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A Fixed Point Result in a DCMS Setting and a Fredholm Integral Equation

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Abstract. In this present work, we prove a fixed point result for generalized contraction mappings in the setting of a double control metric space (DCMS) by using α-orbital admissibility. The uniqueness of the fixed point is established by adding further hypotheses. The presented result is supported by a concrete example. Moreover, we ensure the existence of a solution of a Fredholm type integral equation via a fixed point technique. Some consequences are also presented to make effective the obtained results.

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1. Introduction

The fixed point result due to Banach [1] is considered as the most essential theorem in fixed point theory. It asserts that a contraction mapping on a complete metric space admits a unique fixed point. In 1989, an interesting extension of the metric space was explored by Bakhtin [2] and Czerwik [3] by initiating the concept of b-metric spaces. Later, in 2017, this setting was generalized to extended b-metric spaces initiated by Kiran et al. [4], where the triangular inequality is extended via a controlled function. On the other hand, using two control functions $\varpi, \epsilon : \mho \times \mho \longrightarrow [1, \infty)$, the notion of a double controlled metric space [5] (DCMS) was considred by Abdeljawad et al. [5]. Many related

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works in this direction appeared, see as examples [6–9]. These generalizations led to study several fixed point results for contractions arising in many real applications for several problems in nonlinear analysis. Integral equations appear naturally in several branches of and engineering and science. Particularly, Fredholm integral equations appeared widely in various scientific areas, like computational mathematics, physics, continuum mechanics, medicine, acoustics and approximation theory. There are several analytical and numerical methods to solve Fredholm integral equations. In 2018, Karapinar et al. [10] solved a Fredholm integral equation given as follows:

$$\hbar(\varsigma) = \int_a^b \aleph(\varsigma, \upsilon, \hbar(\upsilon))d\upsilon + \zeta(\varsigma), \tag{1}$$

by using a fixed point method in the context of extended b-metric spaces. Several works in literature dealing with the existence of a solution of a Fredholm type integhral equations arise. For more details, see the papers [11–14]. In this work, by generalizing the equation (1), the following Fredholm functional integral equation is studied:

$$\hbar(t) = \int_a^b K(t, r, \hbar(r), \hbar(g(r)), \hbar(a), \hbar(b))dr + f(t). \tag{2}$$

Under appropriate conditions on functions K, g and f, we aim to resolve the equation (2) via a fixed point technique. Namely, we give some fixed point results in a DCMS via orbital α-admissibility. We also present some illustrated concrete examples. At the end, we solve a Fredhom type integral equation in order to show that our required conditions are applicable.

2. Preliminaries

Definition 1. *[5] Consider a nonempty set $\mho$. A function $\zeta : \mho \times \mho \longrightarrow [0, \infty)$ is termed as a DCM with controlled functions $\varpi, \epsilon : \mho \times \mho \longrightarrow [1, \infty)$ if for all $\eta, y, \daleth \in \mho$, we have*
(μ_1): $\zeta(\eta, y) = 0 \Longleftrightarrow \eta = y$;
(μ_2): $\zeta(\eta, y) = \zeta(y, \eta)$;
(μ_3): $\zeta(\eta, y) \leq \varpi(\eta, \daleth)\zeta(\eta, \daleth) + \epsilon(\daleth, y)\zeta(\daleth, y)$.
Here, $(\mho, \zeta)$ is termed as a DCMS.

Example 1. *Let $\mho = [0, \infty]$. Given $\zeta : \mho \times \mho \longrightarrow [0, \infty)$ as*

$$\zeta(\eta, \iota) = \begin{cases} 0 \quad if \quad \eta = \iota \\ \frac{\eta}{\eta+1} \quad if \quad \eta \neq 0 \quad and \quad \iota = 0 \\ \frac{\iota}{\iota+1} \quad if \quad \iota \neq 0 \quad and \quad \eta = 0 \\ \eta + \iota \ if \quad 0 \neq \eta \neq \iota \neq 0. \end{cases}$$

Take $\varpi, \epsilon : \mho \times \mho \longrightarrow [1, \infty)$ as $\varpi(\eta, \iota) = \epsilon(\eta, \iota) = 2\eta + 2\iota + 2$. Here, $(\mho, \zeta)$ is a DCMS.

Definition 2. *[5] Let $\{\tilde{s}_n\}$ be a sequence in a DCMS $(\mho, \zeta)$. Then,*

(i) $\{\tilde{s}_\jmath\}$ is called a convergent sequence, if, for any $\epsilon > 0$, there is an integer $\jmath_0 = \jmath_0(\epsilon)$ so that $\zeta(\tilde{s}_\jmath, \eta) < \epsilon$, for all $\jmath \geq \jmath_0$. One writes $\lim_{n\to\infty} \tilde{s}_\jmath = \eta$;

(ii) $\{\tilde{s}_\jmath\}$ is named Cauchy if $\lim_{\jmath,m\to\infty} \zeta(\tilde{s}_\jmath, \tilde{s}_m) = 0$;

(iii) $(\mho, \zeta)$ is called complete if each Cauchy sequence converges in $\mho$.

Definition 3. *Let $\top$ be a self-mapping on a DCMS $(\mho, \zeta)$. For $\tilde{s}_0 \in \mho$, the set*

$$O(\tilde{s}_0, \top) = \left\{\tilde{s}_0, \top\tilde{s}_0, \top^2\tilde{s}_0, \top^3\tilde{s}_0, \cdots\right\}$$

is named as an orbit of $\top$ at $\tilde{s}_0$. $\top$ is said to be orbitally continuous at $\wp \in \mho$ if $\lim_{k\to\infty} \top^k\tilde{s}_0 = \wp$ yields that $\lim_{k\to\infty} \top\top^k\tilde{s}_0 = \top\wp$. Also, when each Cauchy sequence having the form $\{\top^k\tilde{s}_0\}_k$ converges in $\mho$, then $(\mho, \zeta)$ is orbitally complete.

In the sequel, $Fix(\top) = \{\wp \in \mho \,/\, \top\wp = \wp\}$.

Remark 1. *The continuity yields the orbital continuity.*

Definition 4. *[15] Let $\alpha : \mho \times \mho \longrightarrow \mathbb{R}$ be a function ($\mho$ is a nonempty set). $\top : \mho \longrightarrow \mho$ is termed an α-admissible mapping, if for every $\imath, \ell \in \mho, \alpha(\imath, \ell) \geq 1$ implies $\alpha(\top\imath, \top\ell) \geq 1$. $\top$ is called α-orbitally admissible if, for each $\imath \in \mho, \alpha(\imath, \top\imath) \geq 1$ leads to $\alpha(\top\imath, \top^2\imath) \geq 1$.*

Definition 5. *$\psi : [0, +\infty) \longrightarrow [0, +\infty)$ is termed as a comparison function, if it is nondecreasing and $\lim_{\jmath\to\infty} \psi^\jmath(\tau) = 0$ for each $\tau > 0$. Here, $\psi^\jmath$ is the n^{th} iteration of ψ.*

Ψ is denoted the set of all comparison functions. FP is the abbreviation of a fixed point.

Lemma 1. *For each $\psi \in \Psi$, we have $\psi(0) = 0$ and $\psi(\Theta) < \Theta$ for any $\Theta > 0$.*

The next result is needful in the sequel.

Proposition 1. *Let $(\mho, \zeta)$ be a DCMS with two controlled functions ϖ, ϵ. Let $\{\tilde{s}_n\}$ be a convergent sequence in $\mho$ so that $\lim_{n\to\infty} \epsilon(\gamma, \tilde{s}_n)$ and $\lim_{n\to\infty} \varpi(\tilde{s}_n, \beta)$ exist and are finite for any $\gamma, \beta \in \mho$, then such a convergent sequence possesses a unique limit.*

Proof. Suppose $\{\tilde{s}_\ell\} \in \mho$ converges to σ and ς in $\mho$. We have $\lim_{\ell\to\infty} \zeta(\tilde{s}_\ell, \sigma) = 0$ and $\lim_{\ell\to\infty} \zeta(\tilde{s}_\ell, \varsigma) = 0$. By the triangle inequality in the DCMS, we obtain

$$\zeta(\sigma, \varsigma) \leq \varpi(\sigma, \tilde{s}_\ell)\zeta(\sigma, \tilde{s}_\ell) + \epsilon(\tilde{s}_\ell, \varsigma)\zeta(\tilde{s}_\ell, \varsigma).$$

Taking $\ell \longrightarrow 0$, we get $\zeta(\sigma, \varsigma) \leq 0$. Thus, $\zeta(\sigma, \varsigma) = 0$, which further implies that $\sigma = \varsigma$.

