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Finite Difference Scheme for One-Dimensional Coupled Parabolic System with Blow-up

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Abstract. This study aims to find an efficient technique to estimate the blow-up time (BUT) of one-dimensional semi-linear coupled parabolic systems. Firstly, a fully discrete finite difference formula is derived with a non-fixed time-stepping formula, based on the Crank-Nicolson method. In addition, the consistency, stability, and convergence of the proposed scheme are considered. Secondly, two numerical experiments are presented. For each experiment, we apply the proposed scheme to calculate the numerical blow-up time, error bounds, and the numerical order of convergence for blow-up times. The obtained results show that the proposed C.N scheme is consistent with the system considered. However, it is conditionally stable, and the Crank-Nicolson scheme converges in the stability region and achieves first- and second-order accuracy in temporal and spatial dimensions, respectively. Also, it helps to increase the order of numerical convergence. Furthermore, the numerical experiments demonstrate that the numerical blow-up simultaneously occurs at only the center point. Finally, the numerical blow-up time sequence is convergent. Moreover, convergence for the blow-up time agrees well with the theoretical order of convergence of the proposed scheme.

2020 Mathematics Subject Classifications: 35K57, 65M06, 65M12, 35B44

Key Words and Phrases: Fully discrete, Crank-Nicolson formula, Numerical blow-up times, Convergence, Blow-up time, Consistency, Stability

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1. Introduction

In recent decades, many real-world problems have been modeled by PDEs, see [1–3]. There are many parabolic-type PDEs whose solutions cannot possibly be extended

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globally over time. This phenomenon is known as blow-up (BU), and it occurs in semi-linear diffusion equations, when the heat source is strong enough [4–8] this phenomenon can be applied to interconnected systems, where each variable that relies on another experiences finite growth within a designated time frame. In this specific situation, it is known that the dependent variables in the equations demonstrate synchronized increases [9, 10].This phenomenon has several applications, especially in combustion theory and heat propagation [11] .
According to [10] , a solution of a diffusion equation defined on $\Omega \times \{t > 0\}$ blows up, if there exist $T < \infty$, such that it is still bounded as long as $0 < t < T$, while it is unbounded as t is close to the blow-up time T,

$$\text{i.e.} \quad \sup_{x\in\Omega} |u(x,t)| \to \infty \text{ as } t \to T^-.$$

For a coupled diffusion system:

$$u_t = u_{xx} + F(u,v), \quad v_t = v_{xx} + G(u,v), \quad (x,t) \in \Omega \times (0,T), F, G : R2 \to R,$$

a solution (u, v) blows up simultaneously if there exists $T < \infty$ such that u and v blow up at T.
i.e. $\sup_{x\in\Omega} |u(x,t)| \to \infty, \quad \& \quad \sup_{x\in\Omega} |v(x,t)| \to \infty, \quad \text{as } t \to T^-,$
while for $t < T, \quad \sup_{x\in\Omega}\{|u(x,t)|, |v(x,t)|\} < \infty,$

In this work, we consider the system:

$$\left.\begin{array}{rr} u_t = u_{xx} + f(v), v_t = v_{xx} + g(u), (x,t) \in (0,1) \times (0,T) & \\ u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1) \end{array}\right\} \quad (1)$$

where $u_0(x) \in C^2(R), v_0(x) \in C^2(R)$, satisfying $u_0(0) = u_0(1) = 0, v_0(0) = v_0(1) = 0; f, g \in C^1(R) \cap C^2(R\backslash\{0\})$, are positive, super-linear differentiable, and increasing functions on $(0, \infty)$.

To demonstrate the local existence and distinguish positive solutions to the problem, some authors used the standard parabolic theory [12]. Moreover, for various nonlinear functions: f, g, when the initial functions (u_0, v_0) are sufficiently large, a blow-up may occur in finite time[9]. In addition, because the system (1) is coupled, only a simultaneous blow-up occurs. Regarding the blow-up set and blow-up rate estimate, the system (1) has been extensively studied in [13–19]. Furthermore, imposing some related suggestions in f, g, it is shown that the (BU) occurs at a single point [15]. Particularly, the following two instances of f, g are mostly studied [9, 13, 15] :

$$\begin{array}{c} f(v) = v^p, g(u) = u^q, \quad p, q > 1 \\ f(v) = e^{pv}, g(u) = e^{qu}, \quad p, q > 0 \end{array}$$

Since the last decades, several numerical approximations have been proposed for many parabolic problems with BU to compute the NBU solution and estimate BUT [20–26].
In [25], the authors proposed an approximate explicit scheme for the system (1) with

nonlinear power-type functions, subject to uniform temporal grids. In addition, the computation of the BU time and BU behaviors is studied. Moreover, the convergence of the NBU time is considered. In addition, the BU set, the BU rate and the BU in the L_p-norm were taken into account. Furthermore, the relationship between the BU of the numerical solution and that of the exact solution has been investigated.

Recently, in [26], the authors have studied the semi-discrete problem of system (1) and its properties regarding convergence and (BUT). Namely, it has been shown that the (NBUT) and NBUS of a semi-discrete form of (1) converge to the theoretical ones during grid refinement. Moreover, they have proposed explicit and implicit finite difference approximation schemes with non-fixed time-step to estimate the (BUT) to the system (1).Moreover, they investigated the stability, convergence, and consistency of the suggested methods. In addition, some numerical examples are presented in the form of tables and figures.

