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On the Degenerate Sadik Transform

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Abstract. In this paper, the authors introduce the degenerate Sadik transform and investigates the transform of some elementary functions. Also, sufficient condition for the existence of the said transform is also presented. Furthermore, this paper concludes that degenerate sadik transform is a unification of some other degenerate transforms such as degenerate Laplace transform, degenerate Sumudu transform, degenerate Elzaki integral transform and Laplace-type integral transform.

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Key Words and Phrases: Degenerate Laplace Transform, Degenerate Elzaki Transform, Degenerate Tarig Integral Transform, Degenerate Sumudu Transform, Degenerate Laplace-type Integral Transform

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1. Introduction

The integral transformation method is widely utilized in solving various types of differential equations due to its ability to simplify complex problems. By converting differential equations into algebraic equations, integral transforms streamline the problem-solving process, making it significantly easier and more efficient [1]. Over time, numerous integral transforms have been developed, including the Sumudu transform [2], Tarig transform [3], Elzaki transform [4], Aboodh transform [5], Kamal transform [6], and Laplace-Carson transform [7].

Among these, the Laplace transform, introduced by Pierre-Simon Laplace, remains one of the most prominent and widely applied tools in mathematics, physics, and engineering [8]. The Laplace transform of a function $f(t)$, denoted by $\mathcal{L}\{f(t)\}$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt,$$

for all real numbers $t \geq 0$ where $s \in \mathbb{C}$.

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Throughout the centuries, integral transforms have played a vital role in the development of new mathematical tools across various scientific disciplines. In 2018, mathematician Sadikali Latif Shaikh introduced a new integral transform, named the Sadik transform [9], which generalizes and unifies several existing integral transforms. This transform is denoted by $\mathcal{S}\{f(t)\}$ and defined by

$$F\{(u^{\alpha}, \beta)\} = \mathcal{S}\{f(t)\} = \frac{1}{u^{\beta}} \int_0^{\infty} e^{-u^{\alpha}}(t) f(t) dt. \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$ and u is a parameter. The Sadik transform provides a flexible framework that encompasses a wide range of existing transforms by assigning specific values to the parameters α and β. That is, when $\alpha = 0$ and $\beta = 1$, we get

$$\mathbb{S}\{f(t)\} = \int_0^{\infty} f(ut) e^{-t} dt, u \in (-\tau_1, \tau_2),$$

popularly known as the Sumudu transform [2], when $\alpha = -2$ and $\beta = 1$, we acquire

$$\mathcal{T}\{f(t)\} = \frac{1}{u} \int_0^{\infty} e^{-\frac{t}{u^2}} f(t) dt,$$

commonly called as the Tarig integral transform [3], when $\alpha = -1$ and $\beta = -1$, we obtain

$$\mathcal{E} f\{(t)\} = u \int_0^{\infty} e^{-\frac{t}{u}} f(t) dt, \quad t \leq 0, \ k_1 \leq u \leq k_2,$$

well-known as the Elzaki integral transform [10], when $\alpha = 1$ and $\beta = 1$, we get

$$A\{f(t)\} = \mathcal{K}(u) = \frac{1}{u} \int_0^{\infty} f(t) e^{-ut} dt, t \geq 0, k_1 \leq u, \leq k_2,$$

familiarly named as the Aboodh transform [5], when $\alpha = -1$ and $\beta = 0$, we derive

$$\mathcal{K}\{F(t)\} = \int_0^{\infty} f(t) e^{\frac{-t}{u} t} dt = G(u), t \geq 0, k_1 \leq u \leq k_2,$$

recognized as the Kamal transform [6]; and when $\alpha = 1$ and $\beta = -1$, we get

$$LC\{f(t)\} = G\{p\} = p \int_0^{\infty} e^{-pt} g(t) dt = pL\{g(t)\},$$

identified as as the Laplace-Carson transform [7]. These instances illustrate the remarkable generality and versatility of the Sadik transform as a unifying structure for various well-known transforms. In mathematics the degenerate refers to the simplification of solutions, particularly in limiting scenarios where specific parameters, denoted as λ, approach critical values like zero.

Recently, researchers have shown growing interest in exploring degenerate versions of classical integral transforms. Some of these degenerate transforms are the degenerate

Laplace integral transform [11], degenerate Laplace-type integral transform [12], degenerate Elzaki integral transform [13] and degenerate Sumudu Transform [14] defined by

$$\mathcal{L}_\lambda\{f(t)\} = \int_0^\infty e_\lambda^{-s}(t)f(t)dt = \int_0^\infty (1+\lambda t)^{-\frac{s}{\lambda}} f(t)dt,$$

$$F_{\alpha\lambda}(u) = \mathcal{G}_{\alpha\lambda}\{f(t)\} = u^\alpha \int_0^\infty e_\lambda^{-\frac{1}{u}}(t)\ f(t)dt = u^\alpha \int_0^\infty (1+\lambda t)^{-\frac{1}{u\lambda}}\ f(t)dt,$$

$$\mathcal{E}_\lambda\{f(t)\} = u\int_0^\infty e_\lambda^{-\frac{1}{u}}(t)f(t)dt = u\int_0^\infty (1+\lambda t)^{\frac{1}{u\lambda}} f(t)dt,$$

$$G_\lambda\{(u)\} = \mathbb{S}_\lambda\{f(t)\} = \frac{1}{u}\int_0^\infty e_\lambda^{\frac{-1}{u}}(t)dt, u \in (-\tau_1, \tau_2),$$

respectively.

In light of these developments, the authors are motivated to introduce and analyze a degenerate version of the Sadik transform and investigates the transform of some elementary functions. Also, sufficient condition for the existence of the said transform is also presented.

2. Preliminaries

Taekyon Kim and Dae S. Kim [11] defined the degenerate Laplace transform by the integral

$$\mathcal{L}_\lambda f(t) = \int_0^\infty e_\lambda^{-s}(t)f(t)dt = \int_0^\infty (1+\lambda t)^{-\frac{s}{\lambda}} f(t)dt$$

if the integral converges.