3. Main results

Our first theorem is stated as follows:

Theorem 1. *Let $\top$ be a self-mapping on a orbitally complete DCMS $(\mho, \zeta)$ with controlled functions $\varpi, \epsilon : \mho \times \mho \longrightarrow [1, +\infty[$. Suppose there are two functions $\alpha : \mho \times \mho \longrightarrow [0, +\infty)$, and $\psi \in \Psi$ so that*

$$\alpha(x,y)\varpi(x,y)\epsilon(x,y)\zeta(\top x, \top y) \leq \psi(R(x,y)) \quad \forall x, y \in \mho, \tag{3}$$

where

$$R(x,y) = \max\left\{\zeta(x,y), \zeta(x,\top x), \zeta(y,\top y), \frac{\zeta(x,\top x)[\varpi(x,y)\epsilon(x,y)+\zeta(y,\top y)]}{\varpi(x,y)\epsilon(x,y)+\zeta(x,y)}, \frac{\zeta(y,\top y)[\varpi(x,y)\epsilon(x,y)+\zeta(x,\top x)]}{\varpi(x,y)\epsilon(x,y)+\zeta(x,y)}\right\}.$$

Assume that:

(i) $\top$ *is α-orbitally admissible;*

(ii) there is $\tilde{s}_0 \in \mho$ satisfying $\alpha(\tilde{s}_0, \top\tilde{s}_0) \geq 1$;

(iii) $\sup_{m\geq 1} \lim_{i\to\infty} \dfrac{\epsilon(\tilde{s}_{i+1}, \tilde{s}_m)\varpi(\tilde{s}_{i+1}, \tilde{s}_{i+2})\psi^{i+1}(\zeta(\tilde{s}_0, \tilde{s}_1))}{\varpi(\tilde{s}_i, \tilde{s}_{i+1})\psi^i(\zeta(\tilde{s}_0, \tilde{s}_1))} < 1$, *where* $\tilde{s}_i = \top^i(\tilde{s}_0)$*;*

(iv) $\top$ *is orbitally continuous on* $\mho$*;*

(v) $\lim_{n\to\infty} \epsilon(\tilde{s}_n, x)$ *and* $\lim_{n\to\infty} \varpi(\tilde{s}_n, x)$ *exist and are finite.*

Therefore, $\top$ *possesses a FP in* $\mho$*. If in addition, we have*

$$(U): \quad x, x^* \in Fix(\top) \quad \textit{implies } \alpha(x, x^*) \geq 1,$$

then such a FP is unique.

Proof. By (ii), define a sequence $\{\tilde{s}_\ell\}$ in $\mho$ such that $\tilde{s}_{\ell+1} = \top\tilde{s}_\ell = \top^{n+1}\tilde{s}_0$, for all $\ell \in \mathbb{N}$. When $\tilde{s}_\ell = \tilde{s}_{\ell+1}$ for some $\ell \in \mathbb{N}$, then $\tilde{s}_\ell$ is a FP of $\top$. Now, suppose that $\tilde{s}_\ell \neq \tilde{s}_{\ell+1}$, for any $\ell \in \mathbb{N}$. Due to (i), $\alpha(\tilde{s}_0, \tilde{s}_1) = \alpha(\tilde{s}_0, \top\tilde{s}_0) \geq 1$ implies that $\alpha(\tilde{s}_1, \tilde{s}_2) = \alpha(\top\tilde{s}_0, \top\tilde{s}_1) \geq 1$. Then, $\alpha(\tilde{s}_2, \tilde{s}_3) = \alpha(\top\tilde{s}_1, \top\tilde{s}_2) \geq 1$. Continuing this process, one gets $\alpha(\tilde{s}_\ell, \tilde{s}_{\ell+1}) \geq 1$, for any $\ell \in \mathbb{N}$.
Letting $x = \tilde{s}_{\ell-1}$ and $y = \tilde{s}_\ell$ in (3), we have

$$\begin{aligned} \zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) &= \zeta(\top\tilde{s}_{\ell-1}, \top\tilde{s}_\ell) \\ &\leq \alpha(\tilde{s}_{\ell-1}, \tilde{s}_\ell)\varpi(\tilde{s}_{\ell-1}, \tilde{s}_\ell)\epsilon(\tilde{s}_{\ell-1}, \tilde{s}_\ell)\zeta(\top\tilde{s}_{\ell-1}, \top\tilde{s}_\ell) \\ &\leq \psi(R(\tilde{s}_{\ell-1}, \tilde{s}_\ell)), \end{aligned} \tag{4}$$

where

$$\begin{aligned} R(\tilde{s}_{\ell-1},\tilde{s}_{\ell}) &= \max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell}),\zeta(\tilde{s}_{\ell-1},\top\tilde{s}_{\ell-1}),\zeta(\tilde{s}_{\ell},\top\tilde{s}_{\ell}); \\ & \tfrac{\zeta(\tilde{s}_{\ell-1},\top\tilde{s}_{\ell-1})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\top\tilde{s}_{\ell})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})},\tfrac{\zeta(\tilde{s}_{\ell},\top\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\top\tilde{s}_{\ell-1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}\} \\ &= \max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell}),\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell}),\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1}); \\ & \tfrac{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})};\tfrac{\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}\} \\ &= \max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell}),\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1}),\tfrac{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}\}. \end{aligned} \tag{5}$$

We should take the following:

case 1 If $R(\tilde{s}_{\ell-1},\tilde{s}_{\ell})=\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})$, so using (4), one writes

$$0<\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})\leq\psi(\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1}))<\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1}).$$

It is a contradiction.

case 2 If $R(\tilde{s}_{\ell-1},\tilde{s}_{\ell})=\frac{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}$, then by (5), we have

$$\max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell});\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})\}\leq\frac{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}.$$

We study two subcases:

subcase 1: If $\max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell});\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})\}=\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})$, then

$$\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})<\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})$$

and

$$\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\leq\frac{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})[\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\epsilon(\tilde{s}_{\ell-1},\tilde{s}_{\ell})+\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})}.$$

that is,

$$\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})\leq\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1}),$$

which is a contradiction.

subcase 2: If $\max\{\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell}),\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})\}=\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})$, then

$$\zeta(\tilde{s}_{\ell},\tilde{s}_{\ell+1})>\zeta(\tilde{s}_{\ell-1},\tilde{s}_{\ell})$$

and

$$\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) \leq \frac{\zeta(\tilde{s}_{\ell-1}, \tilde{s}_\ell)[\varpi(\tilde{s}_{\ell-1}, \tilde{s}_\ell)\epsilon(\tilde{s}_{\ell-1}, \tilde{s}_\ell) + \zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1})]}{\varpi(\tilde{s}_{\ell-1}, \tilde{s}_\ell)\epsilon(\tilde{s}_{\ell-1}, \tilde{s}_\ell) + \zeta(\tilde{s}_{\ell-1}, \tilde{s}_\ell)}.$$

That is,

$$\zeta(\tilde{s}_{\ell-1}, \tilde{s}_\ell) \geq \zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}),$$

which is a contradiction.

