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On (Φ, m)-Homoderivations in Rings

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Abstract. In this article, we examine the commutativity of a ring Ω endowed with a specific kind of mapping called centrally extended (Φ, m)-homoderivation, where Φ is a mapping on Ω, and m is an integer. This mapping is a comprehensive kind of the homoderivation, $\Phi-$homoderivation, and $m-$homoderivation. Besides, we provide some properties of its center.

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1. Introduction

The study of derivations in ring theory has significantly evolved over the years, incorporating various extensions and generalizations to enhance the understanding of algebraic structures and their functional properties. Derivations serve as fundamental tools in analyzing ring behavior, particularly in prime and semiprime rings, where they play a crucial role in investigating commutativity and structural integrity. Over time, researchers have proposed numerous generalizations of derivations, including centrally extended mappings, homoderivations, and m-homoderivations, each contributing to a deeper understanding of ring theory.

A pivotal breakthrough in this field was the introduction of centrally extended derivations (CE-derivations) by Bell and Daif [1], which ensured that certain derivation properties were preserved within the center of the ring, thereby influencing its commutativity.

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Throughout, unless stated otherwise, Ω is an associative ring with center $\zeta(\Omega)$. For a subset S of Ω, the map $\mathcal{D}$ preserves S if $\mathcal{D}(S) \subseteq S$.

Suppose that $\mathcal{D}$ is a mapping of a ring Ω. If $\mathcal{D}(s+u) - \mathcal{D}(s) - \mathcal{D}(u) \in \zeta(\Omega)$ and $\mathcal{D}(su) - \mathcal{D}(s)u - s\mathcal{D}(u) \in \zeta(\Omega)$ for every $s, u \in \Omega$, then $\mathcal{D}$ is called a CE-derivation.

The CE-(Φ, Ψ)-derivation on Ω was described by Tammam et al. [2] as a map $\mathcal{D}$ on Ω such that, for each $s, u \in \Omega$, both $\mathcal{D}(s+u)-\mathcal{D}(s)-\mathcal{D}(u)$ and $\mathcal{D}(su)-\mathcal{D}(s)\Phi(u)-\Psi(s)\mathcal{D}(u)$ are in $\zeta(\Omega)$, for more recent work, see [3] and [4].

In 1998, Filippov's concept of δ-derivations broadened the classical definition by incorporating a mapping δ into a derivation as follows: Suppose $\mathcal{B}$ is an algebra over $\mathcal{H}$, where $\mathcal{H}$ is a unital commutative associative ring. Given an arbitrary $\delta \in \mathcal{H}$, a δ-derivation of $\mathcal{B}$ is defined to be an $\mathcal{F}$-linear mapping $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ satisfying the identity $\mathcal{F}(su) = \delta(\mathcal{F}(s)u + s\mathcal{F}(u))$ for each $s, u \in \mathcal{B}$.

Filippov's work [5] has paved the way for more complex derivation structures. Building on these developments, El-Soufi [6] pioneered the study of homoderivations, which is defined as follows: If $\mathcal{F}(su) = s\mathcal{F}(u)+\mathcal{F}(s)u+\mathcal{F}(s)\mathcal{F}(u)$ holds for each $s, u \in \Omega$, then $\mathcal{F}$ is said to be a homoderivation, provided it is additive. This concept has been instrumental in the study of prime and semiprime rings.

In 2022, further advancements by Tammam et al. [7] led to the introduction of m-homoderivations, integrating an integer parameter into the derivation process to provide a new perspective on non-commutative ring behavior. If, for each $s, u \in \Omega$, the map $\mathcal{F}$ satisfies $\mathcal{F}(su) = s\mathcal{F}(u) + \mathcal{F}(s)u + m\mathcal{F}(s)\mathcal{F}(u)$, then the map $\mathcal{F}$ is said to be an m-homoderivation, provided it is additive.

In 2022, El-Soufi et al. [8] introduced the concept of Φ-homoderivations by incorporating a map Φ defined on a ring Ω into the homoderivation framework. Specifically, an additive map $\mathcal{F}$ on Ω is called a Φ-homoderivation if, for all $s, u \in \Omega$, it satisfies

$$\mathcal{F}(su) = \mathcal{F}(s)\Phi(u) + \Phi(s)\mathcal{F}(u) + \mathcal{F}(s)\mathcal{F}(u).$$

A centrally extended homoderivation $\mathcal{F}$ is defined as a map for which

$$\mathcal{F}(s+u) - \mathcal{F}(s) - \mathcal{F}(u) \quad \text{and} \quad \mathcal{F}(su) - \mathcal{F}(s)\mathcal{F}(u) - \mathcal{F}(s)u - s\mathcal{F}(u)$$

both lie in the center $\zeta(\Omega)$ for all $s, u \in \Omega$. Furthermore, if

$$\mathcal{F}(s+u) - \mathcal{F}(s) - \mathcal{F}(u) \in \zeta(\Omega) \quad \text{and} \quad \mathcal{F}(su) - \mathcal{F}(s)\mathcal{F}(u) - \mathcal{F}(s)\Phi(u) - \Phi(s)\mathcal{F}(u) \in \zeta(\Omega),$$

then $\mathcal{F}$ is called a centrally extended Φ-homoderivation (CE-Φ-homoderivation).

