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Trapezoidal and Midpoint-Type Inequalities Based on Extended Conformable Operators

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Abstract. In this paper, we explore some inequalities derived from twice differentiable functions together with the extended conformable fractional operators. First, we investigate two lemmas using extended conformable fractional operators. Then, we utilize these results to explore some new trapezoidal and midpoint-type inequalities through the use of the convex property of twice differentiable functions. Moreover, using the power mean inequality and Hölder's inequality, we introduce a new class of inequalities. The explored results are validated through different 2D and 3D graphs. This new class extends the results of previous research studies. The present paper seeks to motivate researchers to apply these concepts to other fractional operators.

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Key Words and Phrases: Extended conformable fractional operators, trapezoidal-type inequalities, midpoint-type inequalities, Hölder's inequality, power mean inequality

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1. Introduction

Convex functions form an essential branch in mathematics with considerable applications across diverse fields. By definition, a convex function lies above the straight line connecting any two points in its domain. The study of convex functions has evolved over the past century, with roots deeply embedded in geometry [1]. Their utility spans various disciplines, notably physics [2], chemistry [3], medicine [4], optimization, economics, statistics [5], and bioengineering [6]. Additionally, fields such as DC programming [7],

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convex programming [8], functional analysis [9], monotone operator theory [10], Object Detection Algorithm [11], and complex analysis [12] further underscore the importance of convexity. This characteristic plays a crucial role in solving many real-world problems related to minimizing or maximizing functions subject to certain-constraints.

The exploration of convex functions can be traced back to the ancient Greek mathematician Archimedes (287 BC-212 BC), who, in his work On the Sphere and Cylinder [13], described a convex arc as a curved line in a plane that remains entirely on one side of the straight line connecting its endpoints [14]. The study of mathematical inequalities started in the 18th century with work by Carl Friedrich Gauss. Later, mathematicians like Augustin-Louis Cauchy and Pafnuty Chebyshev explored how inequalities could be used in analysis. A key result came from Viktor Bunyakovsky, who proved an early form of the Cauchy-Schwarz inequality [15]. In the 19th century, Otto Hölder introduced a version of what would later be known as Jensen's inequality, assuming the second derivative of a function is non-negative. The study of inequalities grew more important in the 20th century, with major contributions by mathematicians like Leonhard Euler and Adrien-Marie Legendre. In recent years, there has been a growing interest in exploring new aspects of convex functions, particularly in deriving novel inequalities such as Jensen's inequality [16], the power mean inequality [17], the Cauchy-Schwarz inequality [18], Bell's inequality [19], Boole's inequality [20], the Sobolev inequality [21], Chernoff's inequality [22], the Hermite-Hadamard ($\mathcal{H}-\mathcal{H}$) inequality [23], Ostrowski type Inequalities [24], midpoint and trapezoidal-type inequalities [25, 26]. While various types of inequalities exist, convex inequalities play a vital role in this field. As Fractional calculus is a branch of mathematical analysis that generalizes the concepts of differentiation and integration to non-integer order. There are many important applications of fractional calculus, such as modeling influenza [27] and trajectory tracking of the Stanford robot [28]. Mathematicians have derived different operators in fractional calculus to obtain desired results, such as the Riemann-Liouville fractional operators [29], the Caputo-Fabrizio fractional operator [30], the Hilfer fractional derivative operator [31], Hadamard-type fractional operators [32], and Katugampola fractional integrals [33], according to their needs. One such operator is the conformable fractional operator [34], and $(k, \rho)-$Conformable Fractional Integrals [35]. Moreover mathematicians have made significant efforts to analyze the behavior of inequalities particularly fractional inequalities through computational methods [36].

This work is based on investigating new inequalities of trapezoidal and midpoint type for convex functions. These inequalities are obtained with the aid of the extended conformable fractional operators and twice differentiable functions. The extended conformable fractional operators generalize the concept of fractional calculus to a wide range of functions, and as such, it provides a strong tool for studying inequalities with generalized functions. Also, twice differentiable functions are fundamental in the construction and proof of these inequalities. Additionally, the absolute value function and convex properties of twice differentiable functions are utilized to provide connections between the new inequalities. Also, celebrated inequalities like Hölder inequality [37] and the Power Mean Inequality [37] are utilized in deriving special findings.

This paper is organized into sections as follows: Section 2 offers a detailed review of convex functions, including definitions and some properties. Section 3 introduces the new inequalities of trapezoidal type derived by using extended conformable fractional operators along with generalized twice differentiable functions. Section 4 discusses the midpoint type inequalities similar to previous. Finally, Section 5 wraps up the paper by summarizing the findings and proposing avenues for future research.

2. Preliminaries

In this section, we discuss fundamental results that will later prove beneficial. The first integrals to be described are the Riemann-Liouville integrals [38]. Additionally, we recall the extended conformable operators [39], which are well documented in the literature and serves as a crucial building block for this paper. Since the beta and gamma functions [40] are key components of fractional calculus, both are discussed here.

Definition 1. *For positive real numbers ζ and η, the gamma function $\Gamma(\zeta)$ and incomplete beta function $\beta(\zeta,\eta,r)$ are defined as*

$$\Gamma(\zeta) = \int_0^\infty \varphi^{\zeta-1} e^{-\varphi}\, d\varphi$$

and

$$\beta(\zeta,\eta,r) = \int_0^r \varphi^{\zeta-1}(1-\varphi)^{\eta-1}\, d\varphi,$$

respectively.

The Riemann-Liouville fractional integrals [38] are attributed to mathematicians Bernhard Riemann and Joseph Liouville while Liouville first explored the concept of fractional calculus, Riemann's work significantly developed the integral operator that is widely used today as the standard definition of a fractional integral.

Definition 2. *For $h \in L^1[\nu,\omega]$, the Riemann-Liouville integrals $j^{\alpha}_{\nu+}h(\zeta)$ and $j^{\alpha}_{\omega-}h(\zeta)$ of order $\alpha > 0$ are given as*

$$j^{\alpha}_{\nu+}h(\zeta) = \frac{1}{\Gamma\alpha}\int_\nu^\zeta (\zeta-\varphi)^{\alpha-1}h(\varphi)d\varphi, \zeta > \nu, \tag{1}$$

$$j^{\alpha}_{\omega-}h(\zeta) = \frac{1}{\Gamma\alpha}\int_\zeta^\omega (\varphi-\zeta)^{\alpha-1}h(\varphi)d\varphi, \zeta < \omega, \tag{2}$$

It is evident that setting $\alpha = 1$ causes the Riemann-Liouville integrals reduce to the standard integrals.

Fractional conformable operators [41] were defined by Khalil et al. in 2014. Later extended conformable fractional operators [39] were defined which extend some new concepts to fractional calculus.

Definition 3. *For $h \in L^1[\nu,\omega]$, the extended fractional conformable integrals ${}^{\alpha}_{k}j^{\mu}_{\nu+}h(\zeta)$ and ${}^{\alpha}_{k}j^{\mu}_{\omega-}h(\zeta)$ of order $\alpha \in \mathcal{C}, Re(\alpha) > 0$, $k > 0$ and $\mu \in (0,1]$ are given by*

$$ {}^{\alpha}_{k}j^{\mu}_{\nu+}h(\zeta) = \frac{1}{k\Gamma_k(\alpha)} \int_{\nu}^{\zeta} \left(\frac{(\zeta-\nu)^{\mu} - (\varphi-\nu)^{\mu}}{\mu} \right)^{\frac{\alpha}{k}-1} \frac{h(\varphi)}{(\varphi-\nu)^{1-\mu}} d\varphi, \zeta > \nu \tag{3} $$

and

$$ {}^{\alpha}_{k}j^{\mu}_{\omega-}h(\zeta) = \frac{1}{k\Gamma_k(\alpha)} \int_{\zeta}^{\omega} \left(\frac{(\omega-\zeta)^{\mu} - (\omega-\varphi)^{\mu}}{\mu} \right)^{\frac{\alpha}{k}-1} \frac{h(\varphi)}{(\omega-\varphi)^{1-\mu}} d\varphi, \omega > \zeta, \tag{4} $$

where $\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})$.

It can be observed that if $k = 1$, equations (3) and (4) yield the ordinary conformable operator. Additionally, from usual observations, it can be concluded that when $\mu = 1$ and $k = 1$, equations (3) and (4) coincide with (1) and (2) respectively.

Now, we proceed to define well documented concept of convex function [42] and s-convex function in second sense [43] as fellows.

Definition 4. *A function $h : [a,b] \to \mathbb{R}$ is called convex if following inequality holds for all $x, y \in [a,b]$ and $\rho \in [0,1]$*

$$ h(\rho x + (1-\rho)y) \leq \rho h(x) + (1-\rho)h(y). $$

Definition 5. *A function $h : [0,\omega] \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality*

$$ h(\rho x + (1-\rho)y) \leq \rho^s h(x) + (1-\rho)^s h(y), $$

holds for all $x, y \in [0,\omega]$ and ρ, $s \in [0,1]$. This class is usually denoted by K_s^2.

