ and $c_i = 0$ for $i = 1, 2$. Then we obtain the symmetry invariants

$$\begin{aligned}
&r = p,\\
&\mathcal{H}(p, q) = \mu(r).
\end{aligned} \tag{33}$$

The substitution of (33) in (31) yields the following ODE

$$g\mu'' + g'{\mu'}^2 = 0. \tag{34}$$

If we take $g = b_1\mu + b_2$, then the solution of (34) takes the form

$$\mu(r) = \frac{-b_2 + \sqrt{2c_1b_1r + 2c_2b_1 + {b_2}^2}}{b_1}, \tag{35}$$

which implies

$$\mathcal{H}(p,q) = \frac{-b_2 + \sqrt{2c_1b_1p + 2c_2b_1 + {b_2}^2}}{b_1}. \tag{36}$$

Hence, the solution of (2) is

$$w(x,y,t) = \frac{-b_2 + \sqrt{2c_1b_1y + 2c_2b_1 + {b_2}^2}}{b_1}, \tag{37}$$

where c_1 and c_2 are constants of integration.
This solution is same, as obtained in [22].

Case-b: Consider $\mathcal{Y}_1 + \mathcal{Y}_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.
The solution of the characteristic equation for $\mathcal{Y}_1 + \mathcal{Y}_2$ provides the symmetry invariants

$$\begin{aligned} p &= t, \\ q &= y - x, \\ w(x,y,t) &= \mathcal{H}(p,q). \end{aligned} \tag{38}$$

By inserting (38) in (2), we get

$$\mathcal{H}_{pp} - (f' + g'){\mathcal{H}_q}^2 - (f + g)\mathcal{H}_{qq} = 0. \tag{39}$$

Infinitesimals for (39) are as follows

$$\begin{aligned} \zeta_p &= c_1p + c_2, \\ \zeta_q &= c_1q + c_3, \\ \Phi_{\mathcal{H}} &= 0. \end{aligned} \tag{40}$$

Case-b1: In (40), set $c_2 = 1$ and $c_i = 0$ for $i = 1, 3$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{41}$$

The substitution of (41) in (39) yields the following ODE

$$(f + g)\mu'' + (f' + g'){\mu'}^2 = 0. \tag{42}$$

If we take $f = a_1\mu^2 + a_2,\ g = b_1\mu$, then the solution of (42) takes the form

$$\mu(r) = \frac{1}{2a_1}\Big(12c_1ra_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2r^2a_1{}^2 + 72c_1c_2ra_1{}^2 + 36c_1ra_1a_2b_1 - 6c_1rb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1\Big)^{\frac{1}{3}} - (4a_1a_2 - b_1{}^2)\Big/ \Big(2a_1(12c_1ra_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2r^2a_1{}^2 + 72c_1c_2ra_1{}^2 + 36c_1ra_1a_2b_1 - 6c_1rb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1)^{\frac{1}{3}}\Big) - \frac{b_1}{2a_1}, \tag{43}$$

which implies

$$\mathcal{H}(p,q) = \frac{1}{2a_1}\Big(12c_1qa_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2q^2a_1{}^2 + 72c_1c_2qa_1{}^2 + 36c_1qa_1a_2b_1 - 6c_1qb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1\Big)^{\frac{1}{3}} - (4a_1a_2 - b_1{}^2)\Big/ \Big(2a_1(12c_1qa_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2q^2a_1{}^2 + 72c_1c_2qa_1{}^2 + 36c_1qa_1a_2b_1 - 6c_1qb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1)^{\frac{1}{3}}\Big) - \frac{b_1}{2a_1}. \tag{44}$$

Hence, the solution of (2) is

$$w(x,y,t) = \frac{1}{2a_1}\Big(12c_1qa_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2q^2a_1{}^2 + 72c_1c_2qa_1{}^2 + 36c_1qa_1a_2b_1 - 6c_1qb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1\Big)^{\frac{1}{3}} - (4a_1a_2 - b_1{}^2)\Big/ \Big(2a_1(12c_1qa_1{}^2 + 12c_2a_1{}^2 + 6a_1a_2b_1 - b_1{}^3 + 2(36c_1{}^2q^2a_1{}^2 + 72c_1c_2qa_1{}^2 + 36c_1qa_1a_2b_1 - 6c_1qb_1{}^3 + 36c_2{}^2a_1{}^2 + 36c_2a_1a_2b_1 - 6c_2b_1{}^3 + 16a_1a_2{}^3 - 3a_2{}^2b_1{}^2)^{\frac{1}{2}}a_1)^{\frac{1}{3}}\Big) - \frac{b_1}{2a_1}, \tag{45}$$

where c_1 and c_2 are constants of integration.