The authors proved several results in [8, Corollaries 1 - 6], which appear as special cases of our results and are summarized as follows.

Let Ω be a semiprime ring with center $\zeta(\Omega)$, and let Φ be an epimorphism of Ω. The following hold:

(i) If the unique central ideal of Ω is the zero ideal, then each nilpotent CE-homoderivation is an ordinary homoderivation (Corollary 1).

(ii) Assuming that $\zeta(\Omega)$ contains no nonzero nilpotent elements, then every nilpotent CE-Φ-homoderivation which commute with Φ preserves $\zeta(\Omega)$ (Corollary 2).

(iii) Assuming that $\zeta(\Omega)$ contains no nonzero nilpotent elements, then every nilpotent CE-homoderivations stabilize the center $\zeta(\Omega)$ (Corollary 3).

Moreover, when a prime ring admits a nonzero nilpotent CE-homoderivation satisfying one of the following conditions:

(i) The map is not a homoderivation (Corollary 4),

(ii) The map has a nonzero image of zero (Corollary 5),

(iii) The map annihilates all Lie (Jordan) products (Corollary 6),

then the ring Ω is commutative.

These contributions culminated in the study of centrally extended (ϕ, m)- homoderivations, which elegantly unify prior derivation concepts while preserving essential algebraic properties. This paper focuses on a thorough exploration of centrally extended $(\phi, m)-$homoderivations, which incorporate both a mapping ϕ and an integer m while ensuring that the extended map remains within the center of the ring, and their impact on ring commutativity.

Definition 1. *Let $\mathcal{F}$ be an additive mapping on a ring Ω, Φ a mapping on Ω, m an integer, and s, u any two elements in Ω.*

(i) If $\mathcal{F}$ achieves

$$\mathcal{F}(su) = \mathcal{F}(s)\Phi(u) + \Phi(s)\mathcal{F}(u) + m\mathcal{F}(s)\mathcal{F}(u),$$

then $\mathcal{F}$ is called a (Φ, m)-homoderivation.

(ii) If $\mathcal{F}$ achieves

$$\mathcal{F}(s + u) - \mathcal{F}(s) - \mathcal{F}(u) \in \zeta(\Omega)$$

and

$$\mathcal{F}(su) - \{\mathcal{F}(s)\Phi(u) + \Phi(s)\mathcal{F}(u) + m\mathcal{F}(s)\mathcal{F}(u)\} \in \zeta(\Omega),$$

then $\mathcal{F}$ is called a centrally extended (Φ, m)-homoderivation ($CE - (\Phi, m)$- homoderivation).

The concept of nil and nilpotent derivations was first developed by Chung [9]. Let Ω be a ring endowed with a derivation δ. If $n = n(r) \in \mathbb{Z}^+$ exists for each $r \in \Omega$ with $\delta^n(r) = 0$, then δ is said to be nil. Here, the derivation δ is referred to be nilpotent if the integer n may be freely extracted from r.

Definition 2. *Let $\mathcal{F}$ and Φ be two maps on a ring Ω and $\mathcal{S} \subseteq \Omega$. If $\mathcal{F}^n(\mathcal{S}) = (0)$ for some $n \in \mathbb{Z}^+ - \{1\}$, then $\mathcal{F}$ is said to be nilpotent on $\mathcal{S}$. Two maps $\mathcal{F}$ and Φ are called commuting on $\mathcal{S}$ if $\Phi(\mathcal{F}(s)) = \mathcal{F}(\Phi(s))$, for each $s \in \mathcal{S}$.*

Remark 1. **(1)** *Any homoderivation is an* $(I_{id}, 1)$*-homoderivation.*

(2) *Any* $CE-$*homoderivation on* Ω *is a* $CE-(I_{id}, 1)$*-homoderivation of* Ω*, where* I_{id} *refers to the identity map.*

(3) *According to Definition 1, any* (Φ, m)*-homoderivation is a centrally extended* (Φ, m)*-homoderivation, but the inverse (in general) is not true.*

During our research, we will use the following facts.

Lemma 1. *[10, Lemma 1(b)] For a prime ring* Ω*, with center* $\zeta(\Omega)$*, the centralizer of any nonzero one-sided ideal coincides with* $\zeta(\Omega)$*. Consequently, if there exists a nonzero right ideal lying in* $\zeta(\Omega)$*, then* Ω *must necessarily be commutative.*

Lemma 2. *[11, Lemma 4] In a prime ring* Ω*, if both* p *and* qp *lie in the center and* p *is nonzero, then the element* q *lies in the center* $\zeta(\Omega)$*.*

2. Examples of $CE-(\Phi, m)$-homoderivations

In the following examples, we ensure that there are $CE-(\Phi, m)$-homoderivation maps.