Thus, $R(\tilde{s}_{\ell-1}, \tilde{s}_\ell) = \zeta(\tilde{s}_{\ell-1}, \tilde{s}_\ell)$ and by (4), we get

$$\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) \leq \psi(\zeta(\tilde{s}_{\ell-1}, \tilde{s}_\ell)) \leq ... \leq \psi^\ell(\zeta(\tilde{s}_0, \tilde{s}_1)).$$

When $\ell \longrightarrow \infty$, we find $\lim\limits_{\ell\to\infty} \zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) = 0$. Now, we show that $\{\tilde{s}_\ell\}$ is a Cauchy sequence in $\mho$. For $m, \ell \in \mathbb{N}$ with $m > \ell$, we have

$$\begin{aligned}
\zeta(\tilde{s}_\ell, \tilde{s}_m) &\leq \varpi(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) + \epsilon(\tilde{s}_{\ell+1}, \tilde{s}_m)\zeta(\tilde{s}_{\ell+1}, \tilde{s}_m) \\
&\leq \varpi(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) + \epsilon(\tilde{s}_{\ell+1}, \tilde{s}_m)[\varpi(\tilde{s}_{\ell+1}, \tilde{s}_{\ell+2})\zeta(\tilde{s}_{\ell+1}, \tilde{s}_{\ell+2}) + \\
&\epsilon(\tilde{s}_{\ell+2}, \tilde{s}_m)\zeta(\tilde{s}_{\ell+2}, \tilde{s}_m)] \\
&= \varpi(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) + \epsilon(\tilde{s}_{\ell+1}, \tilde{s}_m)\varpi(\tilde{s}_{\ell+1}, \tilde{s}_{\ell+2})\zeta(\tilde{s}_{\ell+1}, \tilde{s}_{\ell+2}) \\
&+ \epsilon(\tilde{s}_{\ell+1}, \tilde{s}_m)\epsilon(\tilde{s}_{\ell+2}, \tilde{s}_m)\zeta(\tilde{s}_{\ell+2}, \tilde{s}_m) \\
&. \\
&. \\
&. \\
&\leq \varpi(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) + \sum_{i=\ell+1}^{m-2} \left(\prod_{j=\ell+1}^{i} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \varpi(\tilde{s}_i, \tilde{s}_{i+1})\zeta(\tilde{s}_i, \tilde{s}_{i+1}) \\
&+ \left(\prod_{j=\ell+1}^{m-1} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \zeta(\tilde{s}_{m-1}, \tilde{s}_m).
\end{aligned}$$

By using the fact that $\varpi(\sigma, \varsigma) \geq 1$ and $\epsilon(\sigma, \varsigma) \geq 1$ for all $\sigma, \varsigma \in X$, we deduce

$$\begin{aligned}
\zeta(\tilde{s}_\ell, \tilde{s}_m) &\leq \varpi(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^{i} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \varpi(\tilde{s}_i, \tilde{s}_{i+1})\zeta(\tilde{s}_i, \tilde{s}_{i+1}) \\
&\leq \sum_{i=n}^{m-1} \left(\prod_{j=n+1}^{i} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \varpi(\tilde{s}_i, \tilde{s}_{i+1})\zeta(\tilde{s}_i, \tilde{s}_{i+1}) \\
&\leq \sum_{i=n}^{m-1} \left(\prod_{j=n+1}^{i} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \varpi(\tilde{s}_i, \tilde{s}_{i+1})\psi^i(\zeta(\tilde{s}_0, \tilde{s}_1)).
\end{aligned}$$

Choose

$$a_i = \left(\prod_{j=n+1}^{i} \epsilon(\tilde{s}_j, \tilde{s}_m) \right) \varpi(\tilde{s}_i, \tilde{s}_{i+1}) \psi^i(\zeta(\tilde{s}_0, \tilde{s}_1)),$$

and $\Omega_p = \sum_{i=1}^{p} a_i$. Then we have

$$\zeta(\tilde{s}_\ell, \tilde{s}_m) \leq \Omega_{m-1} - \Omega_{n-1}. \tag{6}$$

Since $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} \leq \sup_{m\geq 1} \lim_{i\to\infty} \frac{\epsilon(\tilde{s}_{i+1}, \tilde{s}_m)\varpi(\tilde{s}_{i+1}, \tilde{s}_{i+2})\psi^{i+1}(\zeta(\tilde{s}_0, \tilde{s}_1))}{\varpi(\tilde{s}_i, \tilde{s}_{i+1})\psi^i(\zeta(\tilde{s}_0, \tilde{s}_1))}$, by condition (iii) one gets $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1$, so we conclude that the real sequence $\{\Omega_p\}$ converges, then it is a Cauchy sequence in $\mathbb{R}$. By (6), $\{\tilde{s}_\ell\}$ is a Cauchy sequence in $\mho$. Since $(\mho, \zeta)$ is orbitally complete, $\{\tilde{s}_\ell\}$ converges to $x^* \in \mho$. Next, we show that $\top x^* = x^*$. The orbital continuity of $\top$ on $\mho$ leads to $\tilde{s}_{\ell+1} = \top\tilde{s}_\ell = \top(\top^\ell \tilde{s}_0) \longrightarrow \top x^*$ as $\ell \to +\infty$ and $\zeta(\tilde{s}_{\ell+1}, \top x^*) \longrightarrow 0$ as $\ell \to +\infty$.

The triangle inequality of DCMS implies

$$\zeta(x^*, \tilde{s}_{\ell+1}) \leq \varpi(x^*, \tilde{s}_\ell)\zeta(x^*, \tilde{s}_\ell) + \epsilon(\tilde{s}_\ell, \tilde{s}_{\ell+1})\zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}).$$

Using the condition (v) and the fact that $\lim_{\ell\to\infty} \zeta(\tilde{s}_\ell, \tilde{s}_{\ell+1}) = 0$, one writes $\zeta(x^*, \tilde{s}_{\ell+1}) \longrightarrow_{\ell\to\infty} 0$.

Since we have a unique limit of a convergent sequence in the DCMS, one has $x^* = \top x^*$. Thus, $\top$ admits a FP $x^* \in \mho$, i.e., $Fix(\top)$ is nonempty.

Now, we show its uniqueness. Let ϑ and θ be two distinct FPs of $\top$. Due to condition (U), we have $\alpha(\vartheta, \theta) = \alpha(\top\vartheta, \top\theta) \geq 1$. Taking $x = \vartheta$ and $y = \theta$ in (3), we obtain

$$\begin{array}{rcl} \zeta(\vartheta, \theta) & = & \zeta(\top\vartheta, \top\theta) \\ & \leq & \alpha(\vartheta, \theta)\varpi(\vartheta, \theta)\epsilon(\vartheta, \theta)\zeta(\top\vartheta, \top\theta) \\ & \leq & \psi(\Re(\vartheta, \theta)) \\ & = & \max\{\zeta(\vartheta, \theta); \zeta(\vartheta, \top\vartheta); \zeta(\theta, \top\theta); \\ & & \frac{\zeta(\vartheta,\top\vartheta)[\varpi(\vartheta,\theta)\epsilon(\vartheta,\theta)+\zeta(\theta,\top\theta)]}{\varpi(\vartheta,\theta)\epsilon(\vartheta,\theta)+\zeta(\vartheta,\theta)}; \frac{\zeta(\theta,\top\theta)[\varpi(\vartheta,\theta)\epsilon(\vartheta,\theta)+\zeta(\vartheta,\top\vartheta)]}{\varpi(\vartheta,\theta)\epsilon(\vartheta,\theta)+\zeta(\vartheta,\theta)}\} \\ & = & \psi(\zeta(\vartheta, \theta)) \\ & < & \zeta(\vartheta, \theta), \end{array}$$

which is a contradiction. Therefore, $\top$ possesses a unique FP in $\mho$.

Example 2. *Given the DCMS as in Example 1. Consider* $\top : \mho \longrightarrow \mho$ *as*

$$\top x = \begin{cases} \frac{x}{\beta+\beta x} & \text{if} \quad x \in [0, 1] \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta > 108$. *To prove that the mapping* $\top$ *is continuous, let* $\{\tilde{s}_\ell\}$ *be a convergent sequence in* $\mho$ *to* x.