Let us review Hölder's and power mean inequalities [37] as follows:

Definition 6. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $|h|^p$, $|\phi|^q \in L[\nu,\omega]$ are real functions defined on $[\nu,\omega]$, then*

$$ \int_{\nu}^{\omega} |h(x)\phi(x)|dx \leq \left(\int_{\nu}^{\omega} |h(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\nu}^{\omega} |\phi(x)|^q dx \right)^{\frac{1}{q}}. $$

Definition 7. *For $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $h(x)$, $\phi(x)$ are real functions defined on $[\nu,\omega]$ such that $|h|^p$,$|\phi|^q \in L[\nu,\omega]$,then*

$$ \int_{\nu}^{\omega} |h(x)\phi(x)|dx \leq \left(\int_{\nu}^{\omega} |h(x)|dx \right)^{1-\frac{1}{q}} \left(\int_{\nu}^{\omega} |h(x)||\phi(x)|^q dx \right)^{\frac{1}{q}}. $$

3. Trapezoidal Type Inequalities Based on Extended Conformable Fractional Operators

In this section, we derive trapezoidal-type inequalities for twice-differentiable functions. By employing extended conformable fractional operators, we obtain these inequalities. These results extend classical inequalities by incorporating the flexibility and generality of fractional calculus. To establish the trapezoidal-type inequalities for extended conformable fractional operators, we consider the following lemma, which forms the foundation for our subsequent analysis and results.

Lemma 1. *Consider a function $h : [\nu, \omega] \to \mathbb{R}$ that is twice differentiable on (ν, ω) and satisfies $h'' \in L^1([\nu, \omega])$. In this context, the following equality is established*

$$\begin{aligned}
&\frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1) \\
&\times \left({}^{\alpha}_{k}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}^{\alpha}_{k}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) \\
&= \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3} \Bigg[\int_0^1 \left[\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right] \\
&\times h''\left(\frac{\eta-t}{\eta}\nu + \frac{t}{\eta}\omega \right) dt \\
&+ \int_0^1 \left[\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right] h''\left(\frac{t}{\eta}\nu + \frac{\eta-t}{\eta}\omega \right) dt \Bigg],
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\phi_k(\mu,\alpha) =& \frac{\omega-\nu}{\eta^2}(k\mu)^{\frac{\alpha}{k}} \left[h'\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega \right) - h'\left(\frac{\nu}{\eta} + \frac{\eta-1}{\eta}\omega \right) \right] \\
&\times \left[\int_0^1 \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right].
\end{aligned}$$

Proof. By considering the integral

$$I_1 = \int_0^1 \left[\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right] h''\left(\frac{\eta-t}{\eta}\nu + \frac{t}{\eta}\omega \right) dt.$$

Using the technique of integration by parts, it yields

$$\begin{aligned}
I_1 =& \frac{\eta}{\omega-\nu} h'\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega \right) \int_0^1 \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \\
&- \frac{\eta}{\omega-\nu} \int_0^1 \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-t)^{\mu}}{k\mu} \right)^{\frac{\alpha}{k}} \right) h'\left(\frac{\eta-t}{\eta}\nu + \frac{t}{\eta}\omega \right) dt.
\end{aligned}$$

Again, we apply the technique of integration by parts to the second term; therefore, we obtain the following expression

$$I_1 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2 h(\nu)\frac{1}{(ku)^{\frac{\alpha}{k}}}-\left(\frac{\eta}{\omega-\nu}\right)^2\frac{1}{k}\left(\frac{\alpha}{k}\right)\int_0^1\left[\frac{1-(1-t)^{\mu}}{k\mu}\right]^{\frac{\alpha}{k}-1}(1-t)^{\mu-1}$$
$$\times h\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt, \tag{6}$$

Substituting $x=\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega$, *we can write*

$$I_1 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2 h(\nu)\frac{1}{(ku)^{\frac{\alpha}{k}}}-\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\left(\frac{\Gamma(\frac{\alpha}{k}+1)}{k.k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})}\right)$$
$$\times\int_{\nu}^{\frac{(\eta-1)\nu+\omega}{\eta}}\left[\frac{\left(\frac{\omega-\nu}{\eta}\right)^{\mu}-\left(\frac{(\eta-1)\nu+\omega}{\eta}-x\right)^{\mu}}{\mu}\right]^{\frac{\alpha}{k}-1}\left[\frac{(\eta-1)\nu+\omega}{\eta}-x\right]^{\mu-1}dx.$$

By using the relation $\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})$, *we can write*

$$I_1 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2 h(\nu)\frac{1}{(ku)^{\frac{\alpha}{k}}}-\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\Gamma(\frac{\alpha}{k}+1).\left(\frac{1}{k\Gamma_k\alpha}\right)$$
$$\times\int_{\nu}^{\frac{(\eta-1)\nu+\omega}{\eta}}\left[\frac{\left(\frac{(\eta-1)\nu+\omega}{\eta}-\nu\right)^{\mu}-\left(\frac{(\eta-1)\nu+\omega}{\eta}-x\right)^{\mu}}{\mu}\right]^{\frac{\alpha}{k}-1}\left(\frac{h(x)dx}{\left[\frac{(\eta-1)\nu+\omega}{\eta}-x\right]^{1-\mu}}\right).$$

By using the definition of extended conformable operator (4), *we obtain*

$$I_1 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2 h(\nu)\frac{1}{(ku)^{\frac{\alpha}{k}}}-\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\Gamma\Big(\frac{\alpha}{k}+1\Big)\ {}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega)}{\eta}-}h(\nu). \tag{7}$$

Similarly

$$I_2 = \int_0^1\left[\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right]h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)dt,$$

$$I_2 = -\frac{\eta}{\omega-\nu}h'\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\int_0^1\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2 h(\omega)\frac{1}{(ku)^{\frac{\alpha}{k}}}-\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\Gamma\Big(\frac{\alpha}{k}+1\Big)_k^{\alpha} j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega). \tag{8}$$

Adding equations (7) and (8), we get the required result.

Remark 1. *If we substitute k=1 and $\eta = 2$ in Lemma 5, then obtained result leads to [44, Lemma 3].*

Theorem 1. *Consider $h : [\nu, \omega] \to \mathbb{R}$ as a twice differentiable mapping on (ν, ω) such that $h'' \in L([\nu, \omega])$. If $|h''|$ is s-convex in second sense on $[\nu, \omega]$ then,*

$$\Bigg|\frac{h(\nu)+h(\omega)}{\eta}+\phi_k(\mu,\alpha)-\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{K}}\Gamma(\frac{\alpha}{k}+1)$$
$$\times\left({}_k^{\alpha}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_k^{\alpha}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\Bigg| \le \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}\left(\frac{|h''(\nu)|+|h''(\omega)|}{\eta^s}\right)$$
$$\times\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|(t^s+(\eta-t)^s)\,dt. \tag{9}$$

Proof. Taking absolute value of equation (5), we have

$$\Bigg|\frac{h(\nu)+h(\omega)}{\eta}+\phi_k(\mu,\alpha)-\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1)$$
$$\times\left({}_k^{\alpha}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_k^{\alpha}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\Bigg| \le \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}$$
$$\times\Bigg[\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|dt$$
$$+\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-(\frac{1-(1-\varphi)^{\mu}}{k\mu})^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)\right|dt\Bigg]. \tag{10}$$

By using s-convexity of $|h''|$ in second sense, we get

$$\Bigg|\frac{h(\nu)+h(\omega)}{\eta}+\phi_k(\mu,\alpha)-\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1)$$
$$\times\left({}_k^{\alpha}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_k^{\alpha}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\Bigg| \le \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}$$
$$\times\Bigg[\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left(\left(\frac{t}{\eta}\right)^s|h''(\omega)|+\left(\frac{\eta-t}{\eta}\right)^s|h''(\nu)|\right)\,dt$$

$$+\int_0^1 \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - (\frac{1-(1-\varphi)^\mu}{k\mu})^{\frac{\alpha}{k}} \right) d\varphi \right| \left(\left(\frac{t}{\eta}\right)^s |h''(\nu)| + \left(\frac{\eta-t}{\eta}\right)^s |h''(\omega)| \right) dt \Bigg]$$

$$\Bigg| \frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}} (k\mu)^{\frac{\alpha}{k}} \Gamma(\frac{\alpha}{k}+1)$$
$$\times \left({}_k^\alpha J^\mu_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) +_k^\alpha J^\mu_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) \Bigg| \leq \frac{(\omega-\nu)^2 (k\mu)^{\frac{\alpha}{k}}}{\eta^3}$$
$$\times \left(\frac{|h''(\nu)|+|h''(\omega)|}{\eta^s} \right) \int_0^1 \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right|$$
$$\times (t^s + (\eta-t)^s)\, dt.$$

Which lead us to our required result.

Remark 2. *If we set $\eta = 2$,k=1 and s=1 in* (9)*, then inequality reduced to*

$$\left| \frac{h(\nu)+h(\omega)}{2} - \frac{2^{\mu\alpha}-1}{(\omega-\nu)^{\mu\alpha}} (\mu)^\alpha \Gamma(\alpha+1) \left({}^\alpha J^\mu_{\frac{\nu+\omega}{2}-} h(\nu) +^\alpha J^\mu_{\frac{\nu+\omega}{2}+} h(\omega) \right) \right|$$
$$\leq \frac{(\omega-\nu)^2 \mu^\alpha}{8} (|h''(\nu)|+|h''(\omega)|) \int_0^1 \left| \int_0^t \left(\frac{1}{\mu^\alpha} - \left(\frac{1-(1-\varphi)^\mu}{\mu} \right)^\alpha \right) d\varphi \right| dt. \tag{11}$$

Remark 3. *When we substitute μ=1 in* (11)*, then Theorem 1 leads to [45], corollary 7.*

Remark 4. *By setting $\eta = 2$, $k = 1$, $s = 1$, $\mu = 1$ and $\alpha = 1$ in Theorem 1 then inequality reduced to [46], Proposition 2.*

Example 1. *This example illustrates the verification of the inequality stated in Theorem 1 using graphs and a table. To achieve this, consider the function $h(x) = x^6 + 2x^4$ defined over the interval $h : [2, 7] \to \mathbb{R}$. The parameters are chosen as $k = 3$, $\alpha = 4$,s=1, and $\eta = 8$.*

Explanation:

A 2D plot is created to observe the function's behavior across the interval $\mu \in (0, 1]$ shown in Fig. 1, providing a visual way to analyze the inequality (9)*. To check the validity of the theorem, we proceed to calculate numerical values of the inequality for different μ and k values, which confirm our results. These results are compiled into a table shown in Tables 1 and 2 highlighting the precision and reliability of the inequality.*

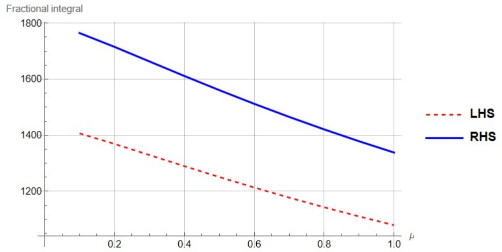

Figure 1: This figure illustrates the graphical representation of (9), corresponding to $\mu \in (0, 1]$ validating our results.