Example 1. *Let* $\Omega = \mathbb{Z}_6$*. Suppose that* $\mathcal{F}, \Phi : \mathbb{Z}_6 \to \mathbb{Z}_6$ *are mappings on* $\mathbb{Z}_6$ *so that* $\mathcal{F}(\alpha) = 4\alpha$ *and* $\Phi(\alpha) = 3\alpha$*, for all* $\alpha \in \mathbb{Z}_6$*. So,* $\mathcal{F}$ *will be a* $(\Phi, 4)$*-homoderivation, where* Φ *is an endomorphism, and* $\mathcal{F}$*,* Φ *are commuting on* $\mathbb{Z}_6$*.*

Example 2. *Let* $\Omega = M_2(\mathbb{Z}) \bigoplus \mathbb{Z}_6$ *be the ring with center* $\zeta(\Omega) = \{(aI, x) : a \in \mathbb{Z}, x \in \mathbb{Z}_6$ *and* $I \in M_2(\mathbb{Z})$ *be the identity matrix*$\}$*. Suppose that the maps* $\Phi, \mathcal{F} : \Omega \to \Omega$ *so that* $\Phi(\sigma, \rho) = (\sigma, 3\rho)$ *and* $\mathcal{F}(\sigma, \rho) = (-\sigma, 3 - \rho)$ *for any* $(\sigma, \rho) \in \Omega$*. For* $(\sigma, \rho), (\tau, \mu) \in \Omega$*, we have* $\Phi((\sigma, \rho) + (\tau, \mu)) = \Phi((\sigma + \tau, \rho + \mu)) = (\sigma + \tau, 3\rho + 3\mu)$*. On the other hand,* $\Phi((\sigma, \rho)) + \Phi((\tau, \mu)) = (\sigma, 3\rho) + (\tau, 3\mu) = (\sigma + \tau, 3\rho + 3\mu)$*. Thus* $\Phi((\sigma, \rho) + (\tau, \mu)) = \Phi((\sigma, \rho)) + \Phi((\tau, \mu))$*. Also, we have* $\Phi((\sigma, \rho)(\tau, \mu)) = \Phi((\sigma\tau, \rho\mu)) = (\sigma\tau, 3\rho\mu)$*. In contrast,* $\Phi((\sigma, \rho))\Phi((\tau, \mu)) = (\sigma, 3\rho)(\tau, 3\mu) = (\sigma\tau, 3\rho\mu)$*. So* $\Phi((\sigma, \rho)(\tau, \mu)) = \Phi((\sigma, \rho))\Phi((\tau, \mu))$*. Furthermore,*

$$\mathcal{F}((\sigma, \rho) + (\tau, \mu)) - \mathcal{F}((\sigma, \rho)) - \mathcal{F}((\tau, \mu)) = (0, 3) \in \zeta(\Omega)$$

and

$$\begin{aligned} &\mathcal{F}((\sigma, \rho)(\tau, \mu)) - \mathcal{F}((\sigma, \rho))\mathcal{F}((\tau, \mu)) \\ &- \mathcal{F}((\sigma, \rho))\Phi((\tau, \mu)) - \Phi((\sigma, \rho))\mathcal{F}((\tau, \mu)) \\ &= (0, 2\rho\mu) \ \in \zeta(\Omega) - \{0\}, \end{aligned}$$

where $2\rho\mu \neq 0$ *for all* $\rho, \mu \in \mathbb{Z}_6$*. Therefore,* Φ *is an endomorphism and* $\mathcal{F}$ *is a* $CE-(\Phi, 3)$*-homoderivation but not a* $(\Phi, 3)$*-homoderivation map.*

3. Rings with centrally extended (Φ, m)-homoderivations

This section provides an answer to the question: When is a $CE-(\Phi, m)$-homoderivation a (Φ, m)-homoderivation? Additionally, we will provide information about a $CE-(\Phi, m)$-homoderivation.

Throughout, $\mathcal{F}$ is a nilpotent centrally extended (Φ, m)-homoderivation of a ring Ω, Φ is an epimorphism on Ω, $\mathcal{F}$ and Φ are commuting on Ω, and $m \in \mathbb{Z}$.

Remark 2. *$\varphi_{\mathcal{F}}(r, s, +)$ and $(\varphi_{\mathcal{F}}(r, s, .))$ refer to the central element generated through the influence of $\mathcal{F}$ on the sum $r + s$ (the product $r.s$) for any two elements $r, s \in \Omega$.*

Theorem 1. *Suppose that Ω is a ring. If the zero ideal is the only ideal of Ω contained in $\zeta(\Omega)$, then $\mathcal{F}$ is additive.*

Proof. Let $s, u \in \Omega$ be two fixed elements. By assumption,

$$\mathcal{F}(s + u) = \mathcal{F}(s) + \mathcal{F}(u) + \varphi_{\mathcal{F}}(s, u, +). \tag{1}$$