This work aims to find an efficient technique to estimate the (BUT) of the system (1). Namely, to show the accuracy and efficiency of the C.N. scheme in computing BU solution and BUT.
This study has six sections. In the second section, some results of the semi-discrete problem on the problem (1) are recalled. The Crank-Nicolson approximation formulas of the system (1) are derived in Section three. In the fourth section, consistency, stability, and convergence are considered. In Section five, two numerical experiments are given to estimate the (NBUT), error bounds, and numerical order of convergence. Lastly, the important conclusions and future work are stated in the last section.

1.1. The semi-discrete problem

Let I be a positive integer and $x_i = ih$, $0 \leq i \leq I$, where $h = \frac{1}{I}$, then we can adjective by the solution: $(U_h(t), V_h(t))$,

$$U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T, \quad V_h(t) = (V_0(t), V_1(t), \ldots, V_I(t))^T. \tag{2}$$

By replacing the second-order space derivative in the problem (1) by the standard second-order central finite difference operator δ^2 [27], we obtain the semi-discrete problem:

$$\frac{d}{dt}U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + f(V_i) \tag{3}$$

$$\frac{d}{dt}V_i = \frac{V_{i+1} - 2V_i + V_{i-1}}{h^2} + g(U_i) \tag{4}$$

$$U_0(t) = U_I(t) = 0,\ V_0(t) = V_I(t) = 0, \tag{5}$$

$$U_i(0) = U_0(x_i),\ V_i(0) = V_0(x_i), \quad 0 \leq i \leq I. \tag{6}$$

Definition 1. *[26] Let (U_h, V_h) be a nonnegative solution to problem (1)–(2). We say that (U_h, V_h) achieves blow-up simultaneously in finite time, if there exists $T_h < \infty$ such* *that:*

$$\|U_h(t)\|_\infty < \infty \text{ and } \|V_h(t)\|_\infty < \infty \text{ for } t \in [0, T_h), \tag{7}$$

$$\|U_h(t)\|_\infty \to \infty, \|V_h(t)\|_\infty \to \infty \text{ as } t \to T_h, \tag{8}$$

where $\|U_h(t)\|_\infty = \max_{0\le i\le 1} |U_i(t)|$ *and* $\|7V_h(t)\|_\infty = \max_{0\le i\le 1} |V_i(t)|$.

Theorem 1. *[26] Let* (U_h, V_h) *be a non-negative solution of the semi-discrete problem, where the reaction functions* f, g *are locally Lipschitz continuous such that* $f, g > 0$ *in* $(0, \infty)$. *If* (U_h, V_h) *achieves blow-up at time:* $T_h < \infty$, *then* T_h *is bounded from below.*

Corollary 1. *[26] Let* (U_h, V_h) *be a non-negative solution of the semi-discrete problem (1)-(2), where* $f = v^p, g = u^q, p, q > 1$. *If* (U_h, V_h) *achieves blow-up at* $T_h < \infty$, *then*

$$\int_{\|V_0\|}^{\infty} \frac{ds}{G_2(s)} = \int_{\|U_0\|}^{\infty} \frac{ds}{G_1(s)} = T \le T_h, \tag{9}$$

where

$$G_1(s) = \left[\frac{p+1}{q+1} S^{q+1} - (p+1)C_0\right]^{\frac{p}{p+1}}, \tag{10}$$

$$G_2(s) = \left[\frac{q+1}{p+1} S^{p+1} - (q+1)C_0\right]^{\frac{q}{q+1}}, \tag{11}$$

$$C_0 = \frac{1}{q+1} u_0^{q+1} - \frac{1}{p+1} v_0^{p+1}. \tag{12}$$

Theorem 2. *[26] Assume the following:*

a) $f, g \in C^1([0, \infty])$ *are convex* $(f'', g'' \ge 0,), f(z) > 0$ *in* $(0, \infty)$, *and* $\forall \varepsilon > 0$, *we have*

$$\int_{\varepsilon}^{\infty} \frac{dZ}{f(Z)} < \infty, \quad \int_{\varepsilon}^{\infty} \frac{dZ}{g(Z)} < \infty. \tag{13}$$

b) $\exists Z_0 \ge 0$ *such that*

$$\left(\frac{f(z)}{Z}\right) \ge \lambda_n, \quad \left(\frac{g(z)}{Z}\right) \ge \lambda_n \text{ for } Z \in [Z_0, \infty), \text{ and}$$
$$\lim_{Z\to\infty} \frac{Z}{f(Z)} = 0, \quad \lim\ {}_{Z\to\infty} \frac{Z}{g(Z)} = 0.$$

c) The initial conditions are non-negative and such that

$$U_h(0) \ne 0, \quad V_h(0) \ne 0, \quad Q_1(0) \ge Z_0, \quad Q_2(0) \ge Z_0, \tag{14}$$

$$f(Q_2(0)) - \lambda n Q_1(0) > 0, \quad g(Q_1(0)) - \lambda n Q_2(0) \ge 0, \tag{15}$$

Where

$$Q_1(t) = \sum_{i=1}^{I-1} hU_i(t)\Upsilon_i, \quad Q_2(t) = \sum_{i=1}^{I-1} hV_i(t), \Upsilon_i, \tag{16}$$

$$\Upsilon_i = \frac{\sin(\pi i h)}{\sum_{m=1}^{I-1} \sin(\pi m h)}, -\delta^2 \Upsilon_i = \lambda_n \Upsilon_i, \quad i = 1, 2, \ldots, I-1, \quad \Upsilon_0 = \Upsilon_I = 0,$$
$$\lambda_n = \left(\frac{4}{h^2}\right) \sin^2\left(\frac{\pi h}{2}\right). \tag{17}$$