Then $\zeta(\tilde{s}_\ell, x) \longrightarrow 0$ *as* $\ell \to \infty$. *We require the following cases:*
Case 1: $\tilde{s}_\ell = x$ *for all but finitely many* ℓ, *then* $\top\tilde{s}_\ell = \top x$. *Here,* $\zeta(\tilde{s}_\ell, x) = 0$ *and* $\zeta(\top\tilde{s}_\ell, \top x) = 0$ *for all but finitely many. So* $\zeta(\top\tilde{s}_\ell, \top x) \longrightarrow 0$ *as* $\ell \to \infty$.
Case 2: $\tilde{s}_\ell \neq 0$ *and* $x \neq 0$ *for all but finitely many* ℓ, *then* $\top\tilde{s}_\ell \neq 0$ *and* $\top x \neq 0$. *We have* $\zeta(\tilde{s}_\ell, x) = \tilde{s}_\ell + x \to 0$ *and*

$$\begin{aligned}
\zeta(\top\tilde{s}_\ell, \top x) &= \top\tilde{s}_\ell + \top x \\
&= \frac{\tilde{s}_\ell}{\beta + \beta\tilde{s}_\ell} + \frac{x}{\beta + \beta x} \\
&= \frac{1}{\beta}\frac{\tilde{s}_\ell + x + 2\tilde{s}_\ell x}{(x+1)(\tilde{s}_\ell + 1)} \\
&\leq \frac{(\tilde{s}_\ell + x) + (\tilde{s}_\ell + x)^2}{\beta(x+1)(\tilde{s}_\ell + 1)} \longrightarrow 0.
\end{aligned}$$

Thus, $\{\top\tilde{s}_\ell\}$ *converges to* $\top x$ *in* $(\mho, \zeta)$.
Case 3: If $\tilde{s}_\ell \neq 0$ *for all but finitely many* ℓ *and* $x = 0$, *we have*

$$\zeta(\tilde{s}_\ell, 0) = \frac{\tilde{s}_\ell}{1 + \tilde{s}_\ell} \longrightarrow 0 \Longrightarrow \tilde{s}_\ell \longrightarrow 0,$$

then

$$\zeta(\top\tilde{s}_\ell, \top 0) = \frac{\top\tilde{s}_\ell}{1 + \top\tilde{s}_\ell} = \frac{\tilde{s}_\ell}{\beta + (\beta+1)\tilde{s}_\ell} \longrightarrow 0.$$

Thus, $\{\top\tilde{s}_\ell\}$ *converges to* $\top x$ *in* $(\mho, \zeta)$.
Case 4: If $\tilde{s}_\ell = 0$ *for all but finitely many* ℓ, *then, when* $\zeta(\tilde{s}_\ell, x) \longrightarrow 0$ *we necessarily have* $x = 0$, *Consequently* $\zeta(\top\tilde{s}_\ell, \top x) \longrightarrow 0$.
In all the cases, if

$$\zeta(\tilde{s}_\ell, x) \longrightarrow 0 \quad \Longrightarrow \quad \zeta(\top\tilde{s}_\ell, \top x) \longrightarrow 0,$$

so $\top$ *is continuous. We have*

$$\top^\ell x = \frac{x}{\beta^\ell + \left(\sum_{k=1}^{\ell} \beta^k\right) x}.$$

It is obvious that $\tilde{s}_\ell = \top^\ell x \longrightarrow 0$ *as* $\ell \longrightarrow \infty$ *and so for each* $x \in X$,

$$\lim_{\ell\to\infty} \varpi(\tilde{s}_\ell, x) = \lim_{\ell\to\infty} \epsilon(\tilde{s}_\ell, x) = 2 + 2x < \infty.$$

In addition, we define a mapping $\top : \mho \times \mho \longrightarrow [0, +\infty[$ *as*

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x, y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{s}_0 \in \mho$ *be a point with* $\alpha(\tilde{s}_0, \top\tilde{s}_0) \geq 1$, *then* $\tilde{s}_0 \in [0,1]$ *and* $\alpha(\top\tilde{s}_0, \top^2\tilde{s}_0) = \alpha(\frac{\tilde{s}_0}{\beta+\beta\tilde{s}_0}, \frac{\tilde{s}_0}{\beta^2+(\beta+\beta^2)\tilde{s}_0}) \geq 1$. *Therefore,* $\top$ *is* α*-orbitally admissible.*

Let $\psi(\chi) = k\chi$, for $\chi > 0$, where $k = \frac{36}{\beta}$. Here, $\psi^{\ell}(\chi) = k^{\ell}\chi$.
For all $x, y \in \mho$, we will show that $\alpha(x,y)\varpi(x,y)\epsilon(x,y)\zeta(\top x, \top y) \leq \psi(\zeta(x,y)$. For this, we consider the following cases:
Case 1: *$x = y$. we have $0 = \alpha(x,y)\varpi(x,y)\epsilon(x,y)\zeta(\top x, \top y) \leq \psi(\zeta(x,y)$.*
Case 2: *($x \neq 0$ and $y = 0$) or ($y \neq 0$ and $x = 0$). Without generality, suppose that $x \neq 0$ and $y = 0$. Here, we have*

$$\begin{aligned}\alpha(x,0)\varpi(x,0)\epsilon(x,0)\zeta(\top x, \top 0) &= (2+2x)^2\zeta(\frac{x}{\beta+\beta x}, 0) = (2+2x)^2\frac{\frac{x}{\beta+\beta x}}{\frac{x}{\beta+\beta x}+1}\\ &= (2+2x)^2\frac{x}{\beta+(\beta+1)x}\\ &\leq \frac{1}{\beta}(2+2x)^2\frac{x}{x+1}\\ &\leq \frac{16}{\beta}\frac{x}{x+1}\\ &\leq k.\zeta(x,y)\\ &= \psi(\zeta(x,y)).\end{aligned}$$

Case 3: *$0 \neq x \neq y \neq 0$. Here, we have $0 \neq \frac{x}{\beta+\beta x} \neq \frac{y}{\beta+\beta y} \neq 0$. One writes*

$$\begin{aligned}\alpha(x,y)\varpi(x,y)\epsilon(x,y)\zeta(\top x, \top y) &= (2+2x+2y)^2\zeta(\frac{x}{\beta+\beta x}, \frac{y}{\beta+\beta y})\\ &= (2+2x+2y)^2\left[\frac{x}{\beta+\beta x} + \frac{y}{\beta+\beta y}\right]\\ &\leq \frac{1}{\beta}(2x+2y+2)^2(x+y)\\ &\leq \frac{36}{\beta}[x+y]\\ &= k.\zeta(x,y)\\ &= \psi(\zeta(x,y)).\end{aligned}$$

Moreover, there is $\tilde{s}_0 \in \mho$ with $\alpha(\tilde{s}_0, \top\tilde{s}_0) \geq 1$, then $\alpha(\top\tilde{s}_0, \top^2\tilde{s}_0) \geq 1$. By induction, we obtain $\alpha(\tilde{s}_\ell, \tilde{s}_{\ell+1}) \geq 1$, where $\tilde{s}_\ell = \top^\ell\tilde{s}_0 = \frac{\tilde{s}_0}{\beta^\ell + (\sum_{k=1}^{\ell}\beta^k)\tilde{s}_0} = \frac{\tilde{s}_0}{\beta^\ell + \beta\frac{\beta^\ell - 1}{\beta - 1}\tilde{s}_0}$, for each $\ell \in \mathbb{N}$. It is easy that $\tilde{s}_\ell \longrightarrow 0$ as $\ell \longrightarrow +\infty$. Thus, $(\mho, \zeta)$ is an orbitally complete DCMS. Recall that

$$\lim_{i,m\to+\infty}\varpi(\tilde{s}_i, \tilde{s}_m) = \lim_{i,m\to+\infty}\epsilon(\tilde{s}_i, \tilde{s}_m) = 2$$

then we have

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\epsilon(\tilde{s}_{i+1}, \tilde{s}_m)\varpi(\tilde{s}_{i+1}, \tilde{s}_{i+2})\psi^{i+1}(\zeta(\tilde{s}_0, \tilde{s}_1))}{\varpi(\tilde{s}_i, \tilde{s}_{i+1})\psi^i(\zeta(\tilde{s}_0, \tilde{s}_1))} = \frac{2(2+\tilde{s}_0)k^{i+1}\zeta(\tilde{s}_0, \tilde{s}_1)}{2.k^i\zeta(\tilde{s}_0, \tilde{s}_1)} \leq 3k < 1,$$

where

$$\tilde{s}_i = \top^i(\tilde{s}_0) \quad \text{and} \quad \psi^i(\zeta(\tilde{s}_0, \tilde{s}_1)) = k^i\zeta(\tilde{s}_0, \tilde{s}_1).$$

Proposition 2. *If we replace the condition* (U) *by*

$$\textbf{\textit{(W)}}: \quad \forall \mu, \nu \in Fix(\top), \quad \textit{there is } \delta \in \mho \quad \textit{so that} \quad \alpha(\mu, \delta) \geq 1 \quad \textit{and} \quad \alpha(\nu, \delta) \geq 1,$$

then the map $\top$ *has a unique FP in* $\mho$.