3.2. Similarity reductions for Case 2

Case-a: Consider $\mathcal{Y}_1 + \mathcal{Y}_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$.
The solution of the characteristic equation for $\mathcal{Y}_1 + \mathcal{Y}_3$ provides the symmetry invariants

$$\begin{aligned} &p = y,\\ &q = -x + t,\\ &w(x,y,t) = \mathcal{H}(p,q). \end{aligned} \tag{46}$$

By inserting (46) in (2), we get

$$\mathcal{H}_{qq} - \alpha\mathcal{H}\mathcal{H}_{qq} - \alpha{\mathcal{H}_q}^2 - \beta\mathcal{H}^2\mathcal{H}_{pp} - 2\beta\mathcal{H}{\mathcal{H}_p}^2 = 0. \tag{47}$$

Infinitesimals for (47) are as follows

$$\begin{aligned} \zeta_p &= c_1 p + c_2, \\ \zeta_q &= c_1 q + c_3, \\ \Phi_{\mathcal{H}} &= 0. \end{aligned} \tag{48}$$

Case-a1: In (48), set $c_2 = 1$ and $c_i = 0$ for $i = 1, 3$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q, \\ \mathcal{H}(p, q) &= \mu(r). \end{aligned} \tag{49}$$

The substitution of (49) in (47) yields the following ODE

$$\mu'' - \alpha\mu\mu'' - \alpha\mu'^2 = 0. \tag{50}$$

The solution of (50) takes the form

$$\mu(r) = \frac{1 - \sqrt{1 + (2c_1 r + 2c_2)\alpha}}{\alpha}, \tag{51}$$

which implies

$$\mathcal{H}(p, q) = \frac{1 - \sqrt{1 + (2c_1 q + 2c_2)\alpha}}{\alpha}. \tag{52}$$

Hence, the solution of (2) is

$$w(x, y, t) = \frac{1 - \sqrt{1 + (2c_1(t - x) + 2c_2)\alpha}}{\alpha}, \tag{53}$$

where c_1 and c_2 are constants of integration.

Case-a2: In (48), set $c_3 = 1$ and $c_i = 0$ for $i = 1, 2$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= p, \\ \mathcal{H}(p, q) &= \mu(r). \end{aligned} \tag{54}$$

The substitution of (54) in (47) yields the following ODE

$$\mu^2\mu'' + 2\mu\mu'^2 = 0. \tag{55}$$

The solution of (55) takes the form

$$\mu(r) = (3c_1 r + 3c_2)^{\frac{1}{3}}, \tag{56}$$

which implies

$$\mathcal{H}(p,q) = (3c_1 p + 3c_2)^{\frac{1}{3}}. \tag{57}$$

Hence, the solution of (2) is

$$w(x,y,t) = (3c_1 y + 3c_2)^{\frac{1}{3}}, \tag{58}$$

where c_1 and c_2 are constants of integration.

Case-a3: In (48), set $c_2 = c_3 = 1$ and $c_1 = 0$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q - p, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{59}$$

The substitution of (59) in (47) yields the following ODE

$$-\beta\mu^2\mu'' - 2\beta\mu\mu'^2 - \alpha\mu\mu'' - \alpha\mu'^2 + \mu'' = 0. \tag{60}$$

The solution of (60) takes the form

$$\mu(r) = \frac{1}{2\beta}\Big(12c_1\beta^2 r + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2 r^2 + 72c_1c_2\beta^2 r - 6c_1\alpha^3 r - 36c_1\alpha\beta r + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta\Big)^{\frac{1}{3}} + \left(\alpha^2 + 4\beta\right) \Big/ \Big(2\beta(12c_1\beta^2 r + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2 r^2 + 72c_1c_2\beta^2 r - 6c_1\alpha^3 r - 36c_1\alpha\beta r + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta)^{\frac{1}{3}}\Big) - \frac{\alpha}{2\beta},$$

which implies

$$\mathcal{H}(p,q) = \frac{1}{2\beta}\Big(12c_1\beta^2(q-p) + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2(q-p)^2 + 72c_1c_2\beta^2(q-p) - 6c_1\alpha^3(q-p) - 36c_1\alpha\beta(q-p) + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta\Big)^{\frac{1}{3}} + \left(\alpha^2 + 4\beta\right) \Big/ \Big(2\beta(12c_1\beta^2(q-p) + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2(q-p)^2 + 72c_1c_2\beta^2(q-p) - 6c_1\alpha^3(q-p) - 36c_1\alpha\beta(q-p) + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta)^{\frac{1}{3}}\Big) - \frac{\alpha}{2\beta}.$$