The following are some results of the degenerate Laplace integral transform of some elementary functions $f(t)$:

$f(t)$	$\mathcal{L}_\lambda\{f(t)\}$
1	$\frac{1}{s-\lambda}$
$\cos_\lambda^{(a)}(t)$	$\frac{s-\lambda}{(s-\lambda)^2+a^2}$
t	$\frac{1}{s^2-3s\lambda+2\lambda^2}$
$\sinh_\lambda^{(a)}(t)$	$\frac{a}{(s-\lambda)^2-a^2}$

$f(t)$	$\mathcal{L}_\lambda\{f(t)\}$
$t^n(n=0,1,...)$	$\frac{n!}{s^{n+1}}\frac{1}{\left(1-\frac{\lambda}{s}\right)...\left(1-\frac{(n+1)\lambda}{s}\right)}$
$\cosh_\lambda^{(a)}(t)$	$\frac{s-\lambda}{(s-\lambda)^2-a^2}$
$e_\lambda^a(t)$	$\frac{1}{s-\lambda-a}$
$\sin_\lambda^{(a)}(t)$	$\frac{a}{(s-\lambda)^2+a^2}$

Now, letting $s = \dfrac{1}{u}$, the degenerate laplace transform can be rewritten as

$$\mathcal{L}_\lambda\{f(t)\} = \int_0^\infty e_\lambda^{\frac{1}{u}}(t)f(t)dt = \int_0^\infty (1+\lambda t)^{\frac{1}{u\lambda}} f(t)dt.$$

Consequently, the degenerate Laplace integral transform [14] of some elementary functions $f(t)$ are as follows:

$f(t)$	$\mathcal{L}_\lambda\{f(t)\}$	$f(t)$	$\mathcal{L}_\lambda\{f(t)\}$
1	$\frac{u}{1-\lambda u}$	$t^n(n=0,1,...)$	$\frac{n!u^{n+1}}{(1-\lambda u)...(1(n+1)\lambda u)}$
t	$\frac{u^2}{(1-\lambda)(1-2\lambda u)}$	$e_\lambda^a(t)$	$\frac{u}{1-u(a+\lambda)}$
$\sinh_\lambda^{(a)}(t)$	$\frac{au^2}{(1-\lambda u)^2-u^2a^2}$	$\cos_\lambda^{(a)}(t)$	$\frac{(1-u\lambda)u}{(1-u\lambda)^2+u^2a^2}$
$\cosh_\lambda^{(a)}(t)$	$\frac{(1-u\lambda)u}{(1-\lambda u)^2-u^2a^2}$	$\sin_\lambda^{(a)}(t)$	$\frac{au^2}{(1-u\lambda)^2+u^2a^2}$

Note that $\lim_{\lambda\to 0}\mathcal{L}_\lambda\{f(t)\}=\mathcal{L}\{f(t)\}$.

In 2023, Jade Bong M. Natuil, Harren J. Campos and Jezer C. Fernandez [12] defined the degenerate of Laplace-type integral transform by the integral

$$F_{\alpha\lambda}(u)=\mathcal{G}_{\alpha\lambda}\{f(t)\}=u^\alpha\int_0^\infty e_\lambda^{-\frac{1}{u}}(t)\ f(t)dt=u^\alpha\int_0^\infty(1+\lambda t)^{-\frac{1}{u\lambda}}\ f(t)dt.$$

Presented below are the Laplace-type integral transform of some elementary functions that are well-discussed in:

$f(t)$	$\mathcal{G}_{\alpha\lambda}\{f(t)\}$	$f(t)$	$\mathcal{G}_{\alpha\lambda}\{f(t)\}$
1	$\frac{u^{\alpha+1}}{1-u\lambda}$	$\sin_\lambda^{(a)}t$	$\frac{au^{\alpha+2}}{(1-\lambda u)^2+u^2a^2}$
$t^n(0,1,2,...)$	$\frac{n!u^{\alpha+1+n}}{(1-u\lambda)...(1-(n+1)u\lambda)}$	$\cos_\lambda^{(a)}t$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2+u^2a^2}$
t	$\frac{u^{\alpha+2}}{(1-u\lambda)(1-2u\lambda)}$	$\sinh_\lambda^{(a)}t$	$\frac{au^{\alpha+2}}{(1-\lambda u)^2-u^2a^2}$
$e_\lambda^a t$	$\frac{u^{\alpha+1}}{1-u(a+\lambda)}$	$\cosh_\lambda^{(a)}t$	$\frac{(1-\lambda u)u^{\alpha+1}}{(1-\lambda u)^2-u^2a^2}$

In 2021, A. Kalavati, T. Kohali, and L.M. Upadhyaya [13] defined the degenerate of Elzaki transform by the integral

$$\mathcal{E}_\lambda\{f(t)\}=u\int_0^\infty e_\lambda^{-\frac{1}{u}}(t)f(t)dt=u\int_0^\infty(1+\lambda t)^{-\frac{1}{u\lambda}}f(t)dt.$$

Below are the degenerate Elzaki integral transform of some elementary functions $f(t)$:

$f(t)$	$\mathcal{E}_\lambda\{f(t)\}$	$f(t)$	$\mathcal{E}_\lambda\{f(t)\}$
1	$\frac{u^2}{1-\lambda u}$	$\cosh_\lambda^{(a)}(t)$	$\frac{(1-u\lambda)u^2}{(1-\lambda u)^2-u^2a^2}$
$\cos_\lambda^{(a)}(t)$	$\frac{(1-u\lambda)u^2}{(1-u\lambda)^2+u^2a^2}$	$e_\lambda^a(t)$	$\frac{u^2}{1-u(a+\lambda)}$
t	$\frac{u^3}{(1-\lambda)(1-2\lambda u)}$	$\sin_\lambda^{(a)}(t)$	$\frac{au^3}{(1-u\lambda)^2+u^2a^2}$
$t^n(n=0,1,...)$	$\frac{n!u^{n+2}}{(1-\lambda u)...(1-(n+1)\lambda u)}$	$\sinh_\lambda^{(a)}(t)$	$\frac{au^3}{(1-\lambda u)^2-u^2a^2}$

Note that $\lim_{\lambda\to 0}\mathcal{E}_\lambda\{f(t)\} = \mathcal{E}\{f(t)\}$.