Table 1: In this figure summary of values confirming (9) corresponding to $\mu \in (0, 1]$.

μ	0.2	0.4	0.6	0.8	1
LHS	1368.99	1289.62	1213.3	1142.93	1078.93
RHS	1715.36	1611.45	1512.02	1420.78	1338.12

Table 2: Summary of values confirming (9) for $k \in [1, 5]$, while keeping $\mu = .5$ fixed.

k	1	2	3	4	5
LHS	1457.94	1365.57	1258.87	1157.15	1065.55
RHS	1775.59	1681.23	1560.81	1441.05	1331.05

To extend the analysis, a 3D plot is generated to evaluate the inequality for parameter ranges $\alpha \in [5, 10]$ and $\mu \in (0, 1]$. Figure 2 displays this 3D visualization, offering further support for the theorem.

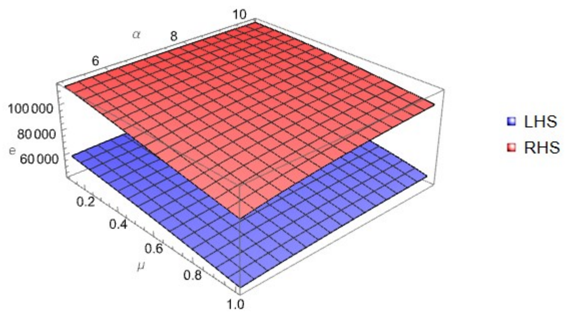

Figure 2: In Fig. 2 three dimensional visualization validating the inequality of Theorem 1 corresponding to $\mu \in (0, 1]$ and $\alpha \in [5, 10]$.

The combination of these methods confirms the robustness and applicability of Theorem 1 in describing the behavior of $h(x)$ under the given conditions.

Theorem 2. *Assume that $h : [\nu, \omega] \to \mathbb{R}$ is a twice differentiable function on (ν, ω) such that $h'' \in L_p([\nu, \omega])$ with $\nu < \omega$. Let $|h''|^q$ be s-convex in second sense on $[\nu, \omega]$ with $q > 1$ then inequality is given by,*

$$\begin{aligned}
&\left|\frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\left(\frac{\alpha}{k}+1\right)\right. \\
&\times \left.\left({}_k^\alpha J^\mu_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}_k^\alpha J^\mu_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega)\right)\right| \le \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3} \\
&\times \left(\int_0^1 \left|\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right) d\varphi\right|^p dt\right)^{\frac{1}{p}} \\
&\times \left(\left[\left(\frac{|h''(\omega)|^q}{\eta^s(s+1)} - \frac{|h''(\nu)|^q}{\eta^s(s+1)}\right)\left((\eta-1)^{s+1} - \eta^{s+1}\right)\right]^{\frac{1}{q}}\right. \\
&+ \left.\left[\left(\frac{|h''(\nu)|^q}{\eta^s(s+1)} - \frac{|h''(\omega)|^q}{\eta^s(s+1)}\right)\left((\eta-1)^{s+1} - \eta^{s+1}\right)\right]^{\frac{1}{q}}\right). \qquad (12)
\end{aligned}$$

Proof. By employing Hölder inequality on (10)*, we get*

$$\begin{aligned}
&\left|\frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{K}}\Gamma(\frac{\alpha}{k}+1)\right. \\
&\times \left.\left({}_k^\alpha J^\mu_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}_k^\alpha J^\mu_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega)\right)\right| \le \frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3} \\
&\times \left[\left(\int_0^1 \left|\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right) d\varphi\right|^p dt\right)^{\frac{1}{p}}\right. \\
&\times \left(\int_0^1 \left|h''\left(\left(\frac{\eta-t}{\eta}\right)\nu + \frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right) d\varphi\right|^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left|h''\left(\frac{t}{\eta}\nu + \left(\frac{\eta-t}{\eta}\right)\omega\right)\right|^q dt\right)^{\frac{1}{q}}. \qquad (13)
\end{aligned}$$

By considering s-convexity in second sense of $|h''(x)|^q$, then

$$\begin{aligned}
&\int_0^1 \left|h''\left(\left(\frac{\eta-t}{\eta}\right)\nu + \frac{t}{\eta}\omega\right)\right|^q dt \le \int_0^1 \left[\left(\frac{\eta-t}{\eta}\right)^s |h''(\nu)|^q + |h''(\omega)|^q\left(\frac{t}{\eta}\right)^s\right] dt \\
&\le \frac{|h''(\omega)|^q}{\eta^s(s+1)} - \frac{|h''(\nu)|^q}{\eta^s(s+1)}\left[(\eta-1)^{s+1} - \eta^{s+1}\right]. \qquad (14)
\end{aligned}$$

Similarly

$$\int_0^1 \left|h''\left(\frac{t}{\eta}\nu + \left(1-\frac{t}{\eta}\right)\omega\right)\right|^q dt \le \frac{|h''(\nu)|^q}{\eta^s(s+1)} - \frac{|h''(\omega)|^q}{\eta^s(s+1)}\left[(\eta-1)^{s+1} - \eta^{s+1}\right]. \qquad (15)$$

Substituting both (14) *and* (15) *in* (13)*, we conclude*

$$\Bigg|\frac{h(\nu)+h(\omega)}{\eta}+\phi_k(\mu,\alpha)-\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\left(\frac{\alpha}{k}+1\right)$$
$$\times\left({}_k^{\alpha}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_k^{\alpha}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\Bigg|\le\frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}$$
$$\times\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\Bigg(\left[\left(\frac{|h''(\omega)|^q}{\eta^s(s+1)}-\frac{|h''(\nu)|^q}{\eta^s(s+1)}\right)\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right]^{\frac{1}{q}}$$
$$+\left[\left(\frac{|h''(\nu)|^q}{\eta^s(s+1)}-\frac{|h''(\omega)|^q}{\eta^s(s+1)}\right)\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right]^{\frac{1}{q}}\Bigg).$$

So inequality is proved.

Remark 5. *If we substitute* $\eta=2,k=1$ *and s=1 in equation* (12)*, then following result is obtained*

$$\left|\frac{h(\nu)+h(\omega)}{2}-\frac{2^{\mu\alpha-1}}{(\omega-\nu)^{\mu\alpha}}\mu^{\alpha}\Gamma(\alpha+1)\left({}^{\alpha}J^{\mu}_{\frac{\nu+\omega}{2}-}h(\nu)+{}^{\alpha}J^{\mu}_{\frac{\nu+\omega}{2}+}h(\omega)\right)\right|$$
$$\le\frac{(\omega-\nu)^2\mu^{\alpha}}{8}\left(\int_0^1\left|\int_0^t\left(\frac{1}{(\mu)^{\alpha}}-\left(\frac{1-(1-\varphi)^{\mu}}{\mu}\right)^{\alpha}\right)d\varphi\right|^p dt\right)^{\frac{1}{p}}$$
$$\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{16}$$

Corollary 1. *If we consider* $\mu=\alpha=1$ *in* (16) *we arrive at the conclusion that is*

$$\left|\frac{h(\nu)+h(\omega)}{2}-\frac{1}{(\omega-\nu)}\int_{\nu}^{\omega}h(x)dx\right|\le\frac{(\omega-\nu)^2}{8}\left(\frac{1}{p+1}-\frac{1}{2^p(2p+1)}\right)^{\frac{1}{p}}$$
$$\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Proof. *Substituting value* $\mu=\alpha=1$ *in* (16)*,*

$$\left|\frac{h(\nu)+h(\omega)}{2}-\frac{1}{(\omega-\nu)}\Gamma(\alpha+1)\int_{\nu}^{\omega}h(x)dx\right|\le\frac{(\omega-\nu)^2\mu^{\alpha}}{8}\left(\int_0^1\left|\int_0^t(1-\varphi)\,d\varphi\right|^p dt\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{17}$$

Consider

$$\left(\int_0^1\left|\int_0^t(1-\varphi)d\varphi\right|^p dt\right)^{\frac{1}{p}}=\left(\int_0^1\left|t-\frac{t^2}{2}\right|^p dt\right)^{\frac{1}{p}}.$$

Under condition $|A-B|^p \leq A^p - B^p$ *when* $A > B > 0$; *and* $p > 1$ *so we can write*

$$\left(\int_0^1 \left|\int_0^t (1-\varphi)d\varphi\right|^p dt\right)^{\frac{1}{p}} \leq \left(\int_0^1 t^p dt - \int_0^1 \frac{t^{2p}}{2^p} dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1 \left|\int_0^t (1-\varphi)d\varphi\right|^p dt\right)^{\frac{1}{p}} \leq \left[\frac{1}{p+1} - \frac{1}{2^p(2p+1)}\right]^{\frac{1}{p}}. \tag{18}$$