Hence, the solution of (2) is

$$w(x,y,t) = \frac{1}{2\beta}\Big(12c_1\beta^2(t-x-y) + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2(t-x-y)^2 + 72c_1c_2\beta^2(t-x-y) - 6c_1\alpha^3(t-x-y) - 36c_1\alpha\beta(t-x-y) + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta\Big)^{\frac{1}{3}} + \left(\alpha^2 + 4\beta\right) \Big/ \Big(2\beta(12c_1\beta^2(t-x-y) + 12c_2\beta^2 - \alpha^3 + 2(36{c_1}^2\beta^2(t-x-y)^2 + 72c_1c_2\beta^2(t-x-y) - 6c_1\alpha^3(t-x-y) - 36c_1\alpha\beta(t-x-y) + 36{c_2}^2\beta^2 - 6c_2\alpha^3 - 36c_2\alpha\beta - 3\alpha^2 - 16\beta)^{\frac{1}{2}}\beta - 6\alpha\beta)^{\frac{1}{3}}\Big) - \frac{\alpha}{2\beta},$$

where c_1 and c_2 are constants of integration.

Case-b: Consider $\mathcal{Y}_1 = \frac{\partial}{\partial x}$.
The solution of the characteristic equation for $\mathcal{Y}_1$ provides the symmetry invariants

$$\begin{aligned} p &= t, \\ q &= y, \\ w(x,y,t) &= \mathcal{H}(p,q). \end{aligned} \tag{61}$$

By inserting (61) in (2), we get

$$\mathcal{H}_{pp} - \beta\mathcal{H}^2\mathcal{H}_{qq} - 2\beta\mathcal{H}{\mathcal{H}_q}^2 = 0. \tag{62}$$

Infinitesimals for (62) are as follows

$$\begin{aligned} \zeta_p &= c_1 p + c_2, \\ \zeta_q &= c_3 q + c_4, \\ \Phi_{\mathcal{H}} &= -\mathcal{H}(c_1 - c_3). \end{aligned} \tag{63}$$

Case-b1: In (63), set $c_4 = 1$ and $c_i = 0$ for $i = 1, 2, 3$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= p, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{64}$$

The substitution of (64) in (62) yields the following ODE

$$\mu'' = 0. \tag{65}$$

The solution of (65) takes the form

$$\mu(r) = c_1 r + c_2, \tag{66}$$

which implies

$$\mathcal{H}(p,q) = c_1 p + c_2. \tag{67}$$

Hence, the solution of (2) is

$$w(x,y,t) = c_1 t + c_2, \tag{68}$$

where c_1 and c_2 are constants of integration.
This solution is also obtained in [22].
Case-b2: In (63), set $c_2 = c_4 = 1$ and $c_i = 0$ for $i = 1, 3$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q - p, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{69}$$

The substitution of (69) in (62) yields the following ODE

$$\mu'' - \beta\mu^2\mu'' - 2\beta\mu{\mu'}^2 = 0. \tag{70}$$

The solution of (70) takes the form

$$\mu(r) = \Big(2^{\frac{1}{3}}(2^{\frac{1}{3}}(\sqrt{-4+9(-c_1 r - c_2)^2\beta} + (3c_1 r + 3c_2)\sqrt{\beta})^{\frac{2}{3}} + 2)\Big) \Big/ \Big(2\sqrt{\beta}(\sqrt{-4+9(-c_1 r - c_2)^2\beta} + (3c_1 r + 3c_2)\sqrt{\beta})^{\frac{1}{3}}\Big). \tag{71}$$

which implies

$$\mathcal{H}(p,q) = \Big(2^{\frac{1}{3}}(2^{\frac{1}{3}}(\sqrt{-4+9(-c_1(q-p) - c_2)^2\beta} + (3c_1(q-p) + 3c_2)\sqrt{\beta})^{\frac{2}{3}} + 2)\Big) \Big/ \Big(2\sqrt{\beta} (\sqrt{-4+9(-c_1(q-p) - c_2)^2\beta} + (3c_1(q-p) + 3c_2)\sqrt{\beta})^{\frac{1}{3}}\Big). \tag{72}$$

Hence, the solution of (2) is

$$w(x,y,t) = \Big(2^{\frac{1}{3}}(2^{\frac{1}{3}}(\sqrt{-4+9(-c_1(y-t) - c_2)^2\beta} + (3c_1(y-t) + 3c_2)\sqrt{\beta})^{\frac{2}{3}} + 2)\Big) \Big/ \Big(2\sqrt{\beta} (\sqrt{-4+9(-c_1(y-t) - c_2)^2\beta} + (3c_1(y-t) + 3c_2)\sqrt{\beta})^{\frac{1}{3}}\Big), \tag{73}$$

where c_1 and c_2 are constants of integration.