So, for each $v \in \Omega$, we obtain

$$\begin{aligned} \mathcal{F}((s + u)v) &= \Phi(s + u)\mathcal{F}(v) + \mathcal{F}(s + u)\Phi(v) + m\mathcal{F}(s + u)\mathcal{F}(v) + \varphi_{\mathcal{F}}(s + u, v, .) \\ &= (\mathcal{F}(s) + \mathcal{F}(u) + \varphi_{\mathcal{F}}(s, u, +))(m\mathcal{F}(v) + \Phi(v)) \\ &\quad + \Phi(u)\mathcal{F}(v) + \Phi(s)\mathcal{F}(v) + \varphi_{\mathcal{F}}(s + u, v, .). \end{aligned} \tag{2}$$

However, we also have

$$\begin{aligned} \mathcal{F}((s + u)v) &= \mathcal{F}(sv + uv) \\ &= \mathcal{F}(sv) + \mathcal{F}(uv) + \varphi_{\mathcal{F}}(sv, uv, +) \\ &= \mathcal{F}(s)\Phi(v) + \Phi(s)\mathcal{F}(v) + m\mathcal{F}(s)\mathcal{F}(v) + \Phi(u)\mathcal{F}(v) + \mathcal{F}(u)\Phi(v) \\ &\quad + m\mathcal{F}(u)\mathcal{F}(v) + \varphi_{\mathcal{F}}(sv, uv, +) + \varphi_{\mathcal{F}}(s, v, .) + \varphi_{\mathcal{F}}(u, v, .). \end{aligned} \tag{3}$$

Comparing (2) and (3), we get

$$(m\mathcal{F}(v) + \Phi(v))\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \ \forall \ v \in \Omega. \tag{4}$$

Due to the fact that $\mathcal{F}$ is nilpotent, $\exists n \in \mathbb{Z}, n > 1$ so that $\mathcal{F}^n(s) = 0$ for all $s \in \Omega$. By putting $\mathcal{F}^{n-1}(v)$ instead of v in (4), the result is

$$\Phi(\mathcal{F}^{n-1}(v))\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \quad \text{for each } v \in \Omega. \tag{5}$$

Since Φ is an epimorphism, then

$$\mathcal{F}^{n-1}(v)\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \quad \text{for each } v \in \Omega. \tag{6}$$

Putting $\mathcal{F}^{n-2}(v)$ instead of v in (4), we get

$$(m\mathcal{F}^{n-1}(v) + \Phi(\mathcal{F}^{n-2}(v)))\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \quad \text{for each } v \in \Omega. \tag{7}$$

Once more, using (6) and the fact that Φ is onto, we get

$$\mathcal{F}^{n-2}(v)\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \quad \text{for each } v \in \Omega. \tag{8}$$

Hence, we may repeat the preceding procedure to achieve

$$\mathcal{F}(v)\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega), \quad \text{for each } v \in \Omega. \tag{9}$$

Using (4) and (9), we get $\Phi(v)\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega)$, $\forall\, v \in \Omega$. But Φ is an epimorphism, thus $v\varphi_{\mathcal{F}}(s, u, +) \in \zeta(\Omega)$, $\forall\, v \in \Omega$. Therefore, $\Omega\varphi_{\mathcal{F}}(s, u, +) \subseteq \zeta(\Omega)$. So, $\Omega\varphi_{\mathcal{F}}(s, u, +) = (0)$. If $Ann(\Omega)$ is the 2-sided annihilator of Ω, then $\varphi_{\mathcal{F}}(s, u, +) \in Ann(\Omega)$. But $Ann(\Omega)$ is an ideal on Ω contained in $\zeta(\Omega)$, thus $\varphi_{\mathcal{F}}(s, u, +) = 0$. Therefore, using (1), $\mathcal{F}(s+u) = \mathcal{F}(s)+\mathcal{F}(u)$.

Theorem 2. *If the only central ideal in a semiprime ring Ω is the zero ideal, then the map $\mathcal{F}$ is a (Φ, m)-homoderivation on Ω.*

Proof. According to Theorem 1, $\mathcal{F}$ is additive. Let $u, s, t \in \Omega$ be any elements in Ω. We have

$$\mathcal{F}(us) = m\mathcal{F}(u)\mathcal{F}(s) + \mathcal{F}(u)\Phi(s) + \Phi(u)\mathcal{F}(s) + \varphi_{\mathcal{F}}(u, s, .), \tag{10}$$

and

$$\mathcal{F}(st) = m\mathcal{F}(s)\mathcal{F}(t) + \Phi(s)\mathcal{F}(t) + \mathcal{F}(s)\Phi(t) + \varphi_{\mathcal{F}}(s, t, .). \tag{11}$$

Using the associativity of Ω, we obtain $\mathcal{F}((us)t) = \mathcal{F}(u(st))$, so

$$\varphi_{\mathcal{F}}(u, s, .)(m\mathcal{F}(t) + \Phi(t)) - \varphi_{\mathcal{F}}(s, t, .)(m\mathcal{F}(u) + \Phi(u)) \in \zeta(\Omega). \tag{12}$$

Therefore,

$$\varphi_{\mathcal{F}}(u, s, .)[m\mathcal{F}(t) + \Phi(t), m\mathcal{F}(u) + \Phi(u)] = 0. \tag{13}$$