Substituting (18) *in* (17), *we get our desired result.*

Example 2. *This example illustrates the applicability of Theorem 2 through graphical and numerical analyses. We consider the function* $h(x) = x^6 + 2x^4$, *defined on the interval* $[2, 7]$, *and verify the inequality using specific parameter values:* $k = 3$, $\alpha = 4$, $s = 1$, $\frac{1}{p} = .6, \frac{1}{q} = .4$, *and* $\eta = 8$.
Explanation:

Figure 3 presents a 2D graph of the inequality over $\mu \in (0, 1]$, *showing the behavior of both the left-hand side and the right-hand side of inequality* (12). *The plot demonstrates that the L.H.S remains within the bounds prescribed by the R.H.S, visually validating the theorem. A numerical analysis is performed by computing the L.H.S. and R.H.S.*

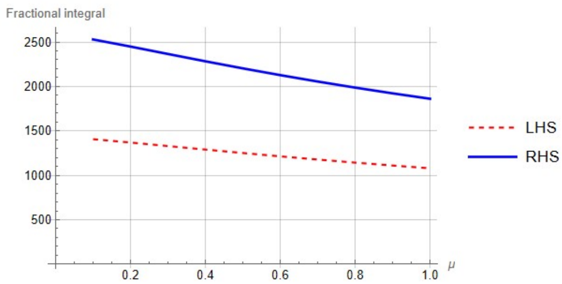

Figure 3: 2D plot for $\mu \in (0, 1]$ of Theorem 2 supporting the validation of theorem.

values for several instances of μ *with* k *fixed, as shown in Table 3. Additionally, by fixing* $\mu = 0.5$, *a table is constructed for different values of* k, *as shown in Table 4. This tabular representation is highlighting the accuracy and consistency of the inequality.*

Table 3: Summary of inequality (12) confirms its validity for $\mu \in (0, 1]$.

μ	0.2	0.4	0.6	0.8	1
LHS	1368.99	1289.62	1213.3	1142.93	1078.93
RHS	2448.47	2282.73	2203.15	1986.66	1861.13

Table 4: Summary of inequality (12) confirms its validity for $k \in [1,5]$ while fixing $\mu = .5$.

μ	1	2	3	4	5
LHS	1415.4	1343.89	1250.8	1157.15	1070.49
RHS	2543.26	2389.52	2203.15	2023.35	1861.33

To extend the analysis, we explore the inequality in three dimensions by varying the parameters $\alpha \in [5, 10]$ and $\mu \in (0, 1]$. Figure 4 illustrates the resulting surface, further confirming the robustness of the inequality across a range of parameter values. These

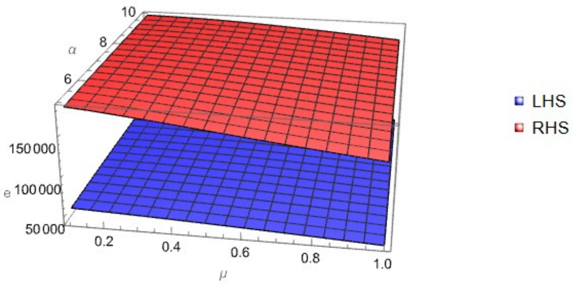

Figure 4: 3D representation verifying the inequality in Theorem 2.

combined analyses establish the reliability and practical utility of Theorem 2 in bounding the behavior of $h(x)$ under the given conditions.

Theorem 3. *Consider $h{:}[\nu,\omega] \to \mathbb{R}$ as a twice differentiable mapping on (ν,ω) such that $h'' \in L_q([\nu,\omega])$. Assume that $|h''|$ admits the s-convexity in second sense on $[\nu,\omega]$ with $q \geq 1$ then,*

$$\begin{aligned}
&\Bigg|\frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1) \\
&\times \left({}_k^{\alpha}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}_k^{\alpha}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega)\right)\Bigg| \leq \frac{(\omega-\nu)^2 (k\mu)^{\frac{\alpha}{k}}}{\eta^3} \\
&\times \left(\int_0^1 \left|\int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}\right) d\varphi\right| dt\right)^{1-\frac{1}{q}} \\
&\times \left(\left[\frac{|h''(v)|^q \Psi_2(\mu,\alpha) + |h''(w)|^q \Psi_1(\mu,\alpha)}{n^s}\right]^{\frac{1}{q}}\right. \\
&+ \left.\left[\frac{|h''(v)|^q \Psi_1(\mu,\alpha) + |h''(w)|^q \Psi_2(\mu,\alpha)}{n^s}\right]^{\frac{1}{q}}\right),
\end{aligned} \tag{19}$$

where

$$\Psi_1(\mu,\alpha)=\int_0^1 t^s\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt,$$
$$\Psi_2(\mu,\alpha)=\int_0^1 (\eta-t)^s\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt.$$

Proof. By employing power mean inequality on equation (10)*, we obtain*

$$\begin{aligned}&\Bigg|\frac{h(\nu)+h(\omega)}{\eta}+\phi_k(\mu,\alpha)-\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1)\\
&\times\left({}_k^\alpha J^\mu_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_k^\alpha J^\mu_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\Bigg|\leq\frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}\\
&\times\Bigg[\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-(\frac{1-(1-\varphi)^\mu}{k\mu})^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg].\end{aligned}\tag{20}$$

By taking advantage of s-convexity of $|h''|$ *on the following terms, we proceed as*

$$\begin{aligned}&\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)\\
&\leq\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left(\left(\frac{\eta-t}{\eta}\right)^s|h''(\nu)|^q+\left(\frac{t}{\eta}\right)^s|h''(\omega)|^q\right)dt\end{aligned}$$

$$\begin{aligned}&\left(\int_0^1\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)\\
&\leq\frac{|h''(\nu)|^q}{\eta^s}\left(\int_0^1(\eta-t)^s\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt\right)\\
&+\frac{|h''(\omega)|^q}{\eta^s}\left(\int_0^1t^s\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt\right).\end{aligned}\tag{21}$$

Assuming

$$\Psi_1(\mu,\alpha)=\int_0^1 t^s\left|\int_0^t\left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}}-\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}\right)d\varphi\right|dt,\tag{22}$$

$$\Psi_2(\mu,\alpha) = \int_0^1 (\eta - t)^s \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right| dt. \tag{23}$$

Substituting equations (22) *and* (23) *in* (21)

$$\left(\int_0^1 \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right| \left| h'' \left(\frac{\eta - t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt \right)$$
$$\leq \left(\frac{|h''(\nu)|^q}{\eta^s} \Psi_2(\mu,\alpha) + \frac{|h''(\omega)|^q}{\eta^s} \Psi_1(\mu,\alpha) \right). \tag{24}$$

Similarly

$$\left(\int_0^1 \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right| \left| h'' \left(\frac{\eta - t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt \right)$$
$$\leq \left(\frac{|h''(\nu)|^q}{\eta^s} \Psi_1(\mu,\alpha) + \frac{|h''(\omega)|^q}{\eta^s} \Psi_2(\mu,\alpha) \right). \tag{25}$$

Substituting (24) *and* (25) *in* (20)

$$\left| \frac{h(\nu)+h(\omega)}{\eta} + \phi_k(\mu,\alpha) - \frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}} (k\mu)^{\frac{\alpha}{k}} \Gamma(\frac{\alpha}{k}+1) \right.$$
$$\left. \times \left({}^{\alpha}_{k}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) +^{\alpha}_{k} J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) \right| \leq \frac{(\omega-\nu)^2 (k\mu)^{\frac{\alpha}{k}}}{\eta^3}$$
$$\times \left(\int_0^1 \left| \int_0^t \left(\frac{1}{(k\mu)^{\frac{\alpha}{k}}} - \left(\frac{1-(1-\varphi)^\mu}{k\mu} \right)^{\frac{\alpha}{k}} \right) d\varphi \right| dt \right)^{1-\frac{1}{q}}$$
$$\times \left(\left[\frac{|h''(v)|^q \Psi_2(\mu,\alpha) + |h''(w)|^q \Psi_1(\mu,\alpha)}{n^s} \right]^{\frac{1}{q}} + \left[\frac{|h''(v)|^q \Psi_1(\mu,\alpha) + |h''(w)|^q \Psi_2(\mu,\alpha)}{n^s} \right]^{\frac{1}{q}} \right). \tag{26}$$

Consequently, we obtain the desired outcome.