Case-b3: In (63), set $c_3 = 1$ and $c_i = 0$ for $i = 1, 2, 4$. Then we obtain the symmetry invariants

$$\begin{aligned} r &= p, \\ \mathcal{H}(p,q) &= q\mu(r). \end{aligned} \tag{74}$$

The substitution of (74) in (62) yields the following ODE

$$\mu'' - 2\beta\mu^3 = 0. \tag{75}$$

The solution of (75) takes the form

$$\mu(r) = c_2 \ JacobiSN\left((\iota\sqrt{\beta}r + c_1)c_2, \iota\right), \tag{76}$$

which implies

$$\mathcal{H}(p,q) = c_2 \ JacobiSN\left((\iota\sqrt{\beta}p + c_1)c_2, \iota\right) q. \tag{77}$$

Hence, the solution of (2) is

$$w(x,y,t) = c_2 \ JacobiSN\left((\iota\sqrt{\beta}t + c_1)c_2, \iota\right) y, \tag{78}$$

where c_1 and c_2 are constants of integration.

3.3. Similarity reductions for Case 3

Case-a: Consider $\mathcal{Y}_2 = \frac{\partial}{\partial y}$.
The solution of the characteristic equation for $\mathcal{Y}_2$ provides the symmetry invariants

$$\begin{aligned} &p = t, \\ &q = x, \\ &w(x, y, t) = \mathcal{H}(p, q). \end{aligned} \tag{79}$$

By inserting (79) in (2), we get

$$\mathcal{H}_{pp} - \alpha \mathcal{H}_{qq} = 0. \tag{80}$$

Infinitesimals for (80) are as follows

$$\begin{aligned} &\zeta_p = f_5(q + \sqrt{\alpha} p) + f_6(q - \sqrt{\alpha} p), \\ &\zeta_q = \sqrt{\alpha} f_5(q + \sqrt{\alpha} p) - \sqrt{\alpha} f_6(q - \sqrt{\alpha} p) + c_2, \\ &\Phi_{\mathcal{H}} = c_1 \mathcal{H} + f_3(q + \sqrt{\alpha} p) + f_4(q - \sqrt{\alpha} p). \end{aligned} \tag{81}$$

Case-a1: In (81), set $c_2 = 1$, $c_1 = 0$ and all arbitrary functions zero. Then we obtain the symmetry invariants

$$\begin{aligned} &r = p, \\ &\mathcal{H}(p, q) = \mu(r). \end{aligned} \tag{82}$$

The substitution of (82) in (80) yields the following ODE

$$\mu'' = 0. \tag{83}$$

The solution of (83) takes the form

$$\mu(r) = c_1 r + c_2, \tag{84}$$

which implies

$$\mathcal{H}(p, q) = c_1 p + c_2. \tag{85}$$

Hence, the solution of (2) is

$$w(x, y, t) = c_1 t + c_2, \tag{86}$$

where c_1 and c_2 are constants of integration.
Case-a2: In (81), set $c_1 = c_2 = 1$ and all arbitrary functions zero. Then we obtain the symmetry invariants

$$\begin{aligned} &r = p, \\ &\mathcal{H}(p, q) = e^q \mu(r). \end{aligned} \tag{87}$$

The substitution of (87) in (80) yields the following ODE

$$\mu'' - \alpha \mu = 0. \tag{88}$$

The solution of (88) takes the form

$$\mu(r) = (c_1 e^{2\sqrt{\alpha}r} + c_2)e^{-\sqrt{\alpha}r}, \tag{89}$$

which implies

$$\mathcal{H}(p,q) = (c_1 e^{2\sqrt{\alpha}p} + c_2)e^{q-\sqrt{\alpha}p}. \tag{90}$$

Hence, the solution of (2) is

$$w(x,y,t) = (c_1 e^{2\sqrt{\alpha}t} + c_2)e^{x-\sqrt{\alpha}t}, \tag{91}$$

where c_1 and c_2 are constants of integration.
Case-b: Consider $\mathcal{Y}_5 + a\mathcal{Y}_3 = y\frac{\partial}{\partial y} + 2w\frac{\partial}{\partial w} + a\frac{\partial}{\partial t}$.
The solution of the characteristic equation for $\mathcal{Y}_5 + a\mathcal{Y}_3$ provides the symmetry invariants

$$\begin{aligned} &p = x, \\ &q = t - a\ln(y), \\ &w(x,y,t) = y^2\mathcal{H}(p,q). \end{aligned} \tag{92}$$