Replacing t by $\mathcal{F}^{n-1}(t)$ in (13), we have

$$\varphi_{\mathcal{F}}(u, s, .)[\Phi(\mathcal{F}^{n-1}(t)), m\mathcal{F}(u) + \Phi(u)] = 0. \tag{14}$$

Since Φ and $\mathcal{F}$ are commuting and Φ is surjective, then

$$\varphi_{\mathcal{F}}(u, s, .)[\mathcal{F}^{n-1}(t), m\mathcal{F}(u) + \Phi(u)] = 0. \tag{15}$$

Replacing t by $\mathcal{F}^{n-2}(t)$ in (13) and using (15), we have

$$\varphi_{\mathcal{F}}(u, s, .)[\mathcal{F}^{n-2}(t), m\mathcal{F}(u) + \Phi(u)] = 0. \tag{16}$$

Repeating the steps above, we get

$$\varphi_{\mathcal{F}}(u, s, .)[\mathcal{F}(t), m\mathcal{F}(u) + \Phi(u)] = 0. \tag{17}$$

From (13) and (17), we get

$$\varphi_{\mathcal{F}}(u,s,.)[\Phi(t),m\mathcal{F}(u)+\Phi(u)]=0. \tag{18}$$

Again, since Φ is surjective, then

$$\varphi_{\mathcal{F}}(u,s,.)[t,m\mathcal{F}(u)+\Phi(u)]=0. \tag{19}$$

Replacing t by tu in (19), we get

$$\varphi_{\mathcal{F}}(u,s,.)t[u,m\mathcal{F}(u)+\Phi(u)]=0.$$

Therefore,

$$\varphi_{\mathcal{F}}(u,s,.)\Omega[u,m\mathcal{F}(u)+\Phi(u)]=0.$$

Let $\mathcal{Q}=\{\mathcal{A}_\alpha|\alpha\in\Lambda,\mathcal{A}_\alpha$ be a prime ideal in $\Omega\}$ and $\cap\mathcal{A}_\alpha=(0)$. Suppose that $\mathcal{A}$ represents a standard $\mathcal{A}_\alpha$ in $\mathcal{Q}$. For each $u\in\Omega$, we have either $\varphi_{\mathcal{F}}(u,s,.)\in\mathcal{A},\ \forall\ s\in\Omega$ or $[x,m\mathcal{F}(u)+\Phi(u)]\in\mathcal{A},\ \forall\ x\in\Omega$. Firstly, if $\varphi_{\mathcal{F}}(u,s,.)\in\mathcal{A},\ \forall\ s\in\Omega$, then $\mathcal{A}+\varphi_{\mathcal{F}}(u,s,.)=\mathcal{A},\ \forall\ s\in\Omega$. Thus, $\mathcal{A}+\Omega\varphi_{\mathcal{F}}(u,s,.)=\mathcal{A},\ \forall\ s\in\Omega$. So, $(\mathcal{A}+\Omega\varphi_{\mathcal{F}}(u,s,.))(\mathcal{A}+w)=(\mathcal{A}+w)(\mathcal{A}+\Omega\varphi_{\mathcal{F}}(u,s,.)),\ \forall\ s,w\in\Omega$. Therefore, $\mathcal{A}+[\Omega\varphi_{\mathcal{F}}(u,s,.),w]=\mathcal{A},\ \forall\ s,w\in\Omega$. Thus, $[\Omega\varphi_{\mathcal{F}}(u,s,.),w]\in\cap\mathcal{A}_\alpha=(0),\ \forall\ s,w\in\Omega$. That is $\Omega\varphi_{\mathcal{F}}(u,s,.)\subseteq\zeta(\Omega),\ \forall\ s\in\Omega$. So, $\varphi_{\mathcal{F}}(u,s,.)=0\ \forall\ s\in\Omega$. In the other case, if $[x,m\mathcal{F}(u)+\Phi(u)]\in\mathcal{A},$ for each $x\in\Omega$, then $[x,\Phi(u)+m\mathcal{F}(u)]+\mathcal{A}=\mathcal{A},$ for each $x\in\Omega$. Therefore,

$$[x+\mathcal{A},\Phi(u)+m\mathcal{F}(u)+\mathcal{A}]=\mathcal{A},\quad \text{for each } x\in\Omega. \tag{20}$$

From (12) and (20), we have

$$\begin{aligned}\mathcal{A}&=[\varphi_{\mathcal{F}}(u,s,.)(m\mathcal{F}(t)+\Phi(t))+\mathcal{A}-\varphi_{\mathcal{F}}(s,t,.)(m\mathcal{F}(u)+\Phi(u))+\mathcal{A},x+\mathcal{A}]\\&=[\varphi_{\mathcal{F}}(u,s,.)(m\mathcal{F}(t)+\Phi(t))+\mathcal{A},x+\mathcal{A}]\ \text{ for each } s,t,x\in\Omega.\end{aligned} \tag{21}$$