In 2020, Duran of Iskenderun Technical University [14] defined the degenerate of Sumudu Transform by the integral

$$G_\lambda\{(u)\} = \mathbb{S}_\lambda\{f(t)\} = \frac{1}{u}\int_0^\infty e_\lambda^{\frac{-1}{u}}(t)dt, u \in (-\tau_1, \tau_2).$$

The degenerate Sumudu transform satisfies the following operational properties and the transform of some elementary function $f(t)$:

$f(t)$	$\mathbb{S}_\lambda\{f(t)\}$
1	$\frac{1}{1-\lambda u}$
t^n	$\frac{n!u^n}{(1-\lambda u)(1-2\lambda u)...(1-(n+1)\lambda u)}$
t	$\frac{u}{(1-\lambda u)(1-2\lambda u)}$
$e_\lambda^a t$	$\frac{1}{1\ -u(a+\lambda)}$

$f(t)$	$\mathbb{S}_\lambda\{f(t)\}$
$\sin_\lambda^{(a)} t$	$\frac{au}{(1-\lambda u)^2+u^2a^2}$
$cos_\lambda^{(a)} t$	$\frac{1-u\lambda}{(1-\lambda u)^2+u^2a^2}$
$\sinh_\lambda^{(a)} t$	$\frac{au}{(1-\lambda u)^2-u^2a^2}$
$\cosh_\lambda^{(a)} t$	$\frac{1-u\lambda}{(1-\lambda u)^2-u^2a^2}$

Furthermore, in 2017, Kim et al [15] defined the degenerate exponential function as

$$\mathrm{e}_\lambda^{\mathrm{x}}(\mathrm{t}) = (1+\lambda\mathrm{t})^{\frac{\mathrm{x}}{\lambda}}, \quad \mathrm{e}_\lambda(\mathrm{t}) = \mathrm{e}_\lambda^1(\mathrm{t}) = (1+\lambda\mathrm{t})^{\frac{1}{\lambda}} \tag{2}$$

for $\lambda \in \mathbb{R},.$ Here, we note that $e_\lambda^x(t) = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}$, where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$ for $n \geq 1$ and that $\lim_{\lambda\to 0} e_\lambda^x(t) = \lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}} = e^{xt}$.

In [16], the degenerate sine and degenerate cosine functions are defined by the relations

$$\sin_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t) - e_\lambda^{-ix}(t)}{2i} = \sin\left(\frac{x}{\lambda}\log(1+\lambda t)\right), \tag{3}$$

$$\cos_\lambda^{(x)}(t) = \frac{e_\lambda^{ix}(t) + e_\lambda^{-ix}(t)}{2} = \cos\left(\frac{x}{\lambda}\log(1+\lambda t)\right), \tag{4}$$

respectively, where $i = \sqrt{-1}$.

In [11], the degenerate Euler function is defined by the relation

$$e_\lambda^{ix}(t) = \cos_\lambda^{(x)}(t) + i\sin_\lambda^{(x)}(t), \tag{5}$$

where $\cos_\lambda^{(x)}(t) = \cos\left(\frac{x}{\lambda}\log(1+\lambda t)\right)$ and $\sin_\lambda^{(x)}(t) = \sin\left(\frac{x}{\lambda}\log(1+\lambda t)\right)$.

In [14], the degenerate hyperbolic sine and degenerate hyperbolic cosine functions are defined by the relations

$$\sinh_\lambda^{(x)}(t) = \frac{e_\lambda^{x}(t) - e_\lambda^{-x}(t)}{2} \tag{6}$$

$$\cosh_\lambda^{(x)}(t) = \frac{e_\lambda^{x}(t) + e_\lambda^{-x}(t)}{2}, \tag{7}$$

respectively.

3. Main Results

This section presents definition of the degenerate Sadik transform. Moreover, discussions on some elementary functions of the degenerate Sadik transform are also provided.

Definition 1. Let $\lambda \in (0, \infty)$ and let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$F_\lambda(u^\alpha, \beta) = \mathcal{S}_\lambda\{f(t)\} = \frac{1}{u^\beta}\int_0^\infty e_\lambda^{-u^\alpha}(t) f(t) dt = \frac{1}{u^\beta}\int_0^\infty (1+\lambda t)^{-\frac{u^\alpha}{\lambda}} f(t) dt \tag{8}$$

is said to be the **degenerate Sadik transform** of $f(t)$. If the improper integral is convergent, then we say that the function $f(t)$ possesses as a degenerate Sadik transform.

Now, observe that, if the degenerate Sadik transform of $f(t)$ exists, then from the above definition,

$$\lim_{\lambda\to 0} \mathcal{S}_\lambda\{f(t)\} = \lim_{\lambda\to 0}\left[\frac{1}{u^\beta}\int_0^\infty e_\lambda^{-u^\alpha}(t) f(t) dt\right] = \frac{1}{u^\beta}\int_0^\infty e^{-u^\alpha}(t) f(t) dt = \mathcal{S}\{f(t)\}.$$

Remark 1. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (8),

$$\mathcal{S}_\lambda\{f(t)\} = \frac{1}{u^0}\int_0^\infty e_\lambda^{-u}(t) f(t)\ dt = \mathcal{L}_\lambda\{f(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (8),

$$\mathcal{S}_\lambda\{f(t)\} = \frac{1}{u^{-1}}\int_0^\infty e_\lambda^{-u^{-1}}(t) f(t)\ dt = \mathcal{E}_\lambda\{f(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (8),

$$\mathcal{S}_\lambda\{f(t)\} = \frac{1}{u^1}\int_0^\infty e_\lambda^{-u^{-1}}(t) f(t)\ dt = \mathbb{S}_\lambda\{f(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (8),

$$\mathcal{G}_{\alpha\lambda}\{f(t)\} = \frac{1}{u^{-\alpha}} \int_0^\infty e_\lambda^{-u^{-1}}(t) f(t)\ dt = \mathcal{G}_{\alpha\lambda}\{f(t)\}.$$

The following theorem is a sufficient condition for the existence of Degenerate Sadik Transform

Theorem 1. *Suppose that $f(t)$ is a piecewise-continuous function on the interval $[0,\infty)$ and of degenerate exponential order at infinity with $|f(t)| \leq Me_\lambda^c(t)$ for $t > P$, where $M \geq 0$ and P, C are constants. Then $\mathcal{S}_\lambda\{f(t)\}$ exists for $\frac{-u^\alpha + c}{\lambda} + 1 < 0$.*

Proof. Suppose that $f(t)$ is a piecewise-continuous function on the interval $[0,\infty)$ and has degenerate exponential order at infinity with $|f(t)| \leq Me_\lambda^c(t)$. Then,

$$\frac{1}{u^\beta} \int_0^\infty e_\lambda^{-u^\alpha}(t) f(t)\ dt = \frac{1}{u^\beta} \int_0^P e_\lambda^{-u^\alpha}(t) f(t)\ dt + \frac{1}{u^\beta} \int_P^\infty e_\lambda^{-u^\alpha}(t) f(t)\ dt. \tag{9}$$

Since the function $f(t)$ is piecewise-continuous in every finite interval $0 \leq t \leq P$, the first integral on the right hand side of equation (9) exists. Now, we will show that the second integral on the right hand side also exists. Note that for $t > P$,

$$\left|e_\lambda^{-u^\alpha}(t) f(t)\right| \leq e_\lambda^{-u^\alpha}(t)\ Me_\lambda^c(t).$$

Thus,

$$\begin{aligned}\frac{1}{u^\beta} \int_P^\infty \left|e_\lambda^{-u^\alpha}(t) f(t)\right|\ dt \leq& \frac{1}{u^\beta} \int_P^\infty e_\lambda^{-u^\alpha}(t) Me_\lambda^c(t)\ dt \\ =& \frac{M}{u^\beta} \lim_{R\to\infty} \int_P^R (1+\lambda t)^{\frac{-u^\alpha+c}{\lambda}}\ dt.\end{aligned}$$

Evaluation the right-hand side of the above equation, we can see that the integral converges for $\frac{-u^\alpha + c}{\lambda} + 1 < 0$. Since both the integrals on the right hand equation (9) converges for $\frac{-u^\alpha + c}{\lambda} + 1 < 0$, $f(t)$ has a degenerate Sadik transform for $\frac{-u^\alpha + c}{\lambda} + 1 < 0$.