By inserting (92) in (2), we get

$$-a^2\beta\mathcal{H}\mathcal{H}_{qq} - 6\beta\mathcal{H}^2 + 7\beta\mathcal{H}\mathcal{H}_q - \beta a^2{\mathcal{H}_q}^2 - \alpha\mathcal{H}_{pp} + \mathcal{H}_{qq} = 0. \tag{93}$$

Infinitesimals for (93) are as follows

$$\begin{aligned} &\zeta_p = c_1, \\ &\zeta_q = c_2, \\ &\Phi_{\mathcal{H}} = 0. \end{aligned} \tag{94}$$

Case-b1: In (94), set $c_2 = 1$ and $c_1 = 0$. Then we obtain the symmetry invariants

$$\begin{aligned} &r = p, \\ &\mathcal{H}(p,q) = \mu(r). \end{aligned} \tag{95}$$

The substitution of (95) in (93) yields the following ODE

$$\alpha\mu'' + 6\beta\mu^2 = 0. \tag{96}$$

The solution of (96) takes the form

$$\mu(r) = -\frac{WeierstrassP(r + c_1, 0, c_2)\alpha}{\beta}, \tag{97}$$

which implies

$$\mathcal{H}(p,q) = -\frac{WeierstrassP(p + c_1, 0, c_2)\alpha}{\beta}. \tag{98}$$

Hence, the solution of (2) is

$$w(x,y,t) = -\frac{WeierstrassP(x+c_1,0,c_2)\alpha y^2}{\beta}, \tag{99}$$

where c_1 and c_2 are constants of integration.

Case-c: Consider $\mathcal{Y}_2 + a\mathcal{Y}_3 = \frac{\partial}{\partial y} + a\frac{\partial}{\partial t}$.
The solution of the characteristic equation for $\mathcal{Y}_2 + a\mathcal{Y}_3$ provides the symmetry invariants

$$\begin{aligned} p &= x, \\ q &= t - ay, \\ w(x,y,t) &= \mathcal{H}(p,q). \end{aligned} \tag{100}$$

By inserting (100) in (2), we get

$$\mathcal{H}_{qq} - \alpha\mathcal{H}_{pp} - a^2\beta\mathcal{H}\mathcal{H}_{qq} - a^2\beta{\mathcal{H}_q}^2 = 0. \tag{101}$$

Infinitesimals for (101) are as follows

$$\begin{aligned} \zeta_p &= c_1 p + c_2, \\ \zeta_q &= c_3 q + c_4, \\ \Phi_{\mathcal{H}} &= -\frac{2(c_1-c_3)(a^2\beta\mathcal{H}-1)}{a^2\beta}. \end{aligned} \tag{102}$$

Case-c1: In (102), set $c_2 = 1$ and other constants zero. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{103}$$

The substitution of (103) in (101) yields the following ODE

$$\mu'' - a^2\beta\mu\mu'' - a^2\beta\mu'^2 = 0. \tag{104}$$

The solution of (104) takes the form

$$\mu(r) = \frac{1-\sqrt{1+2a^2\beta(c_1 r + c_2)}}{a^2\beta}, \tag{105}$$

which implies

$$\mathcal{H}(p,q) = \frac{1-\sqrt{1+2a^2\beta(c_1 q + c_2)}}{a^2\beta}. \tag{106}$$

Hence, the solution of (2) is

$$w(x,y,t) = \frac{1-\sqrt{1+2a^2\beta(c_1(t-ay)+c_2)}}{a^2\beta}, \tag{107}$$

where c_1 and c_2 are constants of integration.

Case-c2: In (102), set $c_4 = 1$ and other constants zero. Then we obtain the symmetry invariants

$$\begin{aligned} r &= p, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{108}$$

The substitution of (108) in (101) yields the following ODE

$$\mu'' = 0. \tag{109}$$

The solution of (109) takes the form

$$\mu(r) = c_1 r + c_2, \tag{110}$$

which implies

$$\mathcal{H}(p,q) = c_1 p + c_2. \tag{111}$$

Hence, the solution of (2) is

$$w(x,y,t) = c_1 x + c_2, \tag{112}$$

where c_1 and c_2 are constants of integration.