Corollary 2. *If we substitute* $k = 1, s = 1$ *and* $\mu = 1$ *in* (26)*, then*

$$\left| \frac{h(\nu)+h(\omega)}{\eta} + \phi_1(1,\alpha) - \frac{\eta^{\alpha-1}}{(\omega-\nu)^{\mu\alpha}} \Gamma(\alpha+1) \left({}^{\alpha}J_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) +^{\alpha} J_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) \right|$$
$$\leq \frac{(\omega-\nu)^2}{\eta^3} \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+3)} \right)^{1-\frac{1}{q}}$$
$$\times \left[\left(\left(\eta \left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)} \right) - \left(\frac{1}{3} - \frac{1}{(\alpha+1)(\alpha+3)} \right) \right) \frac{|h''(\nu)|^q}{\eta} \right. \right.$$
$$+ \left. \left(\frac{1}{3} - \frac{1}{(\alpha+1)(\alpha+3)} \right) \frac{|h''(\omega)|^q}{\eta} \right)^{\frac{1}{q}} + \left(\left(\frac{1}{3} - \frac{1}{(\alpha+1)(\alpha+3)} \right) \frac{|h''(\nu)|^q}{\eta} \right.$$

$$+\left(\eta\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)-\left(\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)}\right)\right)\frac{|h''(\omega)|^q}{\eta}\right)^{\frac{1}{q}}\Bigg]. \tag{27}$$

Proof. Setting $k=\mu=1$ in (22),

$$\Psi_1(1,\alpha)=\int_0^1\left|\int_0^t\left(1-\varphi^\alpha\right)d\varphi\right|dt$$
$$\Psi_1(1,\alpha)=\int_0^1|t-\frac{t^{\alpha+1}}{\alpha+1}|dt$$
$$When\ A>B\,then\,|A-B|=A-B$$
$$\Psi_1(1,\alpha)=\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)} \tag{28}$$
$$Similarly$$
$$\Psi_2(1,\alpha)=\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)} \tag{29}$$

Substituting (28) and (29) in (26),

$$\left|\frac{h(\nu)+h(\omega)}{\eta}+\phi_1(1,\alpha)-\frac{\eta^{\alpha-1}}{(\omega-\nu)^\alpha}\Gamma(\alpha+1)\left({}^\alpha J_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^\alpha J_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{\eta^3}\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+3)}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg[\left(\left(\eta\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)-\left(\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)}\right)\right)\frac{|h''(\nu)|^q}{\eta}\right.$$
$$\left.+\left(\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)}\right)\frac{|h''(\omega)|^q}{\eta}\right)^{\frac{1}{q}}+\left(\left(\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)}\right)\frac{|h''(\nu)|^q}{\eta}\right.$$
$$\left.+\left(\eta\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)-\left(\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)}\right)\right)\frac{|h''(\omega)|^q}{\eta}\right)^{\frac{1}{q}}\Bigg]. \tag{30}$$

Ultimately, the expected outcome has been reached.

Remark 6. *If we assign $\eta=2$ and $\alpha=1$ in (27), our result reduced as follow*

$$\left|\frac{h(\nu)+h(\omega)}{2}-\frac{1}{\omega-\nu}\int_\nu^\omega h(x)dx\right|\leq\frac{(\omega-\nu)^2}{24}\left[\left(\frac{11}{16}|h''(\nu)|^q+\frac{5}{16}|h''(\omega)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5}{16}|h''(\nu)|^q+\frac{11}{16}|h''(\omega)|^q\right)^{\frac{1}{q}}\right].$$

Example 3. *This example demonstrates the application of Theorem 3 through both graphical and numerical methods. We analyze the function $h(x)=x^6+2x^4$ within the interval $[2,7]$ to verify the inequality using the following parameter values: $k=3$, $\alpha=4$, $\frac{1}{p}=0.6$,*

$\frac{1}{q} = 0.4$, *and* $\eta = 8$.

Explanation:

Figure 5 displays a two-dimensional plot illustrating the inequality for $\mu \in (0, 1]$. *The graph of inequality* (19) *is clearly showing that the LHS consistently stays within the bounds set by the RHS. Numerical calculations are performed to compare the inequality at various values of* μ. *These results are summarized in the table provided in Fig. 5, highlighting the precision and consistency of the inequality.*

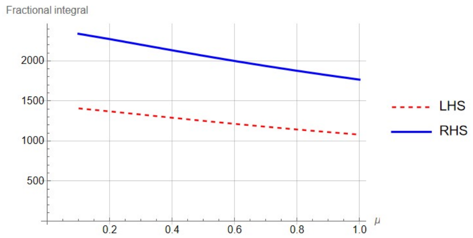

Figure 5: This figure provides a visualization of Theorem 3 using a 2D graph confirming its validity.

Table 5: Table shown the summary of equation (19) corresponding to $\mu \in (0, 1]$ as evidence of inequality.

μ	0.2	0.4	0.6	0.8	1
LHS	1368.99	1289.62	1213.3	1142.93	1078.93
RHS	2271.79	2132.02	1998.51	1876.17	1765.5

To further validate the results, the inequality is analyzed in three dimensions by varying $\alpha \in [5, 10]$ *and* $\mu \in (0, 1]$. *Figure 6 shows a 3D surface plot, confirming the robustness of the inequality across a broader parameter range.*

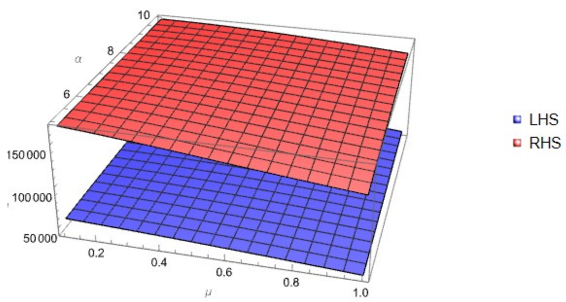

Figure 6: 3D visualization validating the inequality in Theorem 3.

Together, these analyses reinforce the accuracy and practical relevance of Theorem 3 in characterizing the behavior of $h(x)$ *under the specified conditions.*

4. Mid Point Type Inequalities Based on Extended Conformable Fractional Operators

In this section, we delve into the study of midpoint inequalities derived from twice-differentiable functions. By employing extended conformable fractional operators and twice-differentiable operators, we aim to establish new insights and results in this area. Our approach begins with the formulation of a fundamental identity, which serves as the foundation for deriving the desired inequalities.By leveraging the capabilities of extended conformable fractional operators, we extend the applicability of midpoint inequalities to a broader class of functions, offering a more comprehensive understanding of their behavior and potential applications. The results obtained in this study not only contribute to the theoretical framework but also pave the way for future research in related domains.

Lemma 2. *Let $h : [\nu,\omega] \to \mathbb{R}$ be a twice differentiable mapping on (ν,ω) with $h'' \in L_1([\nu,\omega])$. Then the following equality holds:*

$$\begin{aligned}
&\frac{\eta^{\frac{\mu\alpha}{k}-1}}{(\omega-\nu)^{\frac{\mu\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma(\frac{\alpha}{k}+1)\left({}^{\alpha}_{k}J^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}J^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]\\
&-\frac{1}{\eta}\left[h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right]\\
&=\frac{(\omega-\nu)^2(k\mu)^{\frac{\alpha}{k}}}{\eta^3}\Bigg[\int_0^1\left[\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right]h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt\\
&+\int_0^1\left[\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right]h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)dt\Bigg], \qquad (31)
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon_k[\mu,\alpha]&=\frac{(\omega-\nu)}{\eta^2}(k\mu)^{\frac{\alpha}{k}}\left(h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)-h'\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\\
&\times\left[\int_0^1\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right].
\end{aligned}$$

Proof. Consider

$$I_3=\int_0^1\left[\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right]h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt,$$

Utilizing the technique of integration by parts, we obtain

$$\begin{aligned}
I_3&=\frac{\eta}{\omega-\nu}h'\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\Bigg|_0^1\\
&-\frac{\eta}{\omega-\nu}\int_0^1\left(\frac{1-(1-t)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}h'\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt,
\end{aligned}$$

$$I_3 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi$$
$$-\frac{\eta}{\omega-\nu}\int_0^1\left(\frac{1-(1-t)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}h'\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt,$$

Again employing integration by parts on second term

$$I_3 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi$$
$$-\left(\frac{\eta}{\omega-\nu}\right)^2\left(\frac{1-(1-t)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}h\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\Big|_0^1$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2\cdot\frac{1}{k}\left(\frac{\alpha}{k}\right)\int_0^1\left[\frac{1-(1-t)^\mu}{k\mu}\right]^{\frac{\alpha}{k}-1}(1-t)^{\mu-1}$$
$$\times h\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt$$

$$I_3 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi-\left(\frac{\eta}{\omega-\nu}\right)^2\left(\frac{1}{k\mu}\right)^{\frac{\alpha}{k}}h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^2\cdot\frac{1}{k}\frac{1}{k^{\frac{\alpha}{k}-1}}\left(\frac{\alpha}{k}\right)\int_0^1\left[\frac{1-(1-t)^\mu}{k\mu}\right]^{\frac{\alpha}{k}-1}(1-t)^{\mu-1}$$
$$\times h\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)dt.$$

Substituting $x=\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega$, *we can write*

$$I_3 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi$$
$$-\left(\frac{m}{\omega-\nu}\right)^2\left(\frac{1}{k\mu}\right)^{\frac{\alpha}{k}}h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\left(\frac{\Gamma(\frac{\alpha}{k}+1)}{k.k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k})}\right)$$
$$\times\int_\nu^{\frac{(\eta-1)\nu+\omega}{\eta}}\left[\frac{\left(\frac{\omega-\nu}{\eta}\right)^\mu-\left(\frac{(\eta-1)\nu+\omega}{\eta}-x\right)^\mu}{\mu}\right]^{\frac{\alpha}{k}-1}\left[\frac{(\eta-1)\nu+\omega}{\eta}-x\right]^{\mu-1}h(x)dx.$$

By using relation $\Gamma_k\alpha = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right)$, *we can write*

$$I_3 = \frac{\eta}{\omega-\nu}h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi-\left(\frac{\eta}{\omega-\nu}\right)^2\left(\frac{1}{k\mu}\right)^{\frac{\alpha}{k}}h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)$$
$$+\left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2}\Gamma(\frac{\alpha}{k}+1)\left(\frac{1}{k\Gamma_k\alpha}\right)\int_\nu^{\frac{(\eta-1)\nu+\omega}{\eta}}\left[\frac{\left(\frac{(\eta-1)\nu+\omega}{\eta}-\nu\right)^\mu-\left(\frac{(\eta-1)\nu+\omega}{\eta}-x\right)^\mu}{\mu}\right]^{\frac{\alpha}{k}-1} \tag{32}$$

$$\times \left(\frac{h(\nu)dx}{\left[\frac{(\eta-1)\nu+\omega}{\eta} - x\right]^{1-\mu}} \right). \tag{33}$$