The following theorem is the linearity property of the Degenerate Sadik Transform

Theorem 2. *Let $a, b \in \mathbb{R}$ and let $f(t)$ and $g(t)$ be functions whose degenerate Sadik transform exist, then*

$$\mathcal{S}_\lambda[af(t) + bg(t)] = a\mathcal{S}_\lambda[f(t)] + b\mathcal{S}_\lambda[g(t)]$$

Proof. Let $a, b \in \mathbb{R}$ and let $f(t)$ and $g(t)$ be any function whose degenerate Sadik transform exist, then

$$\mathcal{S}_\lambda\{af(t) + bg(t)\} = \frac{1}{u^\beta} \int_0^\infty e_\lambda^{-u^\alpha}\ (t)\ \{af(t) + bg(t)\}\ dt$$

$$
\begin{aligned}
&= a\ \frac{1}{u^{\beta}} \int_0^{\infty} e_{\lambda}^{-u^{\alpha}}(t)\ f(t)\ dt + b\ \frac{1}{u^{\beta}} \int_0^{\infty} e_{\lambda}^{-u^{\alpha}}(t)\ g(t)\ dt \\
&= a\ \mathcal{S}_{\lambda}\{f(t)\} + b\ \mathcal{S}_{\lambda}\{g(t)\}.
\end{aligned}
$$

The following results are some of the elementary functions of the degenerate Sadik Transform.

Theorem 3. *The degenerate Sadik transform of the function $f(t) = 1$ is given by*

$$
\mathcal{S}_{\lambda}\{1\} = \frac{1}{u^{\alpha+\beta} - u^{\beta}\lambda}, \quad \text{for} \quad \frac{-u^{\alpha}}{\lambda} + 1 < 0. \tag{10}
$$

Proof. From equation (8), when $f(t) = 1$, we have

$$
\begin{aligned}
\mathcal{S}_{\lambda}\{1\} &= \frac{1}{u^{\beta}} \int_0^{\infty} e_{\lambda}^{-u^{\alpha}}(t)\{1\}\ dt \\
&= \frac{1}{u^{\beta}} \int_0^{\infty} (1 + \lambda t)^{\frac{-u^{\alpha}}{\lambda}}\ dt \\
&= \frac{1}{u^{\beta}} \lim_{R\to\infty} \int_0^{R} (1 + \lambda t)^{\frac{-u^{\alpha}}{\lambda}}\ dt \\
&= \frac{1}{u^{\beta}} \lim_{R\to\infty} \left[\left(\frac{(1 + \lambda R)^{\frac{-u^{\alpha}}{\lambda}+1}}{-u^{\alpha} + \lambda} \right) - \left(\frac{(1)^{\frac{-u^{\alpha}}{\lambda}+1}}{-u^{\alpha} + \lambda} \right) \right] \\
&= \frac{1}{u^{\beta}} \left[-\left(\frac{1}{-u^{\alpha} + \lambda} \right) \right], \ \text{for} \ \frac{-u^{\alpha}}{\lambda} + 1 < 0 \\
&= \frac{1}{u^{\beta}} \left(\frac{1}{u^{\alpha} - \lambda} \right) \\
&= \frac{1}{u^{\alpha}u^{\beta} - u^{\beta}\lambda}.
\end{aligned}
$$

Hence, the degenerate Sadik transform of the function $f(t) = 1$ is given by

$$
\mathcal{S}_{\lambda}\{1\} = \frac{1}{u^{\alpha+\beta} - u^{\beta}\lambda}, \quad \text{for} \quad \frac{-u^{\alpha}}{\lambda} + 1 < 0.
$$

Remark 2. Observe that as $\lambda \to 0$, $\mathcal{S}_{\lambda}\{1\}$ tends to $\mathcal{S}\{1\}$. That is,

$$
\lim_{\lambda\to 0} \mathcal{S}_{\lambda}\{1\} = \lim_{\lambda\to 0} \left[\frac{1}{u^{\alpha+\beta} - u^{\beta}\lambda} \right] = \frac{1}{u^{\alpha+\beta}} = \mathcal{S}\{1\}.
$$

Remark 3. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (10),

$$
\mathcal{S}_{\lambda}\{1\} = \frac{1}{u^{1+0} - u^{0}\lambda} = \frac{1}{u - \lambda} = \mathcal{L}_{\lambda}\{1\}.
$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (10),

$$\mathcal{S}_\lambda\{1\} = \frac{1}{u^{-1+-1} - u^{-1}\lambda} = \frac{u^2}{1-u\lambda} = \mathcal{E}_\lambda\{1\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (10),

$$\mathcal{S}_\lambda\{1\} = \frac{1}{u^{-1+1} - u^{1}\lambda} = \frac{1}{1-u\lambda} = \mathbb{S}_\lambda\{1\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (10),

$$\mathcal{S}_\lambda\{1\} = \frac{1}{u^{-1+(-\alpha)} - u^{-\alpha}\lambda} = \frac{u^{(\alpha+1)}}{1-u\lambda} = \mathcal{G}_{\alpha\lambda}\{1\}.$$

Theorem 4. *The degenerate Sadik transform of the function $f(t) = t$ is given by*

$$\mathcal{S}_\lambda\{t\} = \frac{u^{-\beta}}{(u^\alpha - 2\lambda)(u^\alpha - \lambda)}, \quad \textit{for} \quad \frac{-u^\alpha}{\lambda} + 2 < 0. \tag{11}$$