Case-c3: In (102), set $c_2 = c_4 = 1$ and other constants zero. Then we obtain the symmetry invariants

$$\begin{aligned} r &= q - p, \\ \mathcal{H}(p,q) &= \mu(r). \end{aligned} \tag{113}$$

The substitution of (113) in (101) yields the following ODE

$$(-a^2\beta\mu - \alpha + 1)\mu'' - a^2\beta\mu'^2 = 0. \tag{114}$$

The solution of (114) takes the form

$$\mu(r) = \frac{-\alpha + 1 - \sqrt{2(c_1 r + c_2)a^2\beta + (\alpha - 1)^2}}{a^2\beta}, \tag{115}$$

which implies

$$\mathcal{H}(p,q) = \frac{-\alpha + 1 - \sqrt{2(c_1(q-p) + c_2)a^2\beta + (\alpha - 1)^2}}{a^2\beta}. \tag{116}$$

Hence, the solution of (2) is

$$w(x,y,t) = \frac{-\alpha + 1 - \sqrt{2(c_1(t - ay - x) + c_2)a^2\beta + (\alpha - 1)^2}}{a^2\beta}, \tag{117}$$

where c_1 and c_2 are constants of integration.

4. Conservation Laws

This section demonstrates how to use the multiplier approach [12] to compute conservation laws for (2). The multipliers $\mathrm{Q}(x,y,t,w)$ for (2) can be computed by solving the equation given by [25]

$$\frac{\delta}{\delta w}\left(\mathrm{Q}\left(w_{tt}-(f(w)w_x)_x-(g(w)w_y)_y\right)\right)=0. \tag{118}$$

Equation (118) yields the following multiplier functions

$$\begin{aligned} &\mathrm{Q}_1=xyt, \quad \mathrm{Q}_2=xt, \quad \mathrm{Q}_3=xy, \quad \mathrm{Q}_4=x,\\ &\mathrm{Q}_5=yt, \quad \mathrm{Q}_6=t, \quad \mathrm{Q}_7=y, \quad \mathrm{Q}_8=1. \end{aligned} \tag{119}$$

The conserved vectors for (2) can be derived by substituting the multipliers (119) in the following relation [25]

$$\mathrm{Q}\left(w_{tt}-(f(w)w_x)_x-(g(w)w_y)_y\right)=D_t\Gamma^t+D_x\Gamma^x+D_y\Gamma^y, \tag{120}$$

and are formulated as follows

$$\Gamma_1=\begin{cases} {\Gamma_1}^t= & xy(tw_t-w),\\ {\Gamma_1}^x= & \frac{t}{2}(x^2g(w)w_y-2xyf(w)w_x+2y\int f dw),\\ {\Gamma_1}^y= & -\frac{xt}{2}g(w)(xw_x+2yw_y). \end{cases} \tag{121}$$

$$\Gamma_2=\begin{cases} {\Gamma_2}^t= & x(tw_t-w),\\ {\Gamma_2}^x= & t(-xf(w)w_x+\int f dw),\\ {\Gamma_2}^y= & -xtg(w)w_y. \end{cases} \tag{122}$$

$$\Gamma_3=\begin{cases} {\Gamma_3}^t= & xyw_t,\\ {\Gamma_3}^x= & y\int f dw-xyf(w)w_x+\frac{x^2}{2}g(w)w_y,\\ {\Gamma_3}^y= & -\frac{x}{2}g(w)(xw_x+2yw_y). \end{cases} \tag{123}$$

$$\Gamma_4=\begin{cases} {\Gamma_4}^t= & xw_t,\\ {\Gamma_4}^x= & -xf(w)w_x+\int f dw,\\ {\Gamma_4}^y= & -xg(w)w_y. \end{cases} \tag{124}$$

$$\Gamma_5=\begin{cases} {\Gamma_5}^t= & y(tw_t-w),\\ {\Gamma_5}^x= & t(xg(w)w_y-yf(w)w_x),\\ {\Gamma_5}^y= & -t(xw_x+yw_y)g(w). \end{cases} \tag{125}$$

$$\Gamma_6=\begin{cases} {\Gamma_6}^t= & tw_t-w,\\ {\Gamma_6}^x= & -tf(w)w_x,\\ {\Gamma_6}^y= & -tg(w)w_y. \end{cases} \tag{126}$$

$$\Gamma_7=\begin{cases} {\Gamma_7}^t= & yw_t,\\ {\Gamma_7}^x= & xg(w)w_y-yf(w)w_x,\\ {\Gamma_7}^y= & -g(w)(xw_x+yw_y). \end{cases} \tag{127}$$

$$\Gamma_8 = \begin{cases} {\Gamma_8}^t = & w_t, \\ {\Gamma_8}^x = & -f(w)w_x, \\ {\Gamma_8}^y = & -g(w)w_y. \end{cases} \tag{128}$$

The conserved vectors $\Gamma_2, \Gamma_4, \Gamma_6, \Gamma_8$ are same, which are derived in [23] by using a partial Lagrangian.