As above in equation (13) we get $\mathcal{A}=[\varphi_{\mathcal{F}}(u,s,.)t+\mathcal{A},x+\mathcal{A}]=[\varphi_{\mathcal{F}}(u,s,.)t,x]+\mathcal{A},$ for each $s,t,x\in\Omega$. Thus, $[\varphi_{\mathcal{F}}(u,s,.)t,x]\in\mathcal{A},$ for each $s,t,x\in\Omega$. So, we achieve $[\varphi_{\mathcal{F}}(u,s,.)t,x]\in\cap\mathcal{A}_\alpha=(0),$ for each $u,s,t,x\in\Omega$. Again, $\varphi_{\mathcal{F}}(u,s,.)=0,\ \forall\ s\in\Omega$. So, we have $\varphi_{\mathcal{F}}(u,s,.)=0,\ \forall\ u,s\in\Omega$. From (10), we have

$$\mathcal{F}(us)=\mathcal{F}(u)\Phi(s)+\Phi(u)\mathcal{F}(s)+m\mathcal{F}(u)\mathcal{F}(s).$$

Therefore, $\mathcal{F}$ is a (Φ,m)-homoderivation of Ω.

Theorem 2 yields the following result from [8, Corollary 1].

Corollary 1. *Any nilpotent $CE-$homoderivation is also a homoderivation if the only central ideal in the semiprime ring is the zero ideal.*

A $CE-(\Phi,m)$-homoderivation preserves the center under certain conditions, according to the following theorem.

Theorem 3. *If $\zeta(\Omega)$ contains no non-zero nilpotent elements, then $\mathcal{F}$ preserves $\zeta(\Omega)$.*

Proof. Suppose that $\xi \in \zeta(\Omega)$ and $r \in \Omega$. Then,

$$\mathcal{F}(\xi r) - m\mathcal{F}(\xi)\mathcal{F}(r) - \mathcal{F}(\xi)\Phi(r) - \Phi(\xi)\mathcal{F}(r) \in \zeta(\Omega), \tag{22}$$

and

$$\mathcal{F}(r\xi) - m\mathcal{F}(r)\mathcal{F}(\xi) - \mathcal{F}(r)\Phi(\xi) - \Phi(r)\mathcal{F}(\xi) \in \zeta(\Omega). \tag{23}$$

From (22) and (23) we get

$$[m\mathcal{F}(r) + \Phi(r), \mathcal{F}(\xi)] \in \zeta(\Omega),\ \forall\, r \in \Omega. \tag{24}$$

Putting $\mathcal{F}^{n-1}(r)$ instead of r in (24) and since Φ is onto, and $\mathcal{F}, \Phi$ are commuting, we get

$$[\mathcal{F}^{n-1}(r), \mathcal{F}(\xi)] \in \zeta(\Omega),\ \forall\, r \in \Omega. \tag{25}$$

Once more, substituting $\mathcal{F}^{n-2}(r)$ for r in (24) and using (25), we achieve $[\Phi(\mathcal{F}^{n-2}(r)), \mathcal{F}(\xi)] \in \zeta(\Omega)$, for each $r \in \Omega$. Based on the features of Φ and $\mathcal{F}$, the result is

$$[\mathcal{F}^{n-2}(r), \mathcal{F}(\xi)] \in \zeta(\Omega), \quad \text{for each } r \in \Omega. \tag{26}$$

Using the same procedure as before, we get

$$[\mathcal{F}(r), \mathcal{F}(\xi)] \in \zeta(\Omega), \quad \text{for each } r \in \Omega. \tag{27}$$

From (24) and (27) we have $[\Phi(r), \mathcal{F}(\xi)] \in \zeta(\Omega),\ \forall\, r \in \Omega$. But Φ is surjective, then

$$[r, \mathcal{F}(\xi)] \in \zeta(\Omega), \quad \text{for each } r \in \Omega. \tag{28}$$

In (28), replacing r with $r\mathcal{F}(\xi)$ gives

$$[r\mathcal{F}(\xi), \mathcal{F}(\xi)] = [r, \mathcal{F}(\xi)]\mathcal{F}(\xi) \in \zeta(\Omega), \quad \text{for each } r \in \Omega. \tag{29}$$

So, we get $[[r, \mathcal{F}(\xi)]\mathcal{F}(\xi), r] = 0,\ \forall\, r \in \Omega$. Therefore,

$$[r, \mathcal{F}(\xi)]^2 = 0, \quad \text{for each } r \in \Omega. \tag{30}$$

However, the nilpotent elements in the center $\zeta(\Omega)$ are zero, therefore we can deduce that $[r, \mathcal{F}(\xi)] = 0,\ \forall\, r \in \Omega$ from (28) and (30). Hence, $\mathcal{F}(\xi) \in \zeta(\Omega)$, i.e., $\mathcal{F}$ preserves the center.

As a direct consequence of the above theorem, we recover the following result previously established in [8, Corollaries 2 and 3].