Proof. By Definition for $f(t) = t$, we have

$$\begin{aligned}
\mathcal{S}_\lambda\{t\} =& \frac{1}{u^\beta}\int_0^\infty e_\lambda^{-u^\alpha}(t)\ \{t\}\ dt \\
=& \frac{1}{u^\beta}\lim_{R\to\infty}\int_0^R (1+\lambda t)^{\frac{-u^\alpha}{\lambda}}\ t\ dt. \\
=& \frac{1}{u^\beta\lambda}\lim_{R\to\infty}\left[\left(\frac{(1+\lambda(R))^{\frac{-u^\alpha}{\lambda}+2}}{-u^\alpha+2\lambda} - \frac{(1+\lambda(R))^{\frac{-u^\alpha}{\lambda}+1}}{-u^\alpha+\lambda}\right) - \left(\frac{(1)^{\frac{-u^\alpha}{\lambda}+2}}{-u^\alpha+2\lambda} - \frac{(1)^{\frac{-u^\alpha}{\lambda}+1}}{-u^\alpha+\lambda}\right)\right] \\
=& \frac{1}{u^\beta\lambda} - \left(\frac{1}{-u^\alpha+2\lambda} - \frac{1}{-u^\alpha+\lambda}\right) \quad \text{for} \quad \frac{-u^\alpha}{\lambda}+2<0 \\
=& \frac{1}{u^\beta\lambda}\left(\frac{1}{u^\alpha-2\lambda} - \frac{1}{u^\alpha-\lambda}\right) \\
=& \frac{u^{-\beta}}{(u^\alpha-2\lambda)(u^\alpha-\lambda)}.
\end{aligned}$$

Remark 4. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{t\}$ tends to $\mathcal{S}\{t\}$. That is,

$$\lim_{\lambda\to 0}\mathcal{S}_\lambda\{t\} = \frac{u^{-\beta}}{(u^\alpha)(u^\alpha)} = \frac{u^{-\beta}}{u^{2\alpha}} = \mathcal{S}\{t\}.$$

Remark 5. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (11),

$$\mathcal{S}_\lambda\{t\} = \frac{u^{-0}}{(u^1-2\lambda)(u^1-\lambda)} = \frac{1}{u^2 - 3u\lambda + 2\lambda^2} = \mathcal{L}_\lambda\{t\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (11),

$$\mathcal{S}_\lambda\{t\} = \frac{u^{-(-1)}}{(u^{-1} - 2\lambda)(u^{-1} - \lambda)} = \frac{u^3}{(1 - u\lambda)(1 - 2u\lambda)} = \mathcal{E}_\lambda\{t\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (11),

$$\mathcal{S}_\lambda\{t\} = \frac{u^{-1}}{(u^{-1} - 2\lambda)(u^{-1} - \lambda)} = \frac{u}{(1 - u\lambda)(1 - 2u\lambda)} = \mathbb{S}_\lambda\{t\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (11),

$$\mathcal{S}_\lambda\{t\} = \frac{u^{-(-\alpha)}}{(u^{-1} - 2\lambda)(u^{-1} - \lambda)} = \frac{u^{\alpha+2}}{(1 - u\lambda)(1 - 2u\lambda)} = \mathcal{G}_{\alpha\lambda}\{t\}.$$

Theorem 5. *The degenerate Sadik transform of the function $f(t) = e_\lambda^a(t)$ is given by*

$$\mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-\beta}}{u^\alpha - a - \lambda} \quad for \quad \frac{-u^\alpha + a}{\lambda} + 1 < 0. \tag{12}$$

Proof. From equation (8), when $f(t) = e_\lambda^a(t)$, we have

$$\begin{aligned}
\mathcal{S}_\lambda\{e_\lambda^a(t)\} =& \frac{1}{u^\beta} \int_0^\infty e_\lambda^{-u^\alpha}(t)\; e_\lambda^a(t)\; dt \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \int_0^R (1 + \lambda t)^{\frac{-u^\alpha + a}{\lambda}}\; dt. \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \left[\left(\frac{(1 + \lambda R)^{\frac{-u^\alpha + a}{\lambda} + 1}}{-u^\alpha + a + \lambda} \right) - \left(\frac{(1)^{\frac{-u^\alpha + a}{\lambda} + 1}}{-u^\alpha + a + \lambda} \right) \right] \\
=& \frac{1}{u^\beta} \left[-\left(\frac{1}{-u^\alpha + a + \lambda} \right) \right], \quad \text{for} \quad \frac{-u^\alpha + a}{\lambda} + 1 < 0 \\
=& \frac{1}{u^\beta} \left(\frac{1}{u^\alpha - a - \lambda} \right) \\
=& \frac{u^{-\beta}}{u^\alpha - a - \lambda}.
\end{aligned}$$

Remark 6. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{e_\lambda^a(t)\}$ tends to $\mathcal{S}\{e^a(t)\}$. That is,

$$\lim_{\lambda\to 0} \mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-\beta}}{u^\alpha - a - 0} = \mathcal{S}\{e^a(t)\}.$$

Remark 7. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (12),

$$\mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-0}}{u^1 - a - \lambda} = \frac{1}{u - a - \lambda} = \mathcal{L}_\lambda\{e_\lambda^a(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (12),

$$\mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-(-1)}}{u^{-1} - a - \lambda} = \frac{u^2}{1 - u(a+\lambda)} = \mathcal{E}_\lambda\{e_\lambda^a(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (12),

$$\mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-1}}{u^{-1} - a - \lambda} = \frac{1}{1 - u(a+\lambda)} = \mathbb{S}_\lambda\{e_\lambda^a(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (12),

$$\mathcal{S}_\lambda\{e_\lambda^a(t)\} = \frac{u^{-(-\alpha)}}{u^{-1} - a - \lambda} = \frac{u^{\alpha+1}}{1 - u(a+\lambda)} = \mathcal{G}_{\alpha\lambda}\{e_\lambda^a(t)\}.$$

Corollary 1. The degenerate Sadik transform of the function is $f(t) = e_\lambda^{-a}(t)$ is given by

$$\mathcal{S}_\lambda\{e_\lambda^{-a}(t)\} = \frac{u^{-\beta}}{u^\alpha + a - \lambda}, \quad for \quad \frac{-u^\alpha - a}{\lambda} + 1 < 0. \tag{13}$$

Theorem 6. *The degenerate Sadik transform of the function $f(t) = e_\lambda^{ia}(t)$ is given by*

$$\mathcal{S}_\lambda\{e_\lambda^{ia}(t)\} = \frac{u^{-\beta}}{u^\alpha - ia - \lambda}, \textit{ for } 0 < u^{-\alpha}\lambda < 1. \tag{14}$$