5. Exact solutions via conservation laws

This section deals with the obtention of exact solutions for (2) by using its conserved vectors. By using the technique explained in [26, 27], the conserved vector $(\Gamma^t, \Gamma^x, \Gamma^y)$ of (2) satisfies the condition stated as

$$\begin{aligned} &D_t\Gamma^t = 0, \\ &D_x\Gamma^x = 0, \\ &D_y\Gamma^y = 0. \end{aligned} \tag{129}$$

For $f(w) = \alpha w,\ g(w) = \beta w^2$, the conserved vector (122) becomes

$$\Gamma_2 = \begin{cases} {\Gamma_2}^t = & x(tw_t - w), \\ {\Gamma_2}^x = & \frac{\alpha t w}{2}(w - 2xw_x), \\ {\Gamma_2}^y = & -\beta x t w^2 w_y. \end{cases} \tag{130}$$

The following system is acquired by the insertion of conserved vector (130) in (129).

$$\begin{aligned} &w_{tt} = 0, \\ &w_x(-2xw_x + w) + w(-w_x - 2xw_{xx}) = 0, \\ &2{w_y}^2 + ww_{yy} = 0. \end{aligned} \tag{131}$$

The exact solution of (2) is founded by solving the system (131) and is described as follows

$$\begin{aligned} w(x,y,t) = &\sqrt{2c_3(c_1y + c_2)^{\frac{2}{3}}x + 2\left(-\frac{(12c_1y + 12c_2)^{\frac{1}{3}}}{4} + \frac{\iota\sqrt{3}(12c_1y + 12c_2)^{\frac{1}{3}}}{4}\right)^2 t} \\ &+ \frac{2c_4(c_1y + c_2)^{\frac{1}{3}}(312^{\frac{1}{3}} - 2c_3(18)^{\frac{1}{3}}x + 4{c_3}^2x^2)^{\frac{1}{4}}}{\sqrt{\frac{-\frac{53073}{6692}\iota c_3 x + 3\iota\sqrt{3} + 9}{6{c_3}^2(18)^{\frac{1}{3}}x^2 - 318^{\frac{2}{3}}c_3x + 27}}}. \end{aligned} \tag{132}$$

For $f(w) = \alpha,\ g(w) = \beta w$, the conserved vector (121) takes the following form

$$\Gamma_1 = \begin{cases} {\Gamma_1}^t = & w_t, \\ {\Gamma_1}^x = & -\alpha w_x, \\ {\Gamma_1}^y = & -\beta w w_y. \end{cases} \tag{133}$$

By substituting the conserved vector (133) in (129), we get

$$\begin{aligned} &w_{tt} = 0, \\ &w_{xx} = 0, \\ &{w_y}^2 + w w_{yy} = 0. \end{aligned} \tag{134}$$

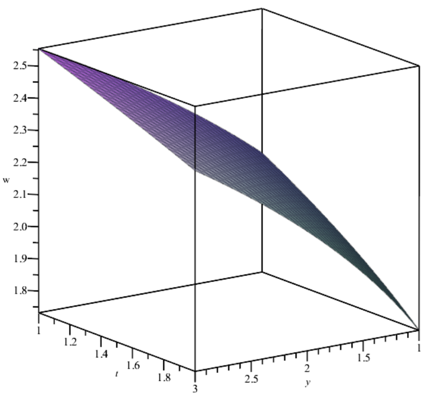
This provides the solution of (2) given by

$$w(x, y, t) = -xt\sqrt{2c_1 y + 2c_2} + \sqrt{c_1 y + c_2}(c_3 x + c_4 t + c_5). \tag{135}$$

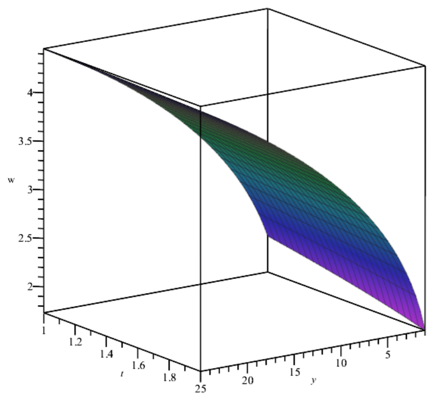
The solutions obtained in this section are not invariant under the Lie algebra.

6. Graphical representation of solutions

In the present section, 3D graphical representations of the obtained findings are given. Geometrical analysis is provided because mathematical expressions are insufficient to describe the physical wave patterns. Figure 1 describes the 3D graphics of (73) in the range of variables $1 \leq t \leq 2,\ 1 \leq y \leq 3$ and $1 \leq t \leq 2,\ 1 \leq y \leq 25$ by assuming $c_1 = c_2 = 1$ and $\beta = 1$. In figure 2, an exponential wave pattern of (91) is shown in the range of variables $1 \leq t \leq 2,\ 1 \leq x \leq 30$ and $1 \leq t \leq 20,\ 1 \leq x \leq 4$ by assuming $c_1 = c_2 = 1$ and $\alpha = 1$. Figure 3 describes the 3D graphics of (135) in the range of variables $1 \leq x \leq 2,\ 1 \leq y \leq 2,\ t = 2$ and $1 \leq x \leq 20,\ 1 \leq y \leq 60,\ t = 40$ by assuming $c_1 = c_2 = c_3 = c_4 = c_5 = 1$.