Proof. Assume there are two distinct FPs, say $x \neq y \in \mho$. According to (W), there is $z \in \mho$, so that

$$\alpha(x, z) \geq 1, \quad \alpha(y, z) \geq 1.$$

The α-admissibility of $\top$ yields that

$$\alpha(x, \top^\ell z) \geq 1; \quad ; \alpha(y, \top^\ell z) \geq 1; \quad \forall \ell \in \mathbb{N}.$$

Consider the sequence $\{z_\ell\} \in \mho$ defined as $z_\ell = \top^\ell z$. We have

$$\begin{aligned} \zeta(x, z_{\ell+1}) = \zeta(\top x, \top z_\ell) &\leq \alpha(x, z_\ell)\zeta(\top x, \top z_\ell) \\ &\leq \psi(R(x, z_\ell)), \end{aligned}$$

where

$$\begin{aligned} R(x, z_\ell) &= \max\{\zeta(x, z_\ell), \zeta(x, \top x), \zeta(z_\ell, \top z_\ell), \\ &\quad \tfrac{\zeta(x,\top x)[\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(z_\ell,\top z_\ell)]}{\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,z_\ell)}, \tfrac{\zeta(z_\ell,\top z_\ell)[\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,\top x)]}{\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,z_\ell)}\} \\ &= \max\left\{\zeta(x, z_\ell), \zeta(z_\ell, z_{\ell+1}), \tfrac{\zeta(z_\ell,z_{\ell+1})\varpi(x,z_\ell)\epsilon(x,z_\ell)}{\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,z_\ell})\right\}. \end{aligned}$$

If $R(x, z_\ell) = \frac{\zeta(z_\ell,z_{\ell+1})\varpi(x,z_\ell)\epsilon(x,z_\ell)}{\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,z_\ell)}$ for some ℓ, then

$$\begin{aligned} \zeta(x, z_{\ell+1}) = \zeta(\top x, \top z_\ell) &\leq \alpha(x, z_\ell)\zeta(\top x, \top z_\ell) \\ &\leq \psi(R(x, z_\ell)) \\ &< R(x, z_\ell) \\ &= \tfrac{\zeta(z_\ell,z_{\ell+1})\varpi(x,z_\ell)\epsilon(x,z_\ell)}{\varpi(x,z_\ell)\epsilon(x,z_\ell)+\zeta(x,z_\ell)}. \end{aligned}$$

Therefore,

$$\zeta(z_\ell, z_{\ell+1}) < \frac{\zeta(z_\ell, z_{\ell+1})\varpi(x, z_\ell)\epsilon(x, z_\ell)}{\varpi(x, z_\ell)\epsilon(x, z_\ell) + \zeta(x, z_\ell)}.$$

That is,

$$\zeta(z_\ell, z_{\ell+1})\varpi(x, z_\ell)\epsilon(x, z_\ell) + \zeta(z_\ell, z_{\ell+1})\zeta(x, z_\ell) < \zeta(z_\ell, z_{\ell+1})\varpi(x, z_\ell)\epsilon(x, z_\ell).$$

Thus,

$$\zeta(z_\ell, z_{\ell+1})\zeta(x, z_\ell) < 0,$$

witch is a contradiction. We have

$$R(x, z_\ell) = \max\{\zeta(x, z_\ell), \zeta(z_\ell, z_{\ell+1})\}.$$

That is,

$$\zeta(z_\ell, z_{\ell+1}) \leq \psi^\ell(\zeta(z, \top z)).$$

Then

$$\lim_{\ell\to+\infty} \zeta(z_\ell, z_{\ell+1}) = 0.$$

So for all $\delta > 0$, there is $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$,

$$\zeta(z_\ell, z_{\ell+1}) < \delta.$$

Now, we show that $\lim_{\ell\to+\infty} \zeta(x, z_\ell) = 0$. Suppose on the contrary that $\lim_{\ell\to+\infty} \zeta(x, z_\ell) \neq 0$, then there exists a subsequence $\{z_{\varphi(\ell)}\}$ of $\{z_\ell\}$ such that $\varphi(\ell) > n$; $\zeta(x, z_{\varphi(\ell)}) \geq \delta$. So for all $\ell \geq \ell_0$ we have $\varphi(\ell) > \ell \geq \ell_0$ and then

$$\begin{cases} \zeta(z_{\varphi(\ell)}, z_{\varphi(\ell)+1}) < \delta \\ \zeta(x, z_{\varphi(\ell)}) \geq \delta. \end{cases}$$

Hence, for all $\ell \geq \ell_0$,

$$R(x, z_{\varphi(\ell)}) = \zeta(x, z_{\varphi(\ell)}).$$

Also, we have for all $\ell \geq \ell_0$,

$$\begin{aligned} \zeta(x, z_{\varphi(\ell)+1}) = \zeta(\top x, \top z_{\varphi(\ell)}) \leq \alpha(x, z_{\varphi(\ell)})\zeta(\top x, \top z_{\varphi(\ell)}) &\leq \psi(\zeta(x, z_{\varphi(\ell)})) \\ &\ \ \vdots \\ &\leq \psi^{\varphi(\ell)-\ell_0}(\zeta(x, z_{\ell_0})). \end{aligned}$$

Letting $\ell \longrightarrow +\infty$ we deduce that $\lim_{\ell\to+\infty} \zeta(x, z_{\varphi(\ell)+1}) = 0$.
So for all $\ell \geq \ell_0$ and by triangle inequality, we obtain

$$\delta \leq \zeta(x, z_{\varphi(\ell)}) \leq \varpi(x, z_{\varphi(\ell)+1})\zeta(x, z_{\varphi(\ell)+1}) + \epsilon(z_{\varphi(\ell)+1}, z_{\varphi(\ell)})\zeta(z_{\varphi(\ell)+1}, z_{\varphi(\ell)}).$$

When $\ell \longrightarrow +\infty$, one finds

$$\delta \leq \lim_{\ell\to+\infty} \zeta(x, z_{\varphi(\ell)}) = 0.$$

It is a contradiction. Therefore, $\lim_{\ell\to+\infty} \zeta(x, z_\ell) = 0$.
Similarly, $\lim_{\ell\to+\infty} \zeta(y, z_\ell) = 0$. By uniqueness of the limit, we have $x = y$.