Proof. When $f(t) = e_\lambda^{ia}(t)$ in equation (8), we have

$$\begin{aligned}
\mathcal{S}_\lambda\{e_\lambda^{ia}(t)\} =& \frac{1}{u^\beta} \int_0^\infty e_\lambda^{-u^\alpha}(t)\ \{e_\lambda^{ia}(t)\}\ dt \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \int_0^R (1+\lambda t)^{\frac{-u^\alpha + ia}{\lambda}}\ dt. \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \left[\frac{(1+\lambda R)^{\frac{-u^\alpha+\lambda}{\lambda}} (1+\lambda R)^{\frac{ia}{\lambda}}}{-u^\alpha + ia + \lambda} - \frac{(1)^{\frac{-u^\alpha+\lambda}{\lambda}} (1)^{\frac{ia}{\lambda}}}{-u^\alpha + ia + \lambda} \right] \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \left[\frac{(1+\lambda R)^{\frac{-u^\alpha+\lambda}{\lambda}}\ e_\lambda^{ia} R}{-u^\alpha + ia + \lambda} - \frac{1}{-u^\alpha + ia + \lambda} \right] \\
=& \frac{1}{u^\beta} \lim_{R\to\infty} \left[\frac{(1+\lambda R)^{\frac{-u^\alpha+\lambda}{\lambda}} \left(\cos\left(\frac{a}{\lambda} \log(1+\lambda R) \right) + i \sin\left(\frac{a}{\lambda} \log(1+\lambda R) \right) \right)}{-u^\alpha + ia + \lambda} \right] \\
&- \left[\frac{1}{-u^\alpha + ia + \lambda} \right]
\end{aligned}$$

If $u^{-\alpha}\lambda = 1$, then the first limit on the right-hand side of the above equation does not exist since the value of the function

$$\cos\left(\frac{a}{\lambda}\log(1+\lambda R)\right) \text{ and } i\sin\left(\frac{a}{\lambda}\log(1+\lambda R)\right)$$

is oscillate between 1 and -1. Thus, $\mathcal{S}_\lambda\{e_\lambda^{ia}\}$ is not defined. Similarly, $\mathcal{S}_\lambda\{e_\lambda^{ia}\}$ is not defined for $u^{-\alpha}\lambda > 1$. However, observe that for $0 < u^{-\alpha}\lambda < 1$, $(1+\lambda(R))^{\frac{-u^\alpha+\lambda}{\lambda}}$ approaches to 0 as R approaches to ∞. Hence,

$$\mathcal{S}_\lambda\{e_\lambda^{ia}\} = \frac{1}{u^\beta}\left[-\left(\frac{1}{-u^\alpha + ia + \lambda}\right)\right] = \frac{u^{-\beta}}{u^\alpha - ia - \lambda}, \quad \text{for} \quad u^{-\alpha}\lambda < 1.$$

Corollary 2. *The degenerate Sadik transform of the function $f(t) = e_\lambda^{-ia}(t)$ is given by*

$$\mathcal{S}_\lambda\{e_\lambda^{-ia}(t)\} = \frac{u^{-\beta}}{u^\alpha + ia - \lambda}, \textit{ for } 0 < u^{-\alpha}\lambda < 1. \tag{15}$$

Theorem 7. *The degenerate Sadik transform of the function $f(t) = \sin_\lambda^a(t)$ is given by*

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2}. \tag{16}$$

Proof. Note that from Definition 3 for $f(t) = \sin_\lambda^{(a)}(t)$, we have

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \mathcal{S}_\lambda\left\{\frac{e_\lambda^{ia}(t) - e_\lambda^{-ia}(t)}{2i}\right\}. \tag{17}$$

Now, applying Theorems 2, 6 and Corollary 2 equation (17) deduces

$$\begin{aligned}
\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} =& \left(\frac{1}{2i}\right)\left[\mathcal{S}_\lambda\left\{e_\lambda^{ia}(t)\right\} - \mathcal{S}_\lambda\left\{e_\lambda^{-ia}(t)\right\}\right] \\
=& \left(\frac{1}{2i}\right)\left[\frac{u^{-\beta}}{u^\alpha - ia - \lambda} - \frac{u^{-\beta}}{u^\alpha + ia - \lambda}\right] \\
=& \left(\frac{u^{-\beta}}{2i}\right)\left[\frac{u^\alpha + ia - \lambda - u^\alpha + ia + \lambda}{(u^\alpha)^2 + u^\alpha ia - u^\alpha\lambda - u^\alpha ia - (ia)^2 + ia\lambda - u^\alpha\lambda - ia\lambda + \lambda^2}\right] \\
=& \left(\frac{u^{-\beta}}{2i}\right)\left[\frac{2ia}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2}\right] \\
=& \left[\frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2}\right].
\end{aligned}$$

Therefore, the degenerate Sadik transform of the function $f(t) = \sin_\lambda^{(a)}(t)$ is given by

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2}.$$

Remark 8. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{\sin_\lambda^{(a)}(t)\}$ tends to $\mathcal{S}\{\sin at\}$. That is,

$$\lim_{\lambda\to 0} \mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \lim_{\lambda\to 0}\left[\frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2}\right] = \frac{au^{-\beta}}{u^{2\alpha} + a^2} = \mathcal{S}\{\sin(at)\}.$$

Remark 9. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (16),

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-0}}{u^{2(1)} - 2u^1\lambda + a^2 + \lambda^2} = \frac{a}{(u-\lambda)^2 + a^2} = \mathcal{L}_\lambda\{\sin_\lambda^a(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (16),

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-(-1)}}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{au^3}{(1-\lambda u)^2 + a^2u^2} = \mathcal{E}_\lambda\{\sin_\lambda^a(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (16),

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-1}}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{au}{(1-\lambda u)^2 + a^2u^2} = \mathbb{S}_\lambda\{\sin_\lambda^a(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (16),

$$\mathcal{S}_\lambda\{\sin_\lambda^a(t)\} = \frac{au^{-(-\alpha)}}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{au^{\alpha+2}}{(1-\lambda u)^2 + a^2u^2} = \mathcal{G}_{\alpha\lambda}\{\sin_\lambda^a(t)\}.$$

Theorem 8. *The degenerate Sadik transform of the function $f(t) = \cos_\lambda^a(t)$ is given by*

$$\mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{au^{\alpha-\beta} - u^{-\beta}\lambda}{u^{2\alpha} - 2u^\alpha\lambda + a^2 + \lambda^2} \tag{18}$$

Remark 10. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{\cos_\lambda^{(a)}(t)\}$ tends to $\mathcal{S}\{\cos at\}$. That is,

$$\lim_{\lambda\to 0} \mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{u^{\alpha-\beta} - u^{-\beta}(0)}{u^{2\alpha} - 2u^\alpha(0) + a^2 + (0)^2} = \mathcal{S}\{\cos(at)\}.$$