By using the definition of extended conformable operator (4)*, we can write*

$$\begin{aligned} I_3 &= \frac{\eta}{\omega-\nu} h'\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega\right) \int_0^1 \left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}} d\varphi \\ &- \left(\frac{\eta}{\omega-\nu}\right)^2 \left(\frac{1}{k\mu}\right)^{\frac{\alpha}{k}} h\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega\right) \\ &+ \left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2} \Gamma(\frac{\alpha}{k}+1) {}_k^{\alpha} j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu). \end{aligned} \tag{34}$$

Similarly

$$\begin{aligned} I_4 &= \int_0^1 \left[\int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}} d\varphi\right] h''\left(\frac{t}{\eta}\nu + \frac{\eta-t}{\eta}\omega\right) dt \\ I_4 &= -\frac{\eta}{\omega-\nu} h'\left(\frac{1}{\eta}\nu + \frac{\eta-1}{\eta}\omega\right) \int_0^1 \left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}} d\varphi \\ &- \left(\frac{\eta}{\omega-\nu}\right)^2 h\left(\frac{\nu}{\eta} + \frac{\eta-1}{\eta}\omega\right) \frac{1}{(ku)^{\frac{\alpha}{k}}} \\ &+ \left(\frac{\eta}{\omega-\nu}\right)^{\mu\frac{\alpha}{k}+2} \Gamma\Big(\frac{\alpha}{k}+1\Big) {}_k^{\alpha} j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega). \end{aligned} \tag{35}$$

Adding equations (34) *and* (35)*, we get our required result.*

Remark 7. *By considering $\eta = 2$ and k=1 simultaneously in equation* (31)*, that directly leads to [[44], Lemma 10].*

Theorem 4. *Assume that $h : [\nu, \omega] \to \mathbb{R}$ as a twice differentiable function on (ν, ω) such that $h'' \in L_1([\nu, \omega])$. By considering the convexity $|h''|$ on $[\nu, \omega]$ the inequality is given as,*

$$\begin{aligned} &\Bigg| \frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}} (k\mu)^{\frac{\alpha}{k}} \Gamma\Big(\frac{\alpha}{k}+1\Big) \left({}_k^{\alpha} j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}_k^{\alpha} j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) + \Upsilon_k[\mu,\alpha] \\ &- \frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega\right) + h\left(\frac{1}{\eta}\nu + \frac{\eta-1}{\eta}\omega\right)\right) \Bigg| \\ &\le \frac{(\omega-\nu)^2}{\eta^3} (k\mu)^{\frac{\alpha}{k}} \frac{1}{\mu} \int_0^1 \left| \frac{1}{(k\mu)^{\frac{\alpha}{k}}} B\left(\frac{\alpha}{k}+1, \frac{1}{\mu}, 1-(1-t)^{\mu}\right) \right| (t^s + (\eta-t)^s)\, dt \\ &\times \left(\frac{|h''(\nu)| + |h''(\omega)|}{\eta^s}\right), \end{aligned} \tag{36}$$

where

$$\Upsilon_k[\mu,\alpha] = \frac{(\omega-\nu)}{\eta^2}(k\mu)^{\frac{\alpha}{k}}\left(h'\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)-h'\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right) \times\left[\int_0^1\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right].$$

Proof. Taking absolute value on both sides of equation (31)

$$\begin{aligned}&\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\Big(\frac{\alpha}{k}+1\Big)\left({}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]\\&-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg|\\&\le\frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}\left[\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|dt\right.\\&\left.+\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|\left|h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)\right|dt\right].\end{aligned}\tag{37}$$

By considering s-convexity of $|h''|$ *in second sense*

$$\begin{aligned}&\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\Big(\frac{\alpha}{k}+1\Big)\left({}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]\\&-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg|\\&\le\frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}\left[\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|\left(\left(\frac{\eta-t}{\eta}\right)^s|h''(\nu)|+\left(\frac{t}{\eta}\right)^s|h''(\omega)|\right)dt\right.\\&\left.+\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|\left(\left(\frac{t}{\eta}\right)^s|h''(\nu)|+\left(\frac{\eta-t}{\eta}\right)^s|h''(\omega)|\right)dt\right]\end{aligned}$$

$$\begin{aligned}&\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\Big(\frac{\alpha}{k}+1\Big)\left({}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]\\&-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg|\end{aligned}\tag{38}$$

$$\begin{aligned}&\le\frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|(t^s+(\eta-t)^s)\,dt\\&\times\left(\frac{|h''(\nu)|+|h''(\omega)|}{\eta^s}\right).\end{aligned}\tag{39}$$

Consider

$$\varpi_k[\mu,\alpha]=\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|$$

Assuming

$$p = 1-(1-\varphi)^{\mu}$$
$$d\varphi = \frac{1}{\mu}(1-p)^{\frac{1}{\mu}-1}dp$$

$$\varpi_k[\mu,\alpha] = \frac{1}{\mu}\left|\frac{1}{(k\mu)^{\frac{\alpha}{k}}}\int_0^{1-(1-t)^{\mu}} p^{(\frac{\alpha}{k}+1)-1}(1-p)^{\frac{1}{\mu}-1}dp\right|$$

By using definition of incomplete beta function , we can write

$$\varpi_k[\mu,\alpha] = \frac{1}{\mu}\left|\frac{1}{(k\mu)^{\frac{\alpha}{k}}}B\left(\frac{\alpha}{k}+1,\frac{1}{\mu},1-(1-t)^{\mu}\right)\right|. \tag{40}$$

Substituting (40) *in equation* (38)*, we obtained*

$$\le \frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}\int_0^1 \frac{1}{\mu}\left|\frac{1}{(k\mu)^{\frac{\alpha}{k}}}B\left(\frac{\alpha}{k}+1,\frac{1}{\mu},1-(1-t)^{\mu}\right)\right|(t^s+(\eta-t)^s)\,dt$$
$$\times\left(\frac{|h''(\nu)|+|h''(\omega)|}{\eta^s}\right). \tag{41}$$

Ultimately, the expected outcome has been reached.

Remark 8. *If we set parameters as $\eta = 2, s = 1$ and k=1 in* (36)*, then Theorem 4 leads to [44] Theorem 11.*

Remark 9. *If we set $\eta = 2, s = 1, k = 1$ and $\mu = 1$ in* (36) *then our result reduced to [46] Theorem 1.5.*

Remark 10. *If we set $\eta = 2, s = 1, k = 1, \mu = 1$ and $\alpha = 1$ simultaneously in 36, then Theorem 4 and [44], Proposition 1 become identical.*

Example 4. *This example demonstrates the application of Theorem 4 using both graphical and numerical methods. We consider the function $h(x) = x^6 + 2x^4$, defined on the interval* $[2, 7]$*, and evaluate the inequality under specific parameter values: $k = 3$,$s = 1$ $\alpha = 4$, and $\eta = 8$.*

Explanation:

Figure 7 presents a 2D plot illustrating the behavior of both sides of the inequality (36) *as μ varies within* $(0, 1]$*. The graph confirms that the left-hand side remains within the bounds set by the right-hand side, visually supporting the theorem's validity. Numerical computations were performed to compare LHS and RHS values for selected values of μ. A table summarizing these results is shown in Figure 6, demonstrating consistency between the computed values and the inequality constraints.*

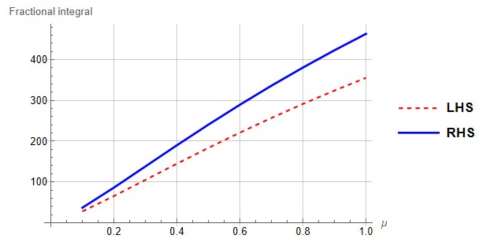

Figure 7: This figure provides a graphical illustration of Theorem 4 for $\mu \in (0, 1]$.

Table 6: The table displays numerical values corresponding to equation (36) for $\mu \in (0, 1]$ which is supporting our results.

μ	0.2	0.4	0.6	0.8	1
LHS	65.30	144.68	221	291.37	355.37
RHS	85.95	189.87	289.29	380.54	463.19

Extending the analysis, a 3D visualization was generated by varying α over $[5, 10]$ and μ within $(0, 1]$. The resulting surface plot, shown in Figure 8, demonstrates that the inequality holds robustly across these parameter ranges. This multi-faceted approach

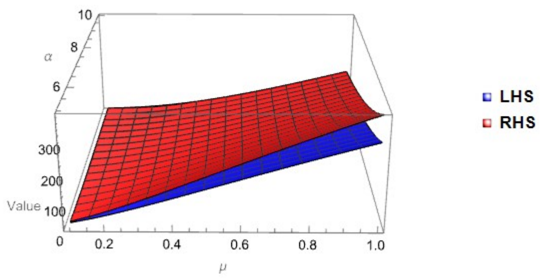

Figure 8: Three-dimensional representation of Theorem 4 validating the inequality across varying α and μ.

highlights the validity and practical applicability of Theorem 4, confirming its effectiveness in bounding the behavior of $h(x)$ under the specified conditions.