4. Applications

Using Theorem 1, we will establish in this section an existence result of a solution of the following nonlinear Fredholm type functional integral equation

$$x(t) = f(t) + \lambda \int_{\tau_1}^{\tau_2} K\left(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)\right) dr, \tag{7}$$

where $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_1 < \tau_2$, $K \in C([\tau_1, \tau_2] \times [\tau_1, \tau_2] \times \mathbb{R}^4)$, $g \in C([\tau_1, \tau_2] \times [\tau_1, \tau_2])$ and $f \in C([\tau_1, \tau_2])$ are given functions and $x \in C[\tau_1, \tau_2]$ is an unknown function.
Take $\mho = C([\tau_1, \tau_2])$. Given $\zeta : \mho \times \mho \longrightarrow [0, +\infty)$ as

$$\zeta(\mu, \nu) = \sup_{t \in [\tau_1, \tau_2]} |\mu(t) - \nu(t)|^p, \qquad \forall \mu, \nu \in \mho.$$

Then $(\mho, \zeta)$ is a complete DCMS with the controlled functions

$$\varpi(\mu, \nu) = 2^{p-1}, \quad \epsilon(\mu, \nu) = 2^{p-1} + \frac{1}{1 + \frac{1}{1+\|\mu\|_\infty}} + \frac{1}{1 + \frac{1}{1+\|\nu\|_\infty}}, \quad \text{where} \quad p > 1$$

where,

$$\|u\|_\infty = \sup_{t \in [\tau_1, \tau_2]} |u(t)|.$$

Define the operator $\top : \mho \longrightarrow \mho$ by

$$\top(x)(t) := f(t) + \lambda \int_{\tau_1}^{\tau_2} K(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2))\, dr; \quad \forall x \in \mho, \quad t \in [\tau_1, \tau_2].$$

In what follows, we will establish the conditions so that the operator $\top$ has at least one FP. For this, define $\alpha : \mho \times \mho \longrightarrow [0, +\infty)$ by

$$\alpha(\mu, \nu) = \begin{cases} 1 & \text{if} \quad \mu(t) \leq \nu(t) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2. *Assume the following conditions hold:*

(i) $\tilde{s}_0(t) \leq f(t) + \lambda \int_{\tau_1}^{\tau_2} K(t, r, \tilde{s}_0(r), \tilde{s}_0(g(r)), \tilde{s}_0(\tau_1), \tilde{s}_0(\tau_2))\, dr, \quad \forall t \in [\tau_1, \tau_2]$;

(ii) For any $x, y \in \mho$ *with* $x(r) \leq y(r)$ *for each* $r \in [\tau_1, \tau_2]$,

$$|K(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)) - K(t, r, y(r), y(g(r)), y(\tau_1), y(\tau_2))| \leq \frac{\gamma(t,r)}{2^{1-\frac{1}{p}} \left(2^{p-1} + \frac{1}{1+\frac{1}{1+\|x\|_\infty}} + \frac{1}{1+\frac{1}{1+\|y\|_\infty}} \right)}$$
$$[|x(r) - y(r)|^p + |x(g(r)) - y(g(r))|^p + |x(\tau_1) - y(\tau_1)|^p + |x(\tau_2) - y(\tau_2)|^p]^{\frac{1}{p}} \quad ; p > 1 \tag{8}$$

where $t, r \in [\tau_1, \tau_2]$ *and* $\gamma : [\tau_1, \tau_2] \times [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ *is continuous so that*

$$\sup_{t \in [\tau_1, \tau_2]} \int_{\tau_1}^{\tau_2} \gamma^p(t, r) dr < \frac{1}{2^{p+2} |\lambda|^p (\tau_2 - 1)^{p-1}}; \tag{9}$$

(iii) For any $\tilde{s}_0 \in \mho$, $\lim_{i, m \to \infty} \epsilon(\tilde{s}_{i+1}, \tilde{s}_m) < 2^p$;

(iv) K *is non-decreasing.*

Then (7) *has a unique solution in* $\mho$.

Proof. For $\tilde{s}_0 \in \mho$, choose $\{\tilde{s}_\ell\}$ in $\mho$ by $\tilde{s}_\ell = \top^\ell \tilde{s}_0, \quad \ell \geq 1$. Consider

$$\tilde{s}_{\ell+1}(t) = \top\tilde{s}_\ell(t) = f(t) + \lambda \int_{\tau_1}^{\tau_2} K\left(t, r, \tilde{s}_\ell(r), \tilde{s}_\ell(g(r)), \tilde{s}_\ell(\tau_1), \tilde{s}_\ell(\tau_2)\right) dr.$$

The function K is non-decreasing in the last four arguments, and so

$$\alpha(\nu, \varsigma) \geq 1 \Longrightarrow \alpha(\top\nu, \top\varsigma) \geq 1.$$

Hence $\top$ is α-admissible, and therefore $\top$ is α-orbitally admissible.
Next, by condition (i), $\alpha(\tilde{s}_0, \top\tilde{s}_0) \geq 1$. Consequently, it follows that $\alpha(\tilde{s}_\ell, \tilde{s}_{\ell+1}) \geq 1$ for each $\ell \in \mathbb{N}$. Let $q > 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Using (8) and the Holder's inequality, one has

$$\begin{aligned}
|\top x(t) - \top y(t)|^p &= \left|\lambda \int_{\tau_1}^{\tau_2} K\left(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)\right) dr - \lambda \int_{\tau_1}^{\tau_2} K\left(t, r, y(r), y(g(r)), y(\tau_1), y(\tau_2)\right) dr\right|^p \\
&\leq \left(\int_{\tau_1}^{\tau_2} |\lambda| \left|K\left(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)\right) - K\left(t, r, y(r), y(g(r)), y(\tau_1), y(\tau_2)\right)\right| dr\right)^p \\
&\leq \left(\int_{\tau_1}^{\tau_2} |\lambda|^q\right)^{\frac{p}{q}} \\
&\quad \left(\left(\int_{\tau_1}^{\tau_2} \left|K\left(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)\right) - K\left(t, r, y(r), y(g(r)), y(\tau_1), y(\tau_2)\right)\right|^p dr\right)^{\frac{1}{p}}\right)^p \\
&= |\lambda|^p (\tau_2 - \tau_1)^{p-1} \\
&\quad \left(\int_{\tau_1}^{\tau_2} \left|K\left(t, r, x(r), x(g(r)), x(\tau_1), x(\tau_2)\right) - K\left(t, r, y(r), y(g(r)), y(\tau_1), y(\tau_2)\right)\right|^p dr\right) \\
&\leq |\lambda|^p (\tau_2 - \tau_1)^{p-1} \int_{\tau_1}^{\tau_2} \frac{\gamma^p(t,r)}{2^{p-1}(2^{p-1} + \frac{1}{1+\frac{1}{1+\|x\|_\infty}} + \frac{1}{1+\frac{1}{1+\|y\|_\infty}})} \\
&\quad \left[|x(r) - y(r)|^p + |x(g(r)) - y(g(r))|^p + |x(\tau_1) - y(\tau_1)|^p + |x(\tau_2) - y(\tau_2)|^p\right] dr \\
&\leq |\lambda|^p (\tau_2 - \tau_1)^{p-1} \int_{\tau_1}^{\tau_2} \frac{\gamma^p(t,r)}{2^{p-1}(2^{p-1} + \frac{1}{1+\frac{1}{1+\|x\|_\infty}} + \frac{1}{1+\frac{1}{1+\|y\|_\infty}})} \left[4\zeta(x, y)\right] dr \\
&\leq 4 |\lambda|^p (\tau_2 - \tau_1)^{p-1} \int_{\tau_1}^{\tau_2} \frac{\gamma^p(t,r)}{2^{p-1}(2^{p-1} + \frac{1}{1+\frac{1}{1+\|x\|_\infty}} + \frac{1}{1+\frac{1}{1+\|y\|_\infty}})} \left[R(x, y)\right] dr \\
&\leq \frac{4|\lambda|^p (\tau_2 - \tau_1)^{p-1} [R(x,y)]}{2^{p-1}(2^{p-1} + \frac{1}{1+\frac{1}{1+\|x\|_\infty}} + \frac{1}{1+\frac{1}{1+\|y\|_\infty}})} \sup_{t \in [\tau_1, \tau_2]} \left(\int_{\tau_1}^{\tau_2} \gamma^p(t, r) dr\right).
\end{aligned}$$

From (9), it results that

$$2^{p-1}\left(2^{p-1} + \frac{1}{1 + \frac{1}{1+\|x\|_\infty}} + \frac{1}{1 + \frac{1}{1+\|y\|_\infty}}\right) |\top x(t) - \top y(t)|^p \leq \frac{1}{2^p} \left(\Re(x, y)\right).$$