Remark 11. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (18),

$$\mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{u^{1-0} - u^{-(1)}\lambda}{u^{2(1)} - 2u^1\lambda + a^2 + \lambda^2} = \frac{u-\lambda}{u^2 - 2u\lambda + \lambda^2 + a^2} = \mathcal{L}_\lambda\{\cos_\lambda^a(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (18),

$$\mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{u^{1-(-1)} - u^{-(-1)}\lambda}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{(1-u\lambda)u^2}{(1-\lambda)^2 + a^2u^2} = \mathcal{E}_\lambda\{\cos_\lambda^a(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (18),

$$\mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{u^{-1-(1)} - u^{-(1)}\lambda}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{(u^{-2} - u^{-1}\lambda)(u^2)}{1 - 2u\lambda + \lambda^2 u^2 + a^2u^2}$$
$$= \frac{1 - u\lambda}{(1-\lambda u)^2 + a^2u^2} = \mathbb{S}_\lambda\{\cos_\lambda^a(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (18),

$$\mathcal{S}_\lambda\{\cos_\lambda^a(t)\} = \frac{u^{-1-(-\alpha)} - u^{-(-\alpha)}\lambda}{u^{2(-1)} - 2u^{-1}\lambda + a^2 + \lambda^2} = \frac{(1-u\lambda)u^{1+\alpha}}{(1-\lambda u)^2 + a^2u^2} = \mathcal{G}_{\alpha\lambda}\{\cos_\lambda^a(t)\}.$$

Theorem 9. *The degenerate Sadik transform of the function $f(t)$ defined by $f(t) = \sinh_\lambda^{(a)}(t)$ is given by*

$$\mathcal{S}_\lambda\{\sinh_\lambda^a(t)\} = \frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}. \tag{19}$$

Remark 12. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{\sinh_\lambda^{(a)}(t)\}$ tends to $\mathcal{S}\{\sinh at\}$. That is,

$$\lim_{\lambda\to 0} \mathcal{S}_\lambda\{\sinh_\lambda^a(t)\} = \lim_{\lambda\to 0}\left[\frac{au^{-\beta}}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}\right] = \mathcal{S}\{\sinh(at)\}.$$

Remark 13. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (19),

$$\mathcal{S}_\lambda\{\sinh_\lambda^a(t)\} = \frac{au^{-0}}{u^{2(1)} - 2u^1\lambda - a^2 + \lambda^2} = \frac{a}{(u-\lambda)^2 - a^2} = \mathcal{L}_\lambda\{\sinh_\lambda^a(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (19),

$$\mathcal{S}_\lambda\{\sinh_\lambda^a(t)\} = \frac{au^{-(-1)}}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{au^3}{(1-\lambda u)^2 - a^2u^2} = \mathcal{E}_\lambda\{\sinh_\lambda^a(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (19),

$$= \frac{au^{-1}}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{au}{(1-\lambda u)^2 - a^2u^2} = \mathbb{S}_\lambda\{\sinh_\lambda^a(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (19),

$$= \frac{au^{-(-\alpha)}}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{au^{\alpha+2}}{(1-\lambda u)^2 - a^2u^2} = \mathcal{G}_{\alpha\lambda}\{\sinh_\lambda^a(t)\}.$$

Theorem 10. *The degenerate Sadik transform of the function $f(t)$ defined by $f(t) = \cosh_\lambda^{(a)}(t)$ is given by*

$$\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \frac{u^{\alpha-\beta} - u^{-\beta}\lambda}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}. \tag{20}$$

Proof. Observe that by Definition 7, for $f(t) = \cosh_\lambda^{(a)}(t)$, we have

$$\mathcal{S}_\lambda\{\cosh_\lambda^{(a)}(t)\} = \mathcal{S}_\lambda\left\{\frac{e_\lambda^a(t) + e_\lambda^{-a}(t)}{2}\right\}.$$

Next, applying Theorems 2, 5 and Corollary 1, simplifying the resulting equation resulted to

$$\begin{aligned}
\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} =& \mathcal{S}_\lambda\left\{\frac{e_\lambda^a(t) + e_\lambda^{-a}(t)}{2}\right\} \\
=& \left(\frac{1}{2}\right)\left[\mathcal{S}_\lambda\left\{e_\lambda^a(t)\right\} + \mathcal{S}_\lambda\left\{e_\lambda^{-a}(t)\right\}\right] \\
=& \left(\frac{1}{2}\right)\left[\frac{u^{-\beta}}{u^\alpha - a - \lambda} + \frac{u^{-\beta}}{u^\alpha + a - \lambda}\right] \\
=& \left(\frac{u^{-\beta}}{2}\right)\left[\frac{2u^\alpha - 2\lambda}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}\right] \\
=& \frac{u^{-\beta}u^\alpha - u^{-\beta}\lambda}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2} \\
=& \frac{u^{\alpha-\beta} - u^{-\beta}\lambda}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}.
\end{aligned}$$

Remark 14. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{\cosh^{(a)}\lambda(t)\}$ tends to $\mathcal{S}\{\cosh at\}$. That is,

$$\lim_{\lambda\to 0}\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \lim_{\lambda\to 0}\left[\frac{u^{\alpha-\beta} - u^{-\beta}\lambda}{u^{2\alpha} - 2u^\alpha\lambda - a^2 + \lambda^2}\right] = \left[\frac{u^{\alpha-\beta} - u^{-\beta}(0)}{u^{2\alpha} - 2u^\alpha(0) - a^2 + (0)^2}\right] = \mathcal{S}\{\cosh(at)\}.$$

Remark 15. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (20),

$$\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \frac{u^{1-0} - u^{-0}\lambda}{u^{2(1)} - 2u^1\lambda - a^2 + \lambda^2} = \frac{u - \lambda}{u^2 - 2u\lambda + \lambda^2 - a^2} = \mathcal{L}_\lambda\{\cosh_\lambda^a(t)\}.$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (20),

$$\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \frac{u^{-1-(-1)} - u^{-(-1)}\lambda}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{(1 - u\lambda)u^2}{1 - 2u\lambda + \lambda^2u^2 - a^2u^2} = \mathcal{E}_\lambda\{\cosh_\lambda^a(t)\}.$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (20),

$$\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \frac{u^{-1-1} - u^{-1}\lambda}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{1 - u\lambda}{(1-\lambda u)^2 - a^2u^2} = \mathbb{S}_\lambda\{\cosh_\lambda^a(t)\}.$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (20),

$$\mathcal{S}_\lambda\{\cosh_\lambda^a(t)\} = \frac{u^{-1-(-\alpha)} - u^{-(-\alpha)}\lambda}{u^{2(-1)} - 2u^{-1}\lambda - a^2 + \lambda^2} = \frac{(1 - u\lambda)u^{\alpha+1}}{(1-\lambda u)^2 - a^2u^2} = \mathcal{G}_{\alpha\lambda}\{\cosh_\lambda^a(t)\}.$$