Then the non-negative solution (U_h, V_h) of the discrete problem achieves blow-up at time T_h with

$$T_h \leq \int_{Q_1(0)}^{\infty} \frac{dZ_1}{f(Z_2) - \lambda_n Z_1}, \quad \forall Z_2 \in [Z_0, \infty), \tag{18}$$

or

$$T_h \leq \int_{Q_2(0)}^{\infty} \frac{dZ_2}{g(Z_1) - \lambda_n Z_2}, \quad \forall Z_1 \in [Z_0, \infty). \tag{19}$$

Theorem 3. *[26] Assume a- The reaction functions: $f, g \in C^1([0,\infty])$, and problem of system (1) has a solution: $(u,v), u, v \in C^{4,1}([0,1] \times [0,T])$.*
b- The initial condition $\left(U_h^0, V_h^0\right)$ satisfies

$$\begin{aligned} \left\|U_h^0 - u_h(0)\right\|_\infty = O(1), \quad h \to 0 \\ \left\|V_h^0 - v_h(0)\right\|_\infty = O(1), \quad h \to 0 \end{aligned} \tag{20}$$

Then, for h sufficiently small, the semi-discrete problem (1) has a unique solution: (U_h, V_h), $\quad U_h, V_h \in C^1([0,T]), R^{J+1}$ such that

$$\begin{aligned} \max_{t\in[o,T]} \|U_h(t) - u_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \quad h \to 0 \\ \max_{t\in[o,T]} \|V_h(t) - v_h(t)\|_\infty = O\left(\|e(0)\|_\infty + h^2\right), \quad h \to 0 \end{aligned} \tag{21}$$

where $\|e(0)\|_\infty = \max\left\{\left\|U_h^0 - u_h(0)\right\|_\infty, \left\|V_h^0 - v_h(0)\right\|_\infty\right\}$

Theorem 4. *[26] Assume*
a) The functions $f, g \in C^1([0,\infty), R)$ and $f(Z), g(Z) > 0$ in $(0,\infty)$.
b) There exists $\bar{Z} \geq 0$, such that

- $\left(\frac{f(Z)}{Z}\right) \geq \pi^2, \quad \left(\frac{g(Z)}{Z}\right) \geq \pi^2,$ *for* $\quad Z \in [\bar{Z}, \infty)$
- $\lim_{Z\to\infty} \frac{Z}{f(Z)} = 0, \quad \lim_{Z\to\infty} \frac{Z}{g(Z)} = 0$
- $f(\bar{Z}) - \pi^2 \bar{Z} > 0 \quad , \quad g(\bar{Z}) - \pi^2 \bar{Z} > 0$

c) There exists $T < \infty$ such that, $u, v \in C^{4,2}([0,1] \times [0,T))$ and

$$\begin{aligned} \lim_{t\to T} \int_0^1 u(x,t) w(x) dx = \infty \\ \lim_{t\to T} \int_0^1 v(x,t) w(x) dx = \infty \end{aligned} \tag{22}$$

If $\|U_h(0) - u_h(0)\|_\infty = 0(1), h \to 0$, *then the solution of the discrete problem achieves blow-up, for* h *sufficiently small at* T_h *and*

$$\lim_{h\to 0} T_h = T$$

2. Crank-Nicolson Method

Define the grid points $x_i = ih, 0 \le i \le I$, where $I \in Z^+, h = 1/I, t_{n+1} = t_n + k_n, k_n > 0, \forall n$.

Set $U_h^n = \left(U_1^n, U_2^n \dots U_{I-1}^n\right)^T, \qquad V_h^n = \left(V_1^n, U_2^n \dots V_{I-1}^n\right)^T$

as the numerical solution of problem (1).

In order to derive the Crank-Nicolson method formally, we approximate the partial derivatives in the semi-discrete problem (3),(4), at the mesh points: $\left(x_i, t_{n+\frac{1}{2}}\right)$.

Firstly, we approximate the time-derivatives using the 1^s order central finite difference formula:

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+\frac{1}{2}} = \frac{1}{k_n}\left(u_i^{n+1} - u_i^n\right) + O\left(K_n\right), \quad \left.\frac{\partial v}{\partial t}\right|_i^{n+\frac{1}{2}} = \frac{1}{k_n}\left(v_i^{n+1} - v_i^n\right) + O\left(k_n\right),$$

and $\delta_x^2 U(t), \delta_x^2 V(t)$ are approximated as follows:

$$\delta_x^2 U_i\left(t_{n+1/2}\right) = \frac{1}{2}\left[\delta_x^2 U_i\left(t_{n+1}\right) + \delta_x^2 U_i\left(t_n\right)\right] + O\left(k_n^2\right)$$

$$\delta_x^2 V_i\left(t_{n+1/2}\right) = \frac{1}{2}\left[\delta_x^2 V_i\left(t_{n+1}\right) + \delta_x^2 V_i\left(t_n\right)\right] + O\left(k_n^2\right)$$

It follows that

$$\delta_x^2 U_i\left(t_{n+1/2}\right) = \frac{1}{2h^2}\left[\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right)\right]$$
$$\delta_x^2 V_i\left(t_{n+1/2}\right) = \frac{1}{2h^2}\left[\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) + \left(V_{i+1}^{n+1} - 2V_i^{n+1} + v_{i-1}^{n+1}\right)\right]$$