Setting $\psi(t) = \frac{1}{2^p} t$, we obtain that

$$\alpha(x, y)\varpi(x, y)\epsilon(x, y)\zeta(\top x, \top y) \leq \psi(\Re(x, y)).$$

Next, using the condition

$$\lim_{i,m \to \infty} \epsilon(\tilde{s}_{i+1}, \tilde{s}_m) < 2^p,$$

we deduce

$$\begin{aligned}\sup_m \lim_{i\to\infty} \frac{\epsilon(\tilde{s}_{i+1},\tilde{s}_m)\varpi(\tilde{s}_{i+1},\tilde{s}_{i+2})\psi^{i+1}(\zeta(\tilde{s}_0,\tilde{s}_1))}{\varpi(\tilde{s}_i,\tilde{s}_{i+1})\psi^{i}(\zeta(\tilde{s}_0,\tilde{s}_1))} &= \sup_m \lim_{i\to\infty} \frac{\epsilon(\tilde{s}_{i+1},\tilde{s}_m)\frac{1}{2^{p(i+1)}}(\zeta(\tilde{s}_0,\tilde{s}_1))}{\frac{1}{2^{pi}}(\zeta(\tilde{s}_0,\tilde{s}_1))} \\ &= \sup_m \lim_{i\to\infty} \frac{1}{2^p}\epsilon(\tilde{s}_{i+1},\tilde{s}_m) \\ &< 1.\end{aligned}$$

Thus, the condition (iii) in Theorem 1 holds. Since f, g and K are continuous, the operator $\top$ is continuous on $\mho$ and so $\top$ is orbitally continuous on $\mho$. The controlled functions $\varpi; \epsilon$ are defined by

$$\varpi(\mu,\nu) = 2^{p-1}; \quad \epsilon(\mu,\nu) = 2^{p-1} + \frac{1}{1+\frac{1}{1+\|\mu\|_\infty}} + \frac{1}{1+\frac{1}{1+\|\nu\|_\infty}}, \quad \text{where} \quad p > 1.$$

Then

$$\lim_{\ell\to\infty} \varpi(\tilde{s}_\ell, x) = 2^{p-1}.$$

On the other hand, $\left\{\top^\ell x(t)\right\}_n$ converges because $\left\{\top^\ell\right\}_\ell$ is a monotonic sequence and so it follows from Dini theorem that $\sup_{t\in[\tau_1,\tau_2]} \left|\top^\ell x(t)\right|$ converges and hence $\lim_{\ell\to\infty} \epsilon(\tilde{s}_\ell, x)$ exists and is finite.

In $\mho = C([\tau_1,\tau_2])$, if $x \le y$, then there is $z \in \mho$ so that $x \le z$ and $y \le z$, and so **(W)** is satisfied. Thus, all the conditions of Theorem 1 are satisfied, and hence $\top$ possesses a unique FP in $\mho$. That is, the nonlinear Fredholm functional integral equation (7) has a unique solution.

Example 3. *Take the following nonlinear Fredholm functional integral equation: For $t \in [0,1]$ and $\lambda = 1$,*

$$x(t) = 1 + \int_0^1 \frac{1}{256}\left(\frac{|x(r)|(1+|x(r)|)}{2+|x(r)|} + \frac{\left|x(\frac{r}{2})\right|(1+\left|x(\frac{r}{2})\right|)}{2+\left|x(\frac{r}{2})\right|} + \frac{tr\,|x(0)|(1+|x(0)|)}{2+|x(0)|} + \frac{tr\,|x(1)|(1+|x(1)|)}{2+|x(1)|}\right) dr, \tag{10}$$

where $K : [0,1]\times[0,1]\times\mathbb{R}^4 \longrightarrow \mathbb{R}$ is defined by

$$K(t,r,x(r),x(\frac{r}{2}),x(0),x(1)) =$$

$$\frac{1}{256}\left(\frac{|x(r)|(1+|x(r)|)}{2+|x(r)|} + \frac{\left|x(\frac{r}{2})\right|(1+\left|x(\frac{r}{2})\right|)}{2+\left|x(\frac{r}{2})\right|} + \frac{tr\,|x(0)|(1+|x(0)|)}{2+|x(0)|} + \frac{tr\,|x(1)|(1+|x(1)|)}{2+|x(1)|}\right).$$

Let $g : [0,1] \longrightarrow [0,1]$ be given as $g(r) = \frac{r}{2}$ and $f : [0,1] \longrightarrow \mathbb{R}$ be defined by $f(t) = 1$. These functions are continuous and $x \in C([0,1])$ is the unknown function.

In this case, we will use the space $\mho = C([0,1])$ endowed with the double controlled metric $\zeta : \mho\times\mho \longrightarrow [0,+\infty[$ defined by

$$\zeta(\mu,\nu) = \sup_{t\in[0,1]} |\mu(t)-\nu(t)|^2, \qquad \forall \mu,\nu \in \mho$$

The space $(\mho, \zeta)$ *is a DCMS with controlled functions*

$$\varpi(\mu,\nu) = 2; \quad \epsilon(\mu,\nu) = 2 + \frac{1}{1+\frac{1}{1+\|\mu\|_\infty}} + \frac{1}{1+\frac{1}{1+\|\nu\|_\infty}}.$$

Take the operator $\top : \mho \longrightarrow \mho$ *defined for* $t \in [0,1]$ *and* $x \in \mho$ *as follows:*

$$\top x(t) = 1+\int_0^1 \frac{1}{256}\left(\frac{|x(r)|\,(1+|x(r)|)}{2+|x(r)|} + \frac{\left|x(\frac{r}{2})\right|(1+\left|x(\frac{r}{2})\right|)}{2+\left|x(\frac{r}{2})\right|} + \frac{tr\,|x(0)|\,(1+|x(0)|)}{2+|x(0)|} + \frac{tr\,|x(1)|\,(1+|x(1)|)}{2+|x(1)|}\right) dr$$

In what follows, the conditions of Theorem 1 are checked. It is observed that the function K *is non-decreasing. Thereafter, we will quote the following lemma that we will use in the sequel.*

Lemma 2. *For all* $\tau_1, \tau_2 \geq 0$, *we have the*

$$a + b \leq 2\left(a^2+b^2\right)^{\frac{1}{2}}.$$

At this moment, for $t, r \in [0,1], x, y \in \mho = C([0,1])$ *with* $x(r) \leq y(r)$ *for all* $r \in [0,1]$, *we estimate the difference*

