EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6145
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Modifying Spectral Conjugate Gradient Method for Solving Unconstrained Optimization Problems

Kamal Jameel Murad[1,*], Salah Gazi Shareef[1]

[1] *Department of Mathematics, College of Science, University of Zakho, Kurdistan Region, Iraq*

Abstract. This paper proposes a new spectral conjugate gradient method for large-scale unconstrained optimization, designed to improve convergence efficiency by reducing both iteration counts and function evaluations. The method introduces a modified spectral coefficient and a new search direction formula that guarantees descent and sufficient descent conditions at every iteration, without increasing the per-iteration computational burden. Unlike existing methods such as the classical conjugate gradient algorithm, the proposed scheme integrates spectral scaling in a way that enhances direction quality and step stability. Theoretical analysis establishes global convergence under standard assumptions. Extensive numerical experiments on a diverse set of test problems demonstrate the superior performance of the proposed method over the classical conjugate gradient and spectral conjugate gradient methods, particularly in scenarios where fast convergence and low evaluation cost are critical. These results suggest that the proposed method offers a robust and computationally efficient alternative for solving unconstrained optimization problems. Future work will explore its application to structured and real-world large-scale problems.

2020 Mathematics Subject Classifications: 65K05, 46N10, 90C26, 90C30
Key Words and Phrases: Unconstrained Optimization, Spectral Conjugate Gradient, Global Convergence, Descent Conditions

1. Introduction

This paper addresses the unconstrained optimization problem:

$$\min\{f(x) : x \in \mathbb{R}^n\}, \tag{1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives, and its gradient $\nabla f(x) = g(x)$ is available for evaluation. Unconstrained optimization plays a crucial role in a wide array of applications, including machine learning [1–3], image processing [4–6], and portfolio optimization, as well as in fields such as economics, engineering, management science, and industrial applications [7–11]. Various methods have been proposed for

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6145

Email addresses: kamal.murad@uoz.edu.krd (K. J. Murad),
salah.shareef@uoz.edu.krd (S. G. Shareef)

solving problem (1), including Newton-type methods, quasi-Newton methods, spectral gradient methods, and conjugate gradient (CG) methods [12–14]. In addition, derivative-free optimization techniques such as the Nelder–Mead simplex method, generalized simulated annealing, and genetic algorithms are also used [15–17]. This paper specifically focuses on CG methods and introduces a new spectral conjugate gradient method designed to efficiently tackle large-scale instances of problem (1).

Recently, geometric optimization approaches have garnered increasing interest. The application of Riemannian geometry to optimization problems, particularly on matrix manifolds, has led to the development of structure-preserving methods. For example, Arif et al. [18] proposed an Extended Hamiltonian Algorithm (EHA) for solving the Algebraic Lyapunov Equation on the manifold of positive-definite Hermitian matrices. This approach utilizes a geodesic-based cost function and exhibits superior convergence compared to traditional gradient-based methods. In addition, studies on the geometry of statistical manifolds, such as the Freund manifold [19] and Gamma exponential manifold [20], have provided insights into the role of geodesic instability and Jacobi fields in understanding optimization paths in curved spaces. These geometric insights inspire us to enhance classical CG methods by incorporating curvature-aware spectral adjustments to improve both convergence and robustness, particularly in high-dimensional and ill-conditioned settings. Motivated by such developments, we propose a new spectral conjugate gradient method tailored to large-scale optimization problems. Our method incorporates adaptive spectral parameters, while maintaining the simplicity and memory efficiency characteristic of traditional CG schemes.

CG methods are widely recognized for their simplicity, efficiency, and low memory requirements, making them highly suitable for large-scale optimization problems. The standard iteration scheme of classical CG methods is given by:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

where d_k denotes the search direction, which is defined as:

$$d_{k+1} = \begin{cases} -g_0, & k = 0, \\ -g_{k+1} + \beta_k d_k, & k \geq 1. \end{cases} \tag{3}$$

Here, $g_k = g(x_k)$, β_k is the conjugate parameter, and α_k is the step size, which can be computed using exact or inexact line search techniques. Due to the computational cost associated with exact line search, inexact line searches such as the Wolfe conditions are often employed:

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k, \end{cases} \tag{4}$$

or the strong Wolfe conditions:

$$\begin{cases} f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k, \\ |g(x_k + \alpha_k d_k)^T d_k| \leq \sigma |g_k^T d_k|. \end{cases} \tag{5}$$

In these conditions, the parameters δ and σ are chosen such that $0 < \delta < \sigma < 1$.

Different choices of the parameter β_k lead to various CG methods, the most well-known of which include Hestenes-Stiefel (HS) [21], Fletcher-Reeves (FR) [22], Polak-Ribière-Polyak (PRP) [23, 24], Liu-Storey (LS) [25], Dai-Yuan (DY) [26], and Conjugate-Descent (CD) [27]. For example:

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{y_k^T d_k}, \tag{6}$$

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \tag{7}$$

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}, \tag{8}$$

$$\beta_k^{LS} = \frac{g_{k+1}^T y_k}{-g_k^T d_k}, \tag{9}$$

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{y_k^T d_k}, \tag{10}$$

$$\beta_k^{CD} = \frac{\|g_{k+1}\|^2}{-g_k^T d_k}, \tag{11}$$

where $y_k = g_{k+1} - g_k$ and $\|\cdot\|$ denotes the Euclidean norm.