Corollary 2. *$\mathcal{F}$ preserves $\zeta(\Omega)$.*

Corollary 3. *If there are no non-zero nilpotent elements in $\zeta(\Omega)$, then every nilpotent centrally extended homoderivation of Ω preserves $\zeta(\Omega)$.*

4. $CE-(\Phi,m)$-homoderivations and commutativity of prime rings

The primary purpose of this section is to demonstrate conditions that assure a prime ring's commutativity when it admits a $CE-(\Phi,m)$-homoderivation. Throughout, Ω will be a prime ring, Φ an epimorphism on Ω, $m\in\mathbb{Z}$, $\mathcal{F}:\Omega\to\Omega$ a nilpotent centrally extended (Φ,m)-homoderivation, and $\mathcal{F},\Phi$ are commuting.

Theorem 4. *If $\mathcal{F}$ is not a (Φ,m)-homoderivation of Ω, then Ω is commutative.*

Proof. If Ω includes no non-zero central ideals, according to Theorem 2, $\mathcal{F}$ is a (Φ,m)-homoderivation on Ω, which is a contradiction. As a consequence, Ω has a non-zero ideal that is contained in the center $\zeta(\Omega)$. So, Ω is commutative, by using Lemma 1.

Theorem 5. *If $\mathcal{F}(0)$ is non-zero, then Ω is commutative.*

Proof. Due to $\mathcal{F}$ is a $CE-(\Phi,m)$-homoderivation, then $\mathcal{F}(0+0)-\mathcal{F}(0)-\mathcal{F}(0)\in\zeta(\Omega)$. It indicates that $\mathcal{F}(0)\in\zeta(\Omega)$. Using the property of $\mathcal{F}$, $\mathcal{F}(0t)-m\mathcal{F}(0)\mathcal{F}(t)-\mathcal{F}(0)\Phi(t)-\Phi(0)\mathcal{F}(t)\in\zeta(\Omega)$, for each $t\in\Omega$, it indicates $\mathcal{F}(0)(m\mathcal{F}(t)+\Phi(t))\in\zeta(\Omega)$, for each $t\in\Omega$. Knowing that $\mathcal{F},\Phi$ are commuting, $\mathcal{F}$ is nilpotent on Ω, and Φ is surjective, then $\mathcal{F}(0)t\in\zeta(\Omega),\forall\ t\in\Omega$. Therefore, $[\mathcal{F}(0)t,v]=0,\ \forall\ t,v\in\Omega$. Since $\mathcal{F}(0)\in\zeta(\Omega)$, we get $\mathcal{F}(0)[t,v]=0,$ for all $v,t\in\Omega$. Replacing t by wt, we arrive at $\mathcal{F}(0)w[t,v]=0,$ for each $v,t,w\in\Omega$. So, $\mathcal{F}(0)\Omega[t,v]=0,$ for all $v,t\in\Omega$. Using the primeness of Ω and $\mathcal{F}(0)\neq 0$, $[t,v]=0,$ for all $v,t\in\Omega$, i.e., Ω is commutative.

Theorem 6. *Let $\mathcal{F}\neq 0$. If $\mathcal{F}([u,s])=0$ (or $\mathcal{F}(u\circ s)=0$), for each $u,s\in\Omega$, then Ω is commutative.*

Proof. If Ω has a non-zero central ideal, by Lemma 1, Ω is commutative. Now, assume that the only central ideal in Ω is the zero ideal. Due to Theorem 1, $\mathcal{F}$ is additive. Firstly, assume that $\mathcal{F}([u,s])=0,\ \forall\ u,s\in\Omega$. Substituting su for u, we get $\mathcal{F}([su,s])=0=\mathcal{F}(s[u,s]),$ for each u,s in Ω. So $\mathcal{F}(s)\Phi([u,s])\in\zeta(\Omega)$. Since Φ is an epimorphism, we get

$$\mathcal{F}(u)[s,\Phi(u)]\in\zeta(\Omega),\ \forall\ u,s\in\Omega. \tag{31}$$

In (31), putting $s\Phi(u)$ instead of s, the result is $\mathcal{F}(u)[s,\Phi(u)]\Phi(u)\in\zeta(\Omega),\ \forall\ u,s\in\Omega$. Thus,

$$[t,\mathcal{F}(u)[s,\Phi(u)]\Phi(u)]=0,\ \forall\ u,s,t\in\Omega,$$

which leads to

$$\mathcal{F}(u)[s,\Phi(u)][t,\Phi(u)]=0,\ \forall\ u,s,t\in\Omega. \tag{32}$$

Putting tw in place t in (32) and using (32), we get

$$\mathcal{F}(u)[s,\Phi(u)]t[w,\Phi(u)]=0,\ \forall\ u,s,w,t\in\Omega. \tag{33}$$

Using the primeness of Ω, for each $u\in\Omega$ either $\Phi(u)\in\zeta(\Omega)$ or $\mathcal{F}(u)[s,\Phi(u)]=0,\ \forall\ s\in\Omega$. Assume that $u\in\Omega$ with $\mathcal{F}(u)[s,\Phi(u)]=0\ \forall\ s\in\Omega$. Replacing s by st, we get