Theorem 5. *Assume $h : [\nu, \omega] \to \mathbb{R}$ is a twice continuously differentiable function over (ν, ω), with $h'' \in L_1([\nu, \omega])$. Let $|h''|^q$ be convex on $[\nu, \omega]$, where $q > 1$, then*

$$\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\left(\frac{\alpha}{k}+1\right)\left({}_{k}^{\alpha}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}_{k}^{\alpha}j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]$$
$$-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg| \le \frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}$$
$$\times\left[\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|^p dt\right]^{\frac{1}{p}}\left[\left(\frac{|h''(\omega)|^q}{\eta^s(s+1)}-\frac{|h''(\nu)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{|h''(\nu)|^q}{\eta^s(s+1)}-\frac{|h''(\omega)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}\Bigg]. \tag{42}$$

Proof. *By employing Hölder inequality on equation* (37)

$$\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\Big(\frac{\alpha}{k}+1\Big)\left({}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}\,j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]$$
$$-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg|\leq\frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}$$
$$\times\Bigg[\left(\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|^p dt\right)^{\frac{1}{p}}\Bigg|\left(\int_0^1\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{43}$$

By taking advantage of s-convexity of $|h''(x)|^q$ *on following equations, consider*

$$\left(\int_0^1\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\leq\left(\int_0^1\left[\left(\frac{\eta-t}{\eta}\right)^s|h''(\nu)|^q+\left(\frac{t}{\eta}\right)^s|h''(\omega)|^q\right]dt\right)^{\frac{1}{q}}$$
$$\left(\int_0^1\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\leq\left(|h''(\omega)|^q\frac{1}{\eta^s}\Big(\frac{t^{s+1}}{s+1}\Big)\Big|_0^1-\frac{|h''(\nu)|^q}{\eta^s(s+1)}(\eta-t)^{s+1}\Big|_0^1\right)^{\frac{1}{q}}$$
$$\left(\int_0^1\left|h''\left(\frac{\eta-t}{\eta}\nu+\frac{t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\leq\left(\frac{|h''(\omega)|^q}{\eta^s(s+1)}-\frac{|h''(\nu)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}. \tag{44}$$

Similarly

$$\left(\int_0^1\left|h''\left(\frac{t}{\eta}\nu+\frac{\eta-t}{\eta}\omega\right)\right|^q dt\right)^{\frac{1}{q}}\leq\left(\frac{|h''(\nu)|^q}{\eta^s(s+1)}-\frac{|h''(\omega)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}. \tag{45}$$

Substituting (44) *and* (45) *in* (43)*, we get*

$$\Bigg|\frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}}(k\mu)^{\frac{\alpha}{k}}\Gamma\Big(\frac{\alpha}{k}+1\Big)\left({}^{\alpha}_{k}j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-}h(\nu)+{}^{\alpha}_{k}\,j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+}h(\omega)\right)+\Upsilon_k[\mu,\alpha]$$
$$-\frac{1}{\eta}\left(h\left(\frac{\eta-1}{\eta}\nu+\frac{1}{\eta}\omega\right)+h\left(\frac{1}{\eta}\nu+\frac{\eta-1}{\eta}\omega\right)\right)\Bigg|\leq\frac{(\omega-\nu)^2}{\eta^3}(k\mu)^{\frac{\alpha}{k}}$$
$$\times\left[\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^\mu}{k\mu}\right)^{\frac{\alpha}{k}}d\varphi\right|^p dt\right]^{\frac{1}{p}}\Bigg[\left(\frac{|h''(\omega)|^q}{\eta^s(s+1)}-\frac{|h''(\nu)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{|h''(\nu)|^q}{\eta^s(s+1)}-\frac{|h''(\omega)|^q}{\eta^s(s+1)}\left((\eta-1)^{s+1}-\eta^{s+1}\right)\right)^{\frac{1}{q}}\Bigg]. \tag{46}$$

Therefore, the calculations produce the anticipated result.

Remark 11. *By substituting k=1,s=1and $\eta = 2$ in (42) we conclude*

$$\left|\frac{2^{\mu\alpha-1}}{(\omega-\nu)^{\mu\alpha}}(\mu)^{\alpha}\Gamma(\alpha+1)\left({}^{\alpha}j^{\mu}_{\frac{\nu+\omega}{2}-}h(\nu)+{}^{\alpha}j^{\mu}_{\frac{\nu+\omega}{2}+}h(\omega)\right)-h\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{8}\mu^{\alpha}\left[\int_0^1\left|\int_0^t\left(\frac{1-(1-\varphi)^{\mu}}{\mu}\right)^{\alpha}d\varphi\right|^p dt\right]^{\frac{1}{p}}\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{47}$$

Remark 12. *By taking values $\mu = 1$ in (47), result reduced to*

$$\left|\frac{2^{\alpha-1}}{(\omega-\nu)^{\alpha}}\Gamma(\alpha+1)\left({}^{\alpha}j_{\frac{\nu+\omega}{2}-}h(\nu)+{}^{\alpha}j_{\frac{\nu+\omega}{2}+}h(\omega)\right)-h\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{8}\left[\int_0^1\left|\int_0^t s^{\alpha}d\varphi\right|^p dt\right]^{\frac{1}{p}}\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{48}$$

Consider the term

$$\left[\int_0^1\left|\int_0^t s^{\alpha}d\varphi\right|^p dt\right]^{\frac{1}{p}}=\left[\int_0^1\left|\frac{t^{\alpha+1}}{\alpha+1}\right|^p dt\right]^{\frac{1}{p}}$$
$$\left[\int_0^1\left|\int_0^t s^{\alpha}d\varphi\right|^p dt\right]^{\frac{1}{p}}=\left(\frac{1}{(\alpha+1)^p}\int_0^1 t^{p\alpha+p}dt\right)^{\frac{1}{p}}$$
$$\left[\int_0^1\left|\int_0^t s^{\alpha}d\varphi\right|^p dt\right]^{\frac{1}{p}}=\left(\frac{1}{(\alpha+1)^p(p\alpha+p+1)}\right)^{\frac{1}{p}}$$

$$\left|\frac{2^{\alpha-1}}{(\omega-\nu)^{\alpha}}\Gamma(\alpha+1)\left({}^{\alpha}j_{\frac{\nu+\omega}{2}-}h(\nu)+{}^{\alpha}j_{\frac{\nu+\omega}{2}+}h(\omega)\right)-h\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{8}\left(\frac{1}{(\alpha+1)^p(p\alpha+p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Remark 13. *Assuming $\alpha = 1$ in (48), we obtain following result*

$$\left|\frac{1}{\omega-\nu}\int_{\nu}^{\omega}h(x)dx-h\left(\frac{\nu+\omega}{2}\right)\right|\leq\frac{(\omega-\nu)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{3|h''(\nu)|^q+|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|h''(\nu)|^q+3|h''(\omega)|^q}{4}\right)^{\frac{1}{q}}\right]. \tag{49}$$

Example 5. *This example illustrates the use of Theorem 5 through a combination of graphical and numerical techniques. The function under consideration is* $h(x) = x^6 + 2x^4$*, defined on the interval* $[2, 7]$*, with the inequality evaluated using the parameters* $k = 3$*,* $\alpha = 4$*,s=1,* $\frac{1}{p} = 0.6$*,* $\frac{1}{q} = 0.4$*, and* $\eta = 8$*.*

Explanation:

Figure 9 provides a 2D visualization of the inequality (42)*, depicting the behavior of the left-hand side and right-hand side as* μ *varies in* $(0, 1]$*. The graph demonstrates that the LHS consistently adheres to the bounds imposed by the RHS, supporting the validity of the theorem. To complement the graphical analysis, numerical evaluations of the LHS and RHS were conducted for specific values of* μ*. The results, displayed in a Table 7, confirm that the inequality holds under the given parameters.*

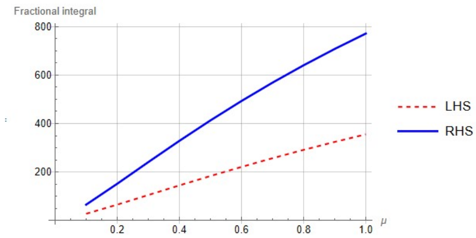

Figure 9: This figure is illustrating the 2D visualization of the inequality (42) confirming the validity its validity.

Table 7: This table summarizes the numerical results of (42).

μ	0.2	0.4	0.6	0.8	1
LHS	65.30	144.68	221	291.37	355.37
RHS	151.38	328.386	492.93	640.33	771.12

Further validation was performed by generating a 3D representation of the inequality as α *ranges within* $[5, 10]$ *and* μ *within* $(0, 1]$*. The resulting surface plot, presented in Figure 10, illustrates that the inequality remains valid across the explored parameter space. This*

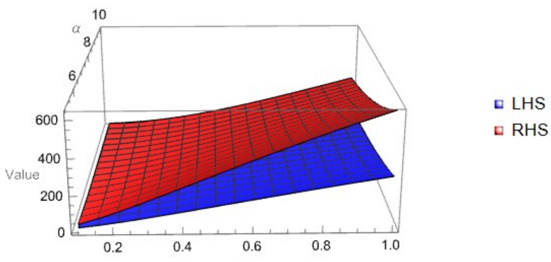

Figure 10: Three-dimensional validation of Theorem 5 across different values of α and μ.

comprehensive analysis underscores the utility and reliability of Theorem 5, verifying its capacity to constrain the behavior of $h(x)$ under the specified conditions.