Theorem 11. *The degenerate Sadik transform of the function $f(t) = t^n$ is given by,*

$$\mathcal{S}_\lambda\{t^n\} = \frac{n!}{u^\beta(u^\alpha - \lambda)(u^\alpha - 2\lambda)....(u^\alpha - n\lambda)(u^\alpha - (n+1)\lambda)} \tag{21}$$

for $\dfrac{(n-k+1)\lambda - u^\alpha}{\lambda} < 0.$

Proof. Directly from the Definition 1, for $f(t) = t^n$, we have

$$\begin{aligned}
\mathcal{S}_\lambda\{t^n\} =& \frac{1}{u^\beta}\int_0^\infty e_\lambda^{-u^\alpha}(t)\ t^n\ dt \\
=& \frac{1}{u^\beta\lambda^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k \lim_{R\to\infty}\left[\frac{\lambda\ \ v^{\frac{-u^\alpha+\lambda n-\lambda k+\lambda}{\lambda}}}{-u^\alpha+\lambda n-\lambda k+\lambda}\right]\Bigg|_1^{1+\lambda R} \\
=& \frac{1}{u^\beta\lambda^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left[-\frac{1}{-u^\alpha+\lambda(n-k+1)}\right],\ \text{for}\ \frac{(n-k+1)\lambda-u^\alpha}{\lambda}<0 \\
=& \frac{1}{u^\beta\lambda^n}\left[\frac{1}{u^\alpha-\lambda(n+1)} - \frac{n}{u^\alpha-n\lambda} + ... + n(-1)^{n-1}\left(\frac{1}{u^\alpha-2\lambda}\right) + (-1)^n\left(\frac{1}{u^\alpha-\lambda}\right)\right] \\
=& \frac{n!\lambda^n}{u^\beta\lambda^n(u^\alpha-\lambda)(u^\alpha-2\lambda)....(u^\alpha-n\lambda)(u^\alpha-(n+1)\lambda)} \\
=& \frac{n!}{u^\beta(u^\alpha-\lambda)(u^\alpha-2\lambda)....(u^\alpha-n\lambda)(u^\alpha-(n+1)\lambda)}
\end{aligned}$$

for $\dfrac{(n-k+1)\lambda - u^\alpha}{\lambda} < 0.$

Remark 16. Observe that as $\lambda \to 0$, $\mathcal{S}_\lambda\{t^n\}$ tends to $\mathcal{S}\{t^n\}$. That is,

$$\lim_{\lambda\to 0}\mathcal{S}_\lambda\{t^n\} = \lim_{\lambda\to 0}\left[\frac{n!}{u^\beta(u^\alpha-\lambda)(u^\alpha-2\lambda)....(u^\alpha-n\lambda)(u^\alpha-(n+1)\lambda)}\right] = \mathcal{S}\{t^n\}.$$

Remark 17. m

1. When $\beta = 0$ and $\alpha = 1$ in equation (21),

$$\begin{aligned}\mathcal{S}_\lambda\{t^n\} =& \frac{n!}{u^0(u^1-\lambda)(u^1-2\lambda)....(u^1-n\lambda)(u^1-(n+1)\lambda)}\\ =& \frac{n!}{u^{n+1}(1-\frac{\lambda}{u})(1-\frac{2\lambda}{u})....(1-\frac{n\lambda}{u})(1-\frac{(n+1)\lambda}{u})}\\ =& \mathcal{L}_\lambda\{t^n\}.\end{aligned}$$

2. When $\beta = -1$ and $\alpha = -1$ in equation (21),

$$\begin{aligned}\mathcal{S}_\lambda\{t^n\} =& \frac{n!}{u^{-1}(u^{-1}-\lambda)(u^{-1}-2\lambda)....(u^{-1}-n\lambda)(u^{-1}-(n+1)\lambda)}\\ =& \frac{n!u^{n+2}}{(1-u\lambda)(1-2u\lambda)....(1-nu\lambda)(1-(n+1)u\lambda)}\\ =& \mathcal{E}_\lambda\{t^n\}.\end{aligned}$$

3. When $\beta = 1$ and $\alpha = -1$ in equation (21),

$$\begin{aligned}\mathcal{S}_\lambda\{t^n\} =& \frac{n!}{u^{1}(u^{-1}-\lambda)(u^{-1}-2\lambda)....(u^{-1}-n\lambda)(u^{-1}-(n+1)\lambda)}\\ =& \frac{n!u^{n}}{(1-u\lambda)(1-2u\lambda)....(1-nu\lambda)(1-(n+1)u\lambda)}\\ =& \mathbb{S}_\lambda\{t^n\}.\end{aligned}$$

4. When $\beta = -\alpha$ and $\alpha = -1$ in equation (21),

$$\begin{aligned}\mathcal{S}_\lambda\{t^n\} =& \frac{n!}{u^{-\alpha}(u^{-1}-\lambda)(u^{-1}-2\lambda)....(u^{-1}-n\lambda)(u^{-1}-(n+1)\lambda)}\\ =& \frac{n!u^{-\alpha+(n+1)}}{(1-u\lambda)(1-2u\lambda)....(1-nu\lambda)(1-(n+1)u\lambda)}\\ =& \mathcal{G}_{\alpha\lambda}\{t^n\}.\end{aligned}$$

Conclusion

This study introduced the Degenerate version of Sadik Transform, a generalization of the Sadik Transform that sum up some known degenerate transforms, including the degenerate Laplace, Sumudu, Tarig, Elzaki, and Laplace-type integral transforms. By defining the degenerate Sadik transform and establishing its existence under specific conditions, the paper demonstrated its capacity to serve as a unifying framework for multiple degenerate transforms. The transform of several elementary functions was derived, and the results validated that degenerate Sadik transform certainly converges to the natural Sadik transform as the degeneracy parameter approaches zero. This highlights the degenerate Sadik transform's flexibility, depth, and potential for broader application across various mathematical and applied areas.

Recommendation

The following are recommended for further investigations:

1. To define degenerate versions of other well-known transforms, such as the Aboodh and Kamal transforms, and derive their forms when applied to elementary functions.

2. To derive and analyze the degenerate Sadik transform of derivatives and integrals of elementary functions, helping to understand how it behaves under basic calculus operations.

3. To explore the inverse of the degenerate Sadik transform, which is a key step for reconstructing original functions and solving applied problems effectively.

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