While the nonlinear terms are taken as follows:

$$f\left(V_i^{n+1/2}\right) = f\left(V_i^n\right) + O\left(K_n\right),$$

$$g\left(U_i^{n+1/2}\right) = g\left(U_i^n\right) + O\left(k_n\right)$$

By substituting all these formulas in (3),(4), we obtain

$$\frac{1}{k_n}\left(U_i^{n+1} - U_i^n\right) = \frac{1}{2h^2}\left[\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right) + \left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right)\right] + f\left(V_i^n\right)$$
$$\frac{1}{k_n}\left(V_i^{n+1} - V_i^n\right) = \frac{1}{2h^2}\left[\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) + \left(V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}\right)\right] + g\left(U_i^n\right).$$

The last difference equation can be written as follows:

$$(1+r_n)\,U_i^{n+1}-\frac{r_n}{2}\left(U_{i+1}^{n+1}+U_{i-1}^{n+1}\right)=(1-r_n)\,U_i^n+\frac{r_n}{2}\left(U_{i+1}^n+U_{i-1}^n\right)+k_nf\left(V_i^n\right) \quad (23)$$

$$(1+r_n)\,V_i^{n+1}-\frac{r_n}{2}\left(V_{i+1}^{n+1}+V_{i-1}^{n+1}\right)=(1-r_n)\,V_i^n+\frac{r_n}{2}\left(V_{i+1}^n+V_{i-1}^n\right)+k_ng\left(U_i^n\right), \quad (24)$$

when $r_n=\frac{k_n}{h^2}$.

Moreover, one can choose the time step:

$$k_n=\min\left(h^2,\frac{h^\alpha}{\left\|U_h^n\right\|_\infty},\frac{h^\alpha}{\left\|V_h^n\right\|_\infty}\right),\quad 1\leq\alpha. \quad (25)$$

System (1) can be presented in matrix form as follows:

$$\left(I+\frac{r_n}{2}H\right)U_h^{n+1}=\left(I-\frac{r_n}{2}H\right)U_h^n+k_nF^n \quad (26)$$

$$\left(I+\frac{r_n}{2}H\right)V_h^{n+1}=\left(I-\frac{r_n}{2}H\right)V_h^n+k_nG^n \quad (27)$$

$$\text{where } H=\begin{bmatrix} 2 & -1 & 0 & \cdots & 0\\ -1 & 2 & -1 & \cdots & 0\\ & & \ddots & & \\ & & & \ddots & \\ 0 & \cdots & 0 & -1 & 2\end{bmatrix}_{(I-1)\times(I-1)}$$

$$F^n=\left(f\left(V_1^n\right),f\left(V_2^n\right),\ldots,f\left(V_{I-1}^n\right)\right)^T \quad , \quad G^n=\left(g\left(U_1^n,\right),g\left(U_2^n,\right),\ldots,g\left(U_{I-1}^n,\right)\right)^T$$

Lemma 1. *Let f,g be differentiable real functions. Then, there exist positive constants $L_1,L_2,>0$, such that*

$$\left|f\left(V_i^n\right)-f\left(\tilde{V}_i^n\right)\right|\leq L_1\left|V_i^n-\tilde{V}_i^n\right| \quad (28)$$

$$\left|g\left(U_i^n\right)-g\left(\widetilde{U}_i^n\right)\right|\leq L_2\left|U_i^n-\widetilde{U}_i^n\right| \quad (29)$$

where $U_i^n,V_i^n,\widetilde{U}_i^n,\tilde{V}_i^n,(i=0,1,2,\ldots I)$ are bounded.

Proof. To end this, we use the mean value theorem:

$$\left|f\left(V_i^n\right)-f\left(\tilde{V}_i^n\right)\right|\leq\left|f'\left(Z_i\right)\right|\left|V_i^n-\tilde{V}_i^n\right| \quad (30)$$

where Z_i is an intermediate value between V_i^n and $\tilde{V}_i^n$.
Sine $V_h^n,\tilde{V}_h^n$ are bounded, then there exists $C_1>0$ such that $|f'(Z_i)|<C_1$.
Thus, (28) is valid. Similarly, one can show that (29) holds true.

2.1. The algorithm steps for Crank Nicolson method

(i) Input $\mathrm{h}, U_h^0, V_h^0, \mathrm{p}, \mathrm{q}, \alpha$

(ii) Put $n = 0$;

(iii) Choose k_n according to (25).

(iv) Calculate the numerical vectors: U_h^{n+1}, V_h^{n+1}, using the Crank Nicolson formula (26) and (27).

(v) For $n = 1, 2, \ldots.$, repeat steps 3,4 until for $n = m$, we get $\|U_h^n\|_\infty \geq 10^{15}$, or $\|V_h^n\|_\infty \geq 10^{15}$

(vi) The numerical blow-up time is $t_m = \sum_{n=0}^{m} k_n$.

3. Consistency, Stability and Convergence of C. N. scheme

In this section, we aim to analyze the Crank-Nicolson scheme in terms of these three core properties. We will first examine its consistency by evaluating the local truncation error. Next, we will explore its stability and conclude the convergence of the method, ensuring that the numerical solution approximates the exact solution.