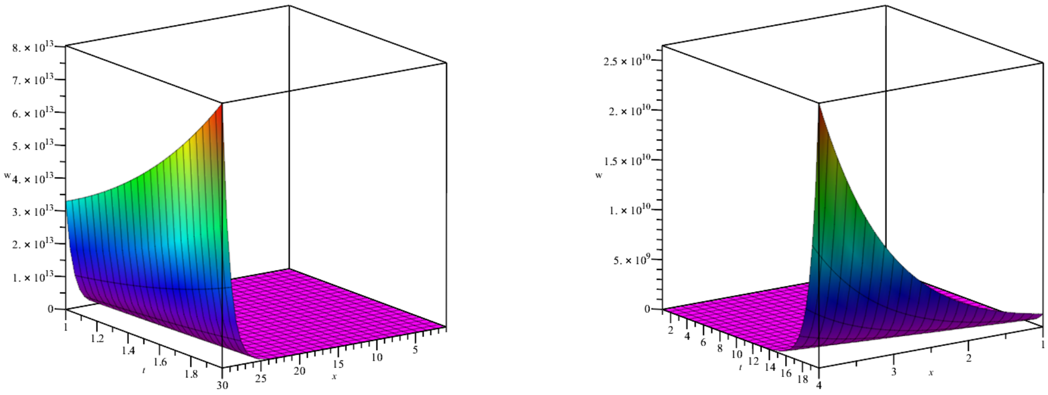


(a)



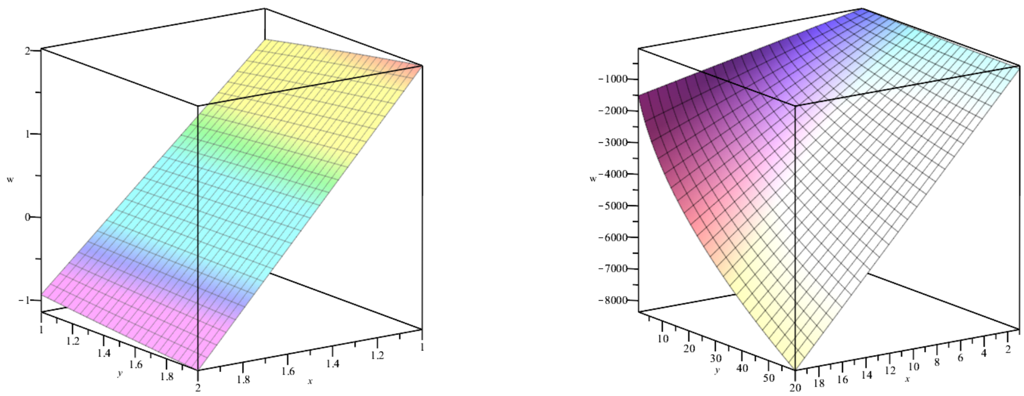
(b)

Figure 1: 3D graphic depictions of (73), by considering $c_1 = c_2 = 1$ and $\beta = 1$.



(a) (b)

Figure 2: 3D graphic depictions of (91) by considering $c_1 = c_2 = 1$ and $\alpha = 1$.



(a) t=2 (b) t=40

Figure 3: 3D graphic depictions of (135) by considering $c_1 = c_2 = c_3 = c_4 = c_5 = 1$.

7. Conclusions

Through the implementation of the Lie group analysis method, we investigated the characteristics of a (2+1)-dimensional nonlinear wave equation. We built an optimal set of subalgebras of Lie algebra, which are one-dimensional, and inspected its representatives in order to investigate invariant solutions and similarity reductions, which provide a multitude of explicit exact solutions using computerized symbolic computation. We applied the Lagrange multiplier approach to generate conserved vectors. We have also obtained precise solutions to the nonlinear wave equation through the use of these conserved vectors. Consequently, graphical representations of the answers are analyzed and illustrated using 3D visuals.

Ethics approval and consent to participate: All the authors approve their consent.

Consent for publication: The authors approve this version for publication.
Availability of data and materials: All data generated or analyzed during this study are included in this published article.
Competing interests: The authors professed no conflicts of interest.
Funding: No external funding received.

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