$\mathcal{F}(u)s[t,\Phi(u)] = 0,\ \forall\ t,s \in \Omega$. Thus, for each $u \in \Omega$ either $\Phi(u) \in \zeta(\Omega)$ or $\mathcal{F}(u) = 0$. Consider that

$$\aleph = \{u \in \Omega : \Phi(u) \in \zeta(\Omega)\}$$

and

$$Æ = \{u \in \Omega : \mathcal{F}(u) = 0\}.$$

Then, $(\aleph, +)$ and $(Æ, +)$ are additive subgroups of the group $(\Omega, +)$, and the union of $\aleph$ and $Æ$ gives the whole ring Ω. So either $\aleph = \Omega$ implies Ω is commutative or $Æ = \Omega$ implies $\mathcal{F} = 0$.

Secondly, let $\mathcal{F}(u \circ s) = 0$, for all u, s in Ω. Putting su instead of u in $\mathcal{F}(u \circ s) = 0$, then $\mathcal{F}(su \circ s) = \mathcal{F}(s(u \circ s)) = 0\ \forall\ u, s \in \Omega$. So,

$$\mathcal{F}(s)\Phi(u \circ s) \in \zeta(\Omega),\ \forall\ u, s \in \Omega. \tag{34}$$

Substituting us for u in (34), we get

$$\mathcal{F}(s)\Phi(u \circ s)\Phi(s) \in \zeta(\Omega),\ \forall\ u, s \in \Omega.$$

By Lemma 2 for each $s \in \Omega$, either $\mathcal{F}(s)\Phi(u \circ s) = 0\ \forall\ u \in \Omega$ or $\Phi(s) \in \zeta(\Omega)$. Assume that $s \in \Omega$ where

$$\mathcal{F}(s)\Phi(u \circ s) = 0\ \forall\ u \in \Omega. \tag{35}$$

Putting su instead of u in (35), we get $\mathcal{F}(s)\Phi(s)\Phi([u,s]) = 0,\ \forall\ u \in \Omega$. Since Φ is an epimorphism, $\mathcal{F}(s)\Phi(s)[u,\Phi(s)] = 0,\ \forall\ u \in \Omega$. Putting ut instead of u, we get $\mathcal{F}(s)\Phi(s)u[t,\Phi(s)] = 0,\ \forall\ u,t \in \Omega$. By the primeness of Ω, either $\mathcal{F}(s)\Phi(s) = 0$ or $\Phi(s) \in \zeta(\Omega)$. Suppose that $\mathcal{F}(s)\Phi(s) = 0$. By (35), we get $\mathcal{F}(s)\Phi(u)\Phi(s) = 0\ \forall\ u \in \Omega$. Again, by the primeness of Ω and Φ is an epimorphism, either $\mathcal{F}(s) = 0$ or $\Phi(s) = 0$. Therefore, for each $s \in \Omega$, there are two cases: either $\mathcal{F}(s) = 0$ or $\Phi(s) \in \zeta(\Omega)$. Thus, as above, $\mathcal{F} = 0$ or Ω is commutative.

The primeness postulate in Theorem 6 cannot be disregarded which is demonstrated by the counterexample that follows.

Example 3. *Let* $\Omega = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : 0, a, b, c \in \mathbb{R}\right\}$ *be the ring of* 2×2 *upper triangular matrices over the field of real numbers* $\mathbb{R}$. *Consider the matrix* $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Omega$, *and define the map* $F : \Omega \to \Omega$ *by* $F(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = (c - a)E_{12}$. *Take* $\Phi = \mathrm{id}_\Omega$, *the identity map on* Ω, *and let* $m \in \mathbb{Z}$, *the set of integers. It is clear that,* Ω *is not a prime ring. Moreover,* $F \neq 0$, $F \circ \Phi = \Phi \circ F$, $F^2 = 0$, *and* $F([A,B]) = 0$ *for all* $A, B \in \Omega$. *The map* F *is a* $\mathrm{CE} - (\Phi, m)$*-homoderivation, while* Ω *is not a commutative ring.*

As an immediate consequence of Theorems 4, 5, and 6, we obtain the following result previously established in [8, Corollaries 4, 5, and 6].

Corollary 4. *A prime ring Ω is commutative if Ω has a nilpotent $CE-$homoderivation $\mathcal{F}$ and any of the following is true:*

(i) $\mathcal{F}$ *is not homoderivation.*

(ii) $\mathcal{F}(0)$ *is non-zero.*

(iii) $\mathcal{F}([p,q]) = 0$ *(or* $\mathcal{F}(p \circ q) = 0$*) for each* $p, q \in \Omega$.

5. Conclusion

This study establishes key structural properties of CE-(Φ, m)-homoderivations and clarifies their reduction to classical (Φ, m)-homoderivations under central ideal constraints. For prime rings, we proved that the presence of such mappings enforces commutativity, while also demonstrating through a counterexample that the primeness condition is indispensable. These results both extend existing derivation theory and provide a solid foundation for further investigations into the interplay between generalized homoderivations and ring commutativity.

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