Theorem 5. *Let (T_{ui}^n, T_{vi}^n) be the local truncation error of the Crank-Nicolson formulas at the grid point $\left(x_i, t_{n+\frac{1}{2}}\right)$. Then*

$$\left|T_{ui}^{n+\frac{1}{2}}\right| \leq C_1 k + C_2 h^2, \left|T_{vi}^{n+\frac{1}{2}}\right| \leq C_3 k + C_4 h^2 \quad , C_1, C_2, C_3, C_4 > 0.$$

Proof. Replacing the precise solution $u_i^n = u\left(x_i, t_{n+\frac{1}{2}}\right), v_i^n = v\left(x_i, t_{n+\frac{1}{2}}\right)$ in the Crank- Nicolson, yields that:

$$T_{ui}^{n+\frac{1}{2}} = \left(u_i^{n+1} - u_i^n\right) - \frac{k_n}{2h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$
$$-k_n f\left(v_i^n\right)$$

It follows that

$$T_{ui}^{n+\frac{1}{2}} = k_n\left[\frac{\partial u_i^{n+\frac{1}{2}}}{\partial t} + O\left(k_n^2\right)\right] - k_n\left[\frac{\partial^2 u_i^{n+1}}{\partial x^2} + O\left(k_n^2 + h^2\right)\right]$$
$$-k_n\left[f\left(v_i^{n+\frac{1}{2}}\right) + O\left(k_n\right)\right]$$

Thus, there exists $C > 0$, such that

$$\left|T_{ui}^{n+\frac{1}{2}}\right| \leq C\left(k + h^2\right), \text{ where } k = \max_{n \in N} k_n,$$

By the same way, one can get $\left|T_{vi}^{n+\frac{1}{2}}\right| \leq C\left(k+h^2\right)$

Clearly, $T_{ui}^{n+\frac{1}{2}}$ and $T_{vi}^{n+\frac{1}{2}}$ approach zero, as the space (time) steps go to zero. Hence, the C.N. formulas are consistent.

Theorem 6. *The Crank-Nicolson scheme is stable if* $(1-r) \geq 0$*, where* $k = \max_{n \in N} k_n$*,*

$$r = \frac{k}{h^2}$$

Proof. Let $E_u^n = \left(e_{ui}^n, i = 1, 2, \ldots I-1\right), E_v^n = \left(e_{vi}^n, i = 1, 2, \ldots I-1\right)$ where $e_{ui}^n = u_i^n - U_i^n$, $e_{vi}^n = v_i^n - V_i^n$, $u_i^n = u\left(x_i, t_n\right)$, $v_i^n = v\left(x_i, t_n\right)$ are the precise solution of problem (1)

Now, For $n=1$, $\left\|E_u^1\right\| = \max_{1 \leq i \leq I}\left|e_{ui}^1\right| = \left|\mathrm{e}_{\mathrm{uj}}^1\right|$, $\left\|E_v^1\right\| = \max_{1 \leq i \leq I}\left|e_{vi}^1\right|$.
Clearly,

$$\begin{aligned}
\left|\mathrm{e}_{\mathrm{uj}}^1\right| &= \left(1+r_0\right)\left|\mathrm{e}_{\mathrm{uj}}^1\right| - \frac{r_0}{2}\left(\left|\mathrm{e}_{\mathrm{uj}}^1\right| + \left|\mathrm{e}_{\mathrm{uj}}^1\right|\right) \\
&\leq \left(1+r_0\right)\left|\mathrm{e}_{\mathrm{uj}}^1\right| - \frac{r_0}{2}\left(\left|\mathrm{e}_{\mathrm{uj+1}}^1\right| + \left|\mathrm{e}_{\mathrm{uj-1}}^1\right|\right) \\
&\leq \left|\left(1+\mathrm{r}_0\right)\mathrm{e}_{\mathrm{uj}}^1 - \frac{r_0}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^1 + \mathrm{e}_{\mathrm{uj-1}}^1\right)\right| \\
&= \left|\left(1-r_0\right)\mathrm{e}_{\mathrm{uj}}^0 + \frac{r_0}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^0 + \mathrm{e}_{\mathrm{uj-1}}^0\right) + k_0\left(f\left(v_j^0\right) - f\left(V_j^0\right)\right)\right|.
\end{aligned}$$

Since $\left(1-r_n\right) \geq 0$, it follows that

$$\begin{gathered}
\left\|E_u^1\right\| \leq \left(1-r_0\right)\left\|E_u^0\right\| + r_0\left\|E_u^0\right\| + L_1 k_0\left|v_j^0 - V_j^0\right| \\
\left\|E_u^1\right\| \leq \left\|E_u^0\right\| + Lk\left\|E_v^0\right\| \\
\leq (1+kL)\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}.
\end{gathered}$$

By the same way, one can get:

$$\begin{gathered}
\left\|E_v^1\right\| \leq \left\|E_v^0\right\| + kL\left\|E_u^0\right\| \\
\leq (1+kL)\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}
\end{gathered}$$

where , $L = \max\left\{L_1, L_2\right\}$.

Assume that:

$$\begin{aligned}
&\left\|E_u^s\right\| \leq (1+kL)^s \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, \quad s = 1, 2, 3, \ldots, n \\
&\left\|E_v^s\right\| \leq (1+kL)^s \max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}, \quad s = 1, 2, 3, \ldots, n.
\end{aligned}$$

Regarding $n+1$,
set $\left\|E_u^{n+1}\right\| = \max_{1 \leq i \leq I}\left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right| = \left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right|$, $\left\|E_v^{n+1}\right\| = \max_{1 \leq i \leq I}\left|\mathrm{e}_{\mathrm{vj}}^{\mathrm{n+1}}\right|$