Theorem 6. *Let $h : [\nu, \omega] \to \mathbb{R}$ be a twice differentiable mapping on (ν, ω) such that $h'' L_1([\nu, \omega])$. Let $|h''|^q$ be a s-convex in second sense on $[\nu, \omega]$ with $q > 1$ then,*

$$\begin{aligned}
&\Bigg| \frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}} (k\mu)^{\frac{\alpha}{k}} \Gamma\Big(\frac{\alpha}{k}+1\Big) \left({}^{\alpha}_{k} j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}^{\alpha}_{k} j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) + \Upsilon_k[\mu, \alpha] \\
&- \frac{1}{\eta} \left(h\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega \right) + h\left(\frac{1}{\eta}\nu + \frac{\eta-1}{\eta}\omega \right) \right) \Bigg| \leq \frac{(\omega-\nu)^2}{\eta^3} (k\mu)^{\frac{\alpha}{k}} \\
&\times \left(\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt \right)^{1-\frac{1}{q}} \left[\left(\frac{|h''(\nu)|^q}{\eta^s} \xi_k^2(\mu, \alpha) + \frac{|h''(\omega)|^q}{\eta^s} \xi_k^1(\mu, \alpha) \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{|h''(\nu)|^q}{\eta^s} \xi_k^1(\mu, \alpha) + \frac{|h''(\omega)|^q}{\eta^s} \xi_k^2(\mu, \alpha) \right)^{\frac{1}{q}} \right] \qquad (50)
\end{aligned}$$

where

$$\begin{aligned}
\xi_k^1(\mu, \alpha) &= \int_0^1 t^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt, \\
\xi_k^2(\mu, \alpha) &= \int_0^1 (\eta - t)^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt.
\end{aligned}$$

Proof. By employing Power mean inequality on equation (37),

$$\begin{aligned}
&\Bigg| \frac{\eta^{\mu\frac{\alpha}{k}-1}}{(\omega-\nu)^{\mu\frac{\alpha}{k}}} (k\mu)^{\frac{\alpha}{k}} \Gamma\Big(\frac{\alpha}{k}+1\Big) \left({}^{\alpha}_{k} j^{\mu}_{\frac{(\eta-1)\nu+\omega}{\eta}-} h(\nu) + {}^{\alpha}_{k} j^{\mu}_{\frac{\nu+(\eta-1)\omega}{\eta}+} h(\omega) \right) + \Upsilon_k[\mu, \alpha] \\
&- \frac{1}{\eta} \left(h\left(\frac{\eta-1}{\eta}\nu + \frac{1}{\eta}\omega \right) + h\left(\frac{1}{\eta}\nu + \frac{\eta-1}{\eta}\omega \right) \right) \Bigg| \leq \frac{(\omega-\nu)^2}{\eta^3} (k\mu)^{\frac{\alpha}{k}} \\
&\times \left[\left(\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt \right)^{1-\frac{1}{q}} \right. \\
&\times \left(\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h''\left(\frac{\eta-t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt \right)^{1-\frac{1}{q}} \\
&\left. \times \left(\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h''\left(\frac{t}{\eta}\nu + \frac{\eta-t}{\eta}\omega \right) \right|^q dt \right)^{\frac{1}{q}} \right]. \qquad (51)
\end{aligned}$$

By considering s-convexity of $|h''(x)|^q$ in second sense, consider

$$\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h''\left(\frac{\eta-t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt$$

$$\leq \int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| \left(|h''(\nu)| \left(\frac{\eta - t}{\eta} \right)^s + \left(\frac{t}{\eta} \right)^s |h''(\omega)| \right) dt$$

$$\begin{aligned}
&\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h'' \left(\frac{\eta - t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt \\
&\leq \frac{|h''(\nu)|^q}{\eta^s} \int_0^1 (\eta - t)^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt \\
&+ \frac{|h''(\omega)|^q}{\eta^s} \int_0^1 t^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt. \qquad (52)
\end{aligned}$$

Assuming

$$\xi_k^1(\mu, \alpha) = \int_0^1 t^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt, \qquad (53)$$

$$\xi_k^2(\mu, \alpha) = \int_0^1 (\eta - t)^s \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| dt. \qquad (54)$$

Substituting (53) *and* (54) *in* (52), *we obtain*

$$\begin{aligned}
&\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h'' \left(\frac{\eta - t}{\eta}\nu + \frac{t}{\eta}\omega \right) \right|^q dt \\
&\leq \frac{|h''(\nu)|^q}{\eta^s} \xi_k^2(\mu, \alpha) + \frac{|h''(\omega)|^q}{\eta^s} \xi_k^1(\mu, \alpha) \qquad (55)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^1 \left| \int_0^t \left(\frac{1-(1-\varphi)^{\mu}}{ku} \right)^{\frac{\alpha}{k}} d\varphi \right| . \left| h'' \left(\frac{t}{\eta}\nu + \frac{\eta - t}{\eta}\omega \right) \right|^q dt \\
&\leq \frac{|h''(\nu)|^q}{\eta^s} \xi_k^1(\mu, \alpha) + \frac{|h''(\omega)|^q}{\eta^s} \xi_k^2(\mu, \alpha). \qquad (56)
\end{aligned}$$

Substituting (55) *and* (56) *in equation* (51), *we obtain required result.*

Remark 14. *By assuming the values of* $\eta = 2$,*s=1 and* $k = 1$ *in* (51), *the resulting outcome is obtained as follow*

$$\begin{aligned}
&\left| \frac{2^{\mu\alpha - 1}}{(\omega - \nu)^{\mu\alpha}} \mu^{\alpha} \Gamma(\alpha + 1) \left({}^{\alpha} j^{\mu}_{\frac{\nu+\omega}{\eta}-} h(\nu) + {}^{\alpha} j^{\mu}_{\frac{\nu+\omega}{\eta}+} h(\omega) \right) - h\left(\frac{\nu + \omega}{2} \right) \right| \leq \frac{(\omega - \nu)^2}{8} \mu^{\alpha} \\
&\times \left(\xi^1(\mu, \alpha) \right)^{1 - \frac{1}{q}} \left[\left(\frac{2\xi^1(\mu, \alpha) - \xi^2(\mu, \alpha)}{2} |h''(\nu)|^q + \frac{\xi^2(\mu, \alpha)}{2} |h''(\omega)|^q \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{\xi^2(\mu, \alpha)}{2} |h''(\nu)|^q + \frac{2\xi^1(\mu, \alpha) - \xi^2(\mu, \alpha)}{2} |h''(\omega)|^q \right)^{\frac{1}{q}} \right]. \qquad (57)
\end{aligned}$$

Remark 15. *Substituting $\alpha = 1$ and $\mu = 1$ in (57), we get the following inequality*

$$\left|\frac{1}{(\omega-\nu)}\int_{\nu}^{\omega} h(x)dx - h'\left(\frac{\nu+\omega}{2}\right)\right| \leq \frac{(\omega-\nu)^2}{8}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\left(\frac{5}{8}|h''(\nu)|^q + \frac{3}{8}|h''(\omega)|^q\right)^{\frac{1}{q}} + \left(\frac{3}{8}|h''(\nu)|^q + \frac{5}{8}|h''(\omega)|^q\right)^{\frac{1}{q}}\right]. \tag{58}$$

Example 6. *This example illustrates the use of Theorem 6 through a combination of graphical and numerical techniques. The function under consideration is $h(x) = x^6 + 2x^4$, defined on the interval $[2, 7]$, with the inequality evaluated using the parameters $k = 3$, $\alpha = 4$,s=1, $\frac{1}{p} = 0.6$, $\frac{1}{q} = 0.4$, and $\eta = 8$.*

Explanation:

Figure 11 provides a 2D visualization of the inequality (51), depicting the behavior of the left-hand side and right-hand side as μ varies in $(0, 1]$. The graph demonstrates the validity of the theorem. To complement the graphical analysis, numerical evaluations of the LHS and RHS were conducted for specific values of μ. The results, displayed in a Table 8, confirm that the inequality holds under the given parameters.

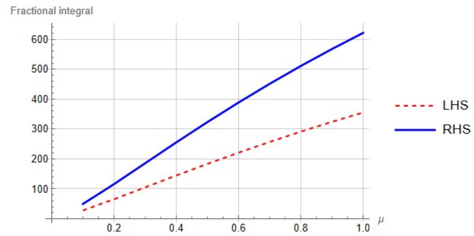

Figure 11: Verification of Theorem 6 using graphical and numerical data for $\mu \in (0, 1]$.

Table 8: This table offers numerical data for comparison of left hand side and right hand side of inequality (51).

μ	0.2	0.4	0.6	0.8	1
LHS	65.30	144.68	221	291.37	355.37
RHS	115.67	255.29	388.67	510.90	621.48

Further validation was performed by generating a 3D representation of the inequality as α ranges within $[5, 10]$ and μ within $(0, 1]$. The resulting surface plot, presented in Figure 12, illustrates that the inequality remains valid across the explored parameter space.

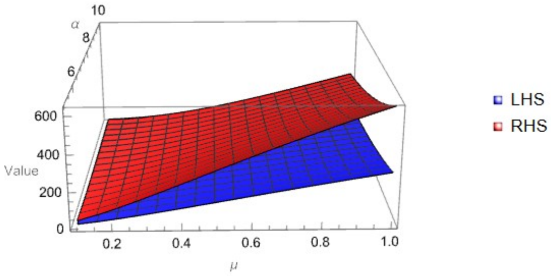

Figure 12: Three-dimensional validation of Theorem 6 across different values of α and μ.

This comprehensive analysis underscores the utility and reliability of Theorem 6, verifying its capacity to constrain the behavior of $h(x)$ under the specified conditions.

5. Conclusion

In this paper, we have successfully extended the understanding of inequalities associated with twice differentiable functions through the lens of extended conformable fractional operators. The establishment of new equalities and the derivation of novel trapezoidal-type and midpoint-type inequalities highlight the significance convexity. The applications of established inequalities, such as the power mean inequality and Hölder's inequality has led to the development of a new class of inequalities, further enriching the existing body of knowledge. These findings not only generalize previous research but also open avenues for future exploration in the field of fractional calculus. Importantly, this research bridges a gap between classical convex analysis and the modern framework of fractional calculus, emphasizing the role of extended operators in advancing mathematical theory. Furthermore, the insights gained from this study may inspire researchers to investigate the application of these concepts to other fractional operators. This study lays a solid foundation for future investigations, particularly in applying extended conformable fractional operators to other forms of inequalities or to different classes of functions beyond the twice differentiable case. This work encourages applying these concepts to other fractional operators and highlights the link between convexity, differentiability, and fractional calculus.

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Declarations:

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final version.

Competing interests

The authors declare that they have no conflicts of interest.

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