$$\begin{aligned}
&|K(t,r,x(r),x(g(r)),x(\tau_1),x(\tau_2)) - K(t,r,y(r),y(g(r)),y(\tau_1),y(\tau_2))| &\leq\\
&\frac{1}{256}\left|\frac{|x(r)|(1+|x(r)|)}{2+|x(r)|} - \frac{|y(r)|(1+|y(r)|)}{2+|y(r)|}\right| + \frac{1}{256}\left|\frac{\left|x(\frac{r}{2})\right|(1+\left|x(\frac{r}{2})\right|)}{2+\left|x(\frac{r}{2})\right|} - \frac{\left|y(\frac{r}{2})\right|(1+\left|y(\frac{r}{2})\right|)}{2+\left|y(\frac{r}{2})\right|}\right|\\
&+\frac{1}{256}\left|\frac{tr|x(0)|(1+|x(0)|)}{2+|x(0)|} - \frac{tr|y(0)|(1+|y(0)|)}{2+|y(0)|}\right| + \frac{1}{256}\left|\frac{tr|x(1)|(1+|x(1)|)}{2+|x(1)|} - \frac{tr|y(1)|(1+|y(1)|)}{2+|y(1)|}\right| &=\\
&\frac{1}{256}\frac{(2+2|x(r)|+2|y(r)|+|x(r)||y(r)|)||x(r)|-|y(r)||}{(2+|x(r)|)(2+|y(r)|)} + \frac{1}{256}\frac{\left(2+2\left|x(\frac{r}{2})\right|+2\left|y(\frac{r}{2})\right|+\left|x(\frac{r}{2})\right|\left|y(\frac{r}{2})\right|\right)\left|\left|x(\frac{r}{2})\right|-\left|y(\frac{r}{2})\right|\right|}{\left(2+\left|x(\frac{r}{2})\right|\right)\left(2+\left|y(\frac{r}{2})\right|\right)}\\
&+\frac{1}{256}\frac{(2+2|x(0)|+2|y(0)|+|x(0)||y(0)|)||x(0)|-|y(0)||}{(2+|x(0)|)(2+|y(0)|)} + \frac{1}{256}\frac{(2+2|x(1)|+2|y(1)|+|x(1)||y(1)|)||x(1)|-|y(1)||}{(2+|x(1)|)(2+|y(1)|)} &\leq\\
&\frac{1}{256}\frac{2(1+|x(r)|)(1+|y(r)|)|x(r)-y(r)|}{(2+|x(r)|)(2+|y(r)|)} + \frac{1}{256}\frac{2(1+\left|x(\frac{r}{2})\right|)(1+\left|y(\frac{r}{2})\right|)\left|x(\frac{r}{2})-y(\frac{r}{2})\right|}{\left(2+\left|x(\frac{r}{2})\right|\right)\left(2+\left|y(\frac{r}{2})\right|\right)}\\
&+\frac{1}{256}\frac{2(1+|x(0)|)(1+|y(0)|)|x(0)-y(0)|}{(2+|x(0)|)(2+|y(0)|)} + \frac{1}{256}\frac{2(1+|x(1)|)(1+|y(1)|)|x(1)-y(1)|}{(2+|x(1)|)(2+|y(1)|)} &=\\
&\frac{1}{128}\frac{(1+|x(r)|)(1+|y(r)|)|x(r)-y(r)|}{(2+|x(r)|)(2+|y(r)|)} + \frac{1}{128}\frac{(1+\left|x(\frac{r}{2})\right|)(1+\left|y(\frac{r}{2})\right|)\left|x(\frac{r}{2})-y(\frac{r}{2})\right|}{\left(2+\left|x(\frac{r}{2})\right|\right)\left(2+\left|y(\frac{r}{2})\right|\right)}\\
&+\frac{1}{128}\frac{(1+|x(0)|)(1+|y(0)|)|x(0)-y(0)|}{(2+|x(0)|)(2+|y(0)|)} + \frac{1}{128}\frac{(1+|x(1)|)(1+|y(1)|)|x(1)-y(1)|}{(2+|x(1)|)(2+|y(1)|)} &=\\
&\frac{1}{64}\frac{1}{2\left(1+\frac{1}{1+|x(r)|}\right)\left(1+\frac{1}{1+|y(r)|}\right)}|x(r)-y(r)| + \frac{1}{64}\frac{1}{2\left(1+\frac{1}{1+\left|x(\frac{r}{2})\right|}\right)\left(1+\frac{1}{1+\left|y(\frac{r}{2})\right|}\right)}\left|x(\tfrac{r}{2})-y(\tfrac{r}{2})\right|\\
&\frac{1}{64}\frac{1}{2\left(1+\frac{1}{1+|x(0)|}\right)\left(1+\frac{1}{1+|y(0)|}\right)}|x(0)-y(0)| + \frac{1}{64}\frac{1}{2\left(1+\frac{1}{1+|x(1)|}\right)\left(1+\frac{1}{1+|y(1)|}\right)}|x(1)-y(1)| &\leq\\
&\frac{1}{64}\frac{1}{2+\frac{1}{1+|x(r)|}+\frac{1}{1+|y(r)|}}|x(r)-y(r)| + \frac{1}{64}\frac{1}{2+\frac{1}{1+\left|x(\frac{r}{2})\right|}+\frac{1}{1+\left|y(\frac{r}{2})\right|}}\left|x(\tfrac{r}{2})-y(\tfrac{r}{2})\right|\\
&\frac{1}{64}\frac{1}{2+\frac{1}{1+|x(0)|}+\frac{1}{1+|y(0)|}}|x(0)-y(0)| + \frac{1}{64}\frac{1}{2+\frac{1}{1+|x(1)|}+\frac{1}{1+|y(1)|}}|x(1)-y(1)| &\leq\\
&\frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}|x(r)-y(r)| + \frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}\left|x(\tfrac{r}{2})-y(\tfrac{r}{2})\right|\\
&\frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}|x(0)-y(0)| + \frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}|x(1)-y(1)| &=\\
&\frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}\left[|x(r)-y(r)| + \left|x(\tfrac{r}{2})-y(\tfrac{r}{2})\right| + |x(0)-y(0)| + |x(1)-y(1)|\right].
\end{aligned}$$

So by lemma 2, one writes

$$\begin{cases} |x(r)-y(r)|+\left|x(\frac{r}{2})-y(\frac{r}{2})\right| & \leq 2\left(|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2\right)^{\frac{1}{2}} \\ |x(0)-y(0)|+|x(1)-y(1)| & \leq 2\left(|x(0)-y(0)|^2+|x(1)-y(1)|^2\right)^{\frac{1}{2}}. \end{cases}$$

Thus, we have

$$\begin{aligned} &|K(t,r,x(r),x(g(r)),x(\tau_1),x(\tau_2))-K(t,r,y(r),y(g(r)),y(\tau_1),y(\tau_2))| \\ &\leq \frac{1}{64}\frac{1}{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}(2(|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2)^{\frac{1}{2}}+2(|x(0)-y(0)|^2+|x(1)-y(1)|^2)^{\frac{1}{2}}) \\ &=\frac{1}{16}\frac{1}{2(2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty})}((|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2)^{\frac{1}{2}}+(|x(0)-y(0)|^2+|x(1)-y(1)|^2)^{\frac{1}{2}}). \end{aligned}$$

We apply another time Lemma 2 to have

$$\begin{aligned} a&=\left(|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2\right)^{\frac{1}{2}} \\ b&=\left(|x(0)-y(0)|^2+|x(1)-y(1)|^2\right)^{\frac{1}{2}}. \end{aligned}$$

That is,

$$\begin{aligned} &|K\left(t,r,x(r),x(g(r)),x(\tau_1),x(\tau_2)\right)-K\left(t,r,y(r),y(g(r)),y(\tau_1),y(\tau_2)\right)| \leq \\ &\frac{1}{8}\frac{1}{2(2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty})}\left(|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2+|x(0)-y(0)|^2+|x(1)-y(1)|^2\right)^{\frac{1}{2}} \leq \\ &\frac{1}{8}\frac{1}{\sqrt{2}\sqrt{2+\frac{1}{1+\|x\|_\infty}+\frac{1}{1+\|y\|_\infty}}}\left(|x(r)-y(r)|^2+\left|x(\frac{r}{2})-y(\frac{r}{2})\right|^2+|x(0)-y(0)|^2+|x(1)-y(1)|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Thus, the second condition (ii) *of Theorem 2 is satisfied with* $\gamma(t,r)=\frac{1}{8}$.
That is,

$$\sup_{t\in[0,1]}\int_0^1 \gamma^2(t,r)dr=\frac{1}{64}<\frac{1}{16}=\frac{1}{2^{2+2}\,|1|^p\,(1-0)^{2-1}}.$$

Next, Using the properties of the functions K, g *and* f*, it is observed that the hypothesis* (i) *is satisfied for* $\tilde{s}_0=f\in\mho=C([0,1])$.
On the other hand, by definition of the function ϵ *the condition* (iii) *is satisfied. It remains to check that the operator* K *is non-decreasing. It is clear that the function* $\sigma: s\longmapsto \frac{s(1+s)}{2+s};\quad s\geq 0$ *is increasing, so on the space of continuous functions on* $[0,1]$ *with value in* $[0,+\infty[$*, we easily see that the operator is indeed non-decreasing with respect to the variable* x*. The conditions of Theorem 2 are fulfilled and it results that the integral equation* (10) *has a unique solution.*

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