It is clear that

$$\begin{aligned}\left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right| &= (1+r_n)\left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right| - \frac{r_n}{2}\left(\left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right| + \left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right|\right)\\ &\leq (1+r_0)\left|\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}}\right| - \frac{r_0}{2}\left(\left|\mathrm{e}_{\mathrm{uj+1}}^{\mathrm{n+1}}\right| + \left|\mathrm{e}_{\mathrm{uj-1}}^{\mathrm{n+1}}\right|\right)\\ &\leq \left|(1+\mathrm{r_n})\,\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}} - \frac{r_{\mathrm{n}}}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^{\mathrm{n+1}} + \mathrm{e}_{\mathrm{uj-1}}^{\mathrm{n+1}}\right)\right|\\ &= \left|(1-r_n)\,\mathrm{e}_{\mathrm{uj}}^{\mathrm{n}} + \frac{r_n}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^{\mathrm{n}} + \mathrm{e}_{\mathrm{uj-1}}^{\mathrm{n}}\right) + k_n\left(f\left(v_j^n\right) - f\left(V_j^n\right)\right)\right|.\end{aligned}$$

Since $(1-r_n) \geq 0$, it follows that

$$\begin{aligned}&\left\|E_u^{n+1}\right\| \leq (1-r_n)\left\|E_u^n\right\| + r_n\left\|E_u^n\right\| + L_1 k_n\left|v_j^n - V_j^n\right|\\ &\left\|E_u^{n+1}\right\| \leq \left\|E_u^n\right\| + Lk\left\|E_v^n\right\|\\ &\leq (1+kL)^{n+1}\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}\\ &\leq \exp((n+1)kL)\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}\end{aligned}$$

Thus

$$\left\|E_u^{n+1}\right\| \leq \exp\left(t_{n+1}L\right)\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}.$$

Similarly, we one can show that

$$\left\|E_v^{n+1}\right\| \leq \exp\left(t_{n+1}L\right)\max\left\{\left\|E_u^0\right\|, \left\|E_v^0\right\|\right\}.$$

Based on the stability definition [22] the Crank-Nicolson scheme is stable provided that $r \leq 1$

Theorem 7. *Under the condition of theorem (2), the Crank-Nicolson scheme converges with the order* $O\left(k+h^2\right)$, *where* $k = \max_{n\in N} k_n$.

Proof. Let $E_u^n = \left(e_{ui}^n, i = 1, 2, \ldots I-1\right), E_v^n = \left(e_{vi}^n, i = 1, 2, \ldots I-1\right)$ where $e_{ui}^n = u_i^n - U_i^n \quad , \quad e_{vi}^n = v_i^n - V_i^n,\ u_i^n = u\left(x_i, t_n\right) \quad , \quad v_i^n = v\left(x_i, t_n\right)$ is the exact solution of problem (1)

Suppose that $e_{ui}^0 = 0, e_{vi}^0 = 0 \quad \forall i = 0, 1, \ldots.$

To prove this theorem, the following inequalities should be held

$$\mathrm{e}_{\mathrm{uj}}^{\mathrm{n+1}} \leq C\left(k+h^2\right) \quad , \quad \mathrm{e}_{\mathrm{vj}}^{\mathrm{n+1}} \leq C\left(k+h^2\right) \quad , \quad n = 0, 1, \ldots.$$

Now, For $n = 1$, set $\left|\mathrm{e}_{\mathrm{uj}}^{1}\right| = \max_{1\leq i\leq I}\left|\mathrm{e}_{\mathrm{ui}}^{1}\right|$.

It clear that

$$\begin{aligned}\left|\mathrm{e}_{\mathrm{uj}}^{1}\right| &= (1+r_0)\left|\mathrm{e}_{\mathrm{uj}}^{1}\right| - \frac{r_0}{2}\left(\left|\mathrm{e}_{\mathrm{uj}}^{1}\right| + \left|\mathrm{e}_{\mathrm{uj}}^{1}\right|\right)\\ &\leq (1+r_0)\left|\mathrm{e}_{\mathrm{uj}}^{1}\right| - \frac{r_0}{2}\left(\left|\mathrm{e}_{\mathrm{uj+1}}^{1}\right| + \left|\mathrm{e}_{\mathrm{uj-1}}^{1}\right|\right)\\ &\leq \left|(1+\mathrm{r_0})\,\mathrm{e}_{\mathrm{uj}}^{1} - \frac{r_0}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^{1} + \mathrm{e}_{\mathrm{uj-1}}^{1}\right)\right|\\ &= \left|(1-r_0)\,\mathrm{e}_{\mathrm{uj}}^{0} + \frac{r_0}{2}\left(\mathrm{e}_{\mathrm{uj+1}}^{0} + \mathrm{e}_{\mathrm{uj-1}}^{0}\right) + k_0\left(f\left(v_j^0\right) - f\left(V_j^0\right)\right) + T_{ui}^0\right|.\end{aligned}$$

It follows that $\left|e_{\mathrm{uj}}^{1}\right| \leq Lk\left|e_{\mathrm{uj}}^{0}\right| + \left|T_{ui}^{0}\right| = \left|T_{ui}^{0}\right| \leq C\left(k+h^{2}\right)$,
where $L = \max\{L_1, L_2\}$.

Hence $\left|e_{\mathrm{uj}}^{1}\right| \leq C_1\left(k+h^{2}\right), i = 0,1,\ldots,I-1, \quad C_1 = C$.

Suppose that $\|E_u^s\| \leq C_s\left(k+h^2\right) \quad \|E_v^s\| \leq C_s\left(k+h^2\right) \quad s = 0,1,2\ldots,n\ C_s > 0$.

For $n+1$, set $\|E_u^n\| = \left|e_{\mathrm{uj}}^{\mathrm{n+1}}\right| = \max_{1\leq i\leq I}\left|e_{\mathrm{ui}}^{\mathrm{n+1}}\right|$

$$\begin{aligned}
\left|e_{uj}^{\mathrm{n+1}}\right| &= (1+r_n)\left|e_{\mathrm{uj}}^{\mathrm{n+1}}\right| - \frac{r_n}{2}\left(\left|e_{\mathrm{uj}}^{\mathrm{n+1}}\right| + \left|e_{\mathrm{uj}}^{\mathrm{n+1}}\right|\right)\\
&\leq (1+r_n)\left|e_{\mathrm{uj}}^{\mathrm{n+1}}\right| - \frac{r_n}{2}\left(\left|e_{\mathrm{uj+1}}^{\mathrm{n+1}}\right| + \left|e_{\mathrm{uj-1}}^{\mathrm{n+1}}\right|\right)\\
&\leq \left|(1+r_{\mathrm{n}})\, e_{\mathrm{uj}}^{\mathrm{n+1}} - \frac{r_{\mathrm{n}}}{2}\left(e_{\mathrm{uj+1}}^{\mathrm{n+1}} + e_{\mathrm{uj-1}}^{\mathrm{n+1}}\right)\right|\\
&= \left|(1-r_n)\, e_{uj}^{\mathrm{n}} + \frac{r_n}{2}\left(e_{\mathrm{uj+1}}^{\mathrm{n}} + e_{\mathrm{uj-1}}^{\mathrm{n}}\right) + k_n\left(f\left(v_j^n\right) - f\left(V_j^n\right)\right) + T_{ui}^{n+1/2}\right|.
\end{aligned}$$

Since $(1-r_n) \geq 0$, it follows that

$$\begin{aligned}
\left\|E_u^{n+1}\right\| &\leq (1-r_n)\left\|E_u^n\right\| + r_n\left\|E_u^n\right\| + L_1 k_n\left|v_j^n - V_j^n\right| + \left|T_{ui}^{n+1/2}\right|\\
\left\|E_u^{n+1}\right\| &\leq \left\|E_u^n\right\| + Lk\left\|E_v^n\right\| + \left|T_{ui}^{n+1/2}\right|\\
&\leq C_n\left(k+h^2\right) + kLC_n\left(k+h^2\right) + C\left(k+h^2\right)\\
&\leq (1+kL)C_n\left(k+h^2\right) + C\left(k+h^2\right)\\
&= \left[(1+kL)C_n + C\right]\left(k+h^2\right).
\end{aligned}$$

It follows that $\left\|E_u^{n+1}\right\| \leq C_{n+1}\left(k+h^2\right), n = 0,1,\ldots.$

In the same way, one can get

$$\left\|E_v^{n+1}\right\| \leq C_{n+1}\left(k+h^2\right), \quad n = 0,1,\ldots$$

Remark

- Clearly, the matrix $\left(I + \frac{r_n}{2}H\right)$ is diagonally dominated; hence, it is non-singular. It follows that the linear systems (26), (27) are uniquely solvable [28]
- The value of (NBUT) is dependent on the space and time steps.
- As it was proved in theorem (3), the Crank-Nicolson numerical scheme gives approximate solutions of the order of convergence $\mathrm{O}\left(k+h^2\right); k = \max_n k_n$, while, with the formula of time steps (25), the convergence rate becomes $\mathrm{O}\left(h^\alpha\right)$, as the space step approaches zero for $1 \leq \alpha \leq 2$. A similar convergence order is expected for (NBUT).

4. Numerical Experiments

In this section, the Crank-Nicolson scheme is used for two numerical experiments, with various grid sizes: $I = \{20, 40, 80, 160, 320\}$, taking $\alpha = 1, 2$. In addition, Matlab (R2020a) software is used for writing all the numerical computations codes. For each example, some tables and figures are presented to show the numerical results obtained by using the proposed scheme. Each table includes:

- For the first $m \in N$ such that the condition $\|U_h^m\|_\infty \geq 10^{15}$ and $\|V_h^m\|_\infty \geq 10^{15}$ holds, the value $T_h = t_m = \sum_{n=0}^{m} k_n$ is considered the (NBUT) for the studied problems.
- $E_h = |T_{2h} - T_h|$ is the error bond between T_{2h} and T_h.
- The order of numerical convergence to the (BUT), using the formula: $S_h = \log(E_{2h}/E_h)/\log 2$.
- The unit time of the central processor (CPUT) in seconds.

4.1. Examples

Example 1.

$$\left.\begin{array}{rl} u_t = u_{xx} + v^5, & v_t = v_{xx} + u^6,\ x \in (0,1),\ t \in (0,T) \\ u(0,t) = u(1,t) = 0, & v(0,t) = v(1,t) = 0,\ t \in (0,T) \\ u(x,0) = 70\left(x - x^2\right), & v(x,0) = 80\left(x - x^2\right),\ x \in (0,1) \end{array}\right\} \tag{31}$$

Example 2.

$$\left.\begin{array}{rl} u_t = u_{xx} + v^6 & v_t = v_{xx} + u^7,\ x \in (0,1),\ t \in (0,T) \\ u(0,t) = u(1,t) = 0, & v(0,t) = v(1,t) = 0,\ t \in (0,T) \\ u(x,0) = 60\left(1 - \mathrm{e}^{x^2 - x}\right), & v(x,0) = 70\left(1 - \mathrm{e}^{x^2 - x}\right),\ x \in (0,1) \end{array}\right\} \tag{32}$$

Table 1: Example 1, C.N. scheme: $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	3	7.6650E-04	0.362506	$\cdots$	$\cdots$
1/40	4	2.0966E-04	0.288462	5.5684E-04	$\cdots$
1/80	4	5.3939E-05	0.346336	1.5572E-04	1.8383
1/160	4	1.5557E-05	0.877538	3.8382E-05	2.0205
1/320	4	6.4259E-06	2.608468	9.1311E-06	2.0716

Table 2: Example 1, C.N. scheme: $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	3.9272E-05	0.230858	$\cdots$	$\cdots$
1/40	4	7.6677E-06	0.335248	3.1604E-05	$\cdots$
1/80	5	1.9252E-06	0.326475	5.7425E-06	2.4604
1/160	8	7.8273E-07	0.789320	1.1425E-06	2.3295
1/320	21	5.2762E-07	2.642523	2.5511E-07	2.1630

Table 3: Example 2, C.N. scheme: $\alpha = 1$

h	m	T_h	$CPUT$	E_h	S_h
1/20	3	8.3371E-04	0.401556	$\cdots$	$\cdots$
1/40	3	2.0886E-04	0.395284	6.2485E-04	$\cdots$
1/80	3	5.2830E-05	0.327884	1.5603E-04	2.0017
1/160	4	1.4065E-05	0.953267	3.8765E-05	2.0090
1/320	4	4.6714E-06	2.653295	9.3936E-06	2.0450

Table 4: Example 2, C.N. scheme: $\alpha = 2$

h	m	T_h	$CPUT$	E_h	S_h
1/20	4	4.8891E-05	0.223913	$\cdots$	$\cdots$
1/40	4	8.8974E-06	0.233469	3.9994E-05	$\cdots$
1/80	4	1.7888E-06	0.329078	7.1086E-06	2.4921
1/160	5	4.4920E-07	1.044031	1.3396E-06	2.4078
1/320	7	1.7424E-07	2.458527	2.7496E-07	2.2845

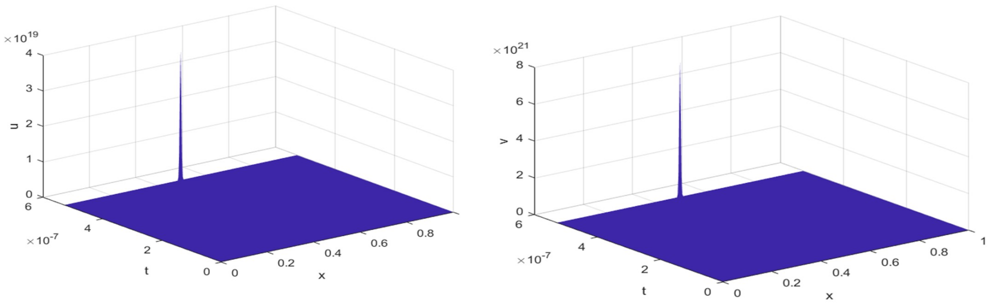

Figure 1: For example 1, (NBU) solution evolution over time using C.N. scheme, with $h = 320, \alpha = 2$.

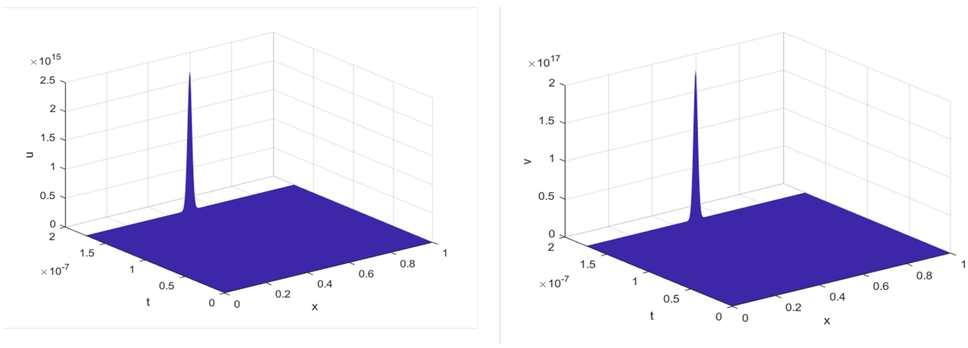

Figure 2: For example 2, (NBU) solution evolution over time using C.N. scheme, with $h = 320, \alpha = 2$.

4.2. Results and Discussion

In Tables 1-4 and Figures 1-2, the following observations can be made: Firstly,the (NBU) simultaneously occurs near the center point ($x = 0.5$), and this is consistent with the known (BU) results of the problem (1), see [15]. In addition, when the space steps are refined, the (BUT) error bounds decrease. In other words, the (NBUT) sequence T_h is convergent, since the space step is close to zero. Furthermore, the numerical convergence order of (NBUT) is $\mathrm{O}\left(h^{\alpha+\epsilon}\right)$, where $\epsilon > 0$.Also,a large value of α requires many iterations to achieve (BU) compared to a small value. Finally, when spatial steps are refined, one can see an increase in CPUT times.

5. Conclusions

This study employs the Crank-Nicolson scheme with a non-fixed time-stepping formula to identify (NBU) solutions and estimate the (NBUT) for a system of two coupled semi-linear heat equations subject to homogeneous Dirichlet boundary conditions. The consistency, stability, and convergence properties of the Crank-Nicolson scheme are rigorously examined. To validate the theoretical findings and determine the convergence rate of the NBUT, two numerical experiments are conducted. The numerical results, presented through tables and figures, demonstrate that the proposed method achieves a high degree of convergence and exhibits good computational efficiency. Future research directions may include exploring special approximations for nonlinear terms in the system to enhance the accuracy of the scheme, potentially achieving second-order truncation errors in both temporal and spatial dimensions.

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