Theoretically, for strongly convex quadratic functions, all choices of β_k are equivalent when exact minimization is applied. However, for non-quadratic objective functions, the performance varies significantly depending on the choice of β_k [28]. Among these methods, the PRP method has been shown to be more computationally efficient than the FR method and has global convergence properties under exact line search when the objective function is convex [24]. Several variations of CG methods have been proposed to improve their performance further [29–34].

Building on the spectral gradient approach [35], Bergin et al. [36] introduced the spectral conjugate gradient (B-SCG) method, where the search direction is defined by:

$$d_{k+1} = \begin{cases} -g_0, & k = 0, \\ -\theta_k g_{k+1} + \beta_k d_k, & k \geq 1. \end{cases} \tag{12}$$

Here, θ_k is the spectral parameter, and β_k is the conjugate parameter defined as:

$$\theta_k = \frac{v_k^T v_k}{v_k^T y_k}, \quad \beta_k = \frac{(\theta_k y_k - v_k)^T g_{k+1}}{d_k^T y_k}. \tag{13}$$

The SCG method reduces to the classical CG method when $\theta_k = 1$ and to the spectral gradient method when $\beta_k = 0$. The SCG [37] was modified by Yu et al. [38] in order to achieve the descent directions. Other modifications based on different descent conditions have been suggested by Wan et al. [39] and Zhang et al. [40], focusing on PRP and FR

variants. Moreover, leveraging the strong convergence of the Newton method, Andrei [41] introduced an accelerated CG approach that integrates Newton's method to improve CG performance.

Recently, modern CG variants and alternative optimization methods have been proposed to enhance robustness and efficiency, particularly for large-scale or noisy problems. Hybrid CG methods combine conjugate directions with quasi-Newton updates or restart strategies [28, 42], while stochastic CG techniques address gradient uncertainty in data-driven applications [43]. Moreover, quasi-Newton approaches such as the limited-memory BFGS (L-BFGS) method [44] are widely used due to their fast convergence and low memory requirements. Compared to these methods, our approach preserves the simplicity and memory efficiency of classical CG methods while incorporating spectral features that improve convergence behavior without relying on Hessian approximations or stochastic estimates.

In this paper, we propose a novel spectral conjugate gradient method that incorporates a modified gradient and an alternative search direction, as given by Eq. (12). Our method ensures that the descent and sufficient descent properties are satisfied per iteration, without adding computational overhead. Under appropriate assumptions, we establish the global convergence of the proposed method.

The remainder of this paper is organized as follows. In Section 2, we introduce the proposed spectral conjugate gradient algorithm. Section 3 presents the analysis of the sufficient descent property and the global convergence of the method. In Section 4, we provide numerical experiments that compare the proposed method with the HS-CG and B-SCG methods. Finally, Section 5 summarizes the main contributions and outlines directions for future research.

2. The New Spectral Conjugate Gradient Algorithm

In this section, we introduce a new spectral conjugate gradient (New-SCG) method to solve unconstrained optimization problems of the form (1), utilizing a modified gradient-difference vector inspired by [45].

We define the adjusted gradient-difference vector as follows:

$$y_k^* = y_k + \frac{(0.2 - \rho_k)}{(1 - \rho_k)} \cdot \frac{\|v_k\| - 2\sqrt{\epsilon_m}(1 + \|x_{k+1}\|)}{2\sqrt{\epsilon_m}(1 + \|x_{k+1}\|)} \cdot y_k, \tag{14}$$

where $v_k = x_{k+1} - x_k$, and $y_k = g_{k+1} - g_k$. Here, $0.2 < \rho_k < 1$ and ϵ_m represents machine precision, typically around 10^{-16}.

The new search direction is given by:

$$d_{k+1}^{\text{New-SCG}} = \begin{cases} -g_1, & k = 0, \\ -\theta_k g_{k+1} + \beta_k d_k, & k \geq 1. \end{cases} \tag{15}$$

where θ_k is a spectral scaling parameter, and β_k is a conjugate gradient parameter.

Dai and Liao [46] proposed the classical conjugacy condition:

$$d_{k+1}^T y_k = -t g_{k+1}^T v_k, \quad \text{where } t \geq 0, \tag{16}$$

which we modify by substituting y_k with y_k^*, yielding:

$$d_{k+1}^T y_k = \frac{-t g_{k+1}^T v_k}{1 + M_1 M_2}, \tag{17}$$

with:

$$M_1 = \frac{0.2 - \rho_k}{1 - \rho_k}, \quad M_2 = \frac{\|v_k\| - 2\sqrt{\epsilon_m}(1 + \|x_{k+1}\|)}{2\sqrt{\epsilon_m}(1 + \|x_{k+1}\|)}. \tag{18}$$

Multiplying both sides of Eq. (15) by y_k and using Eq. (17), we define the spectral parameter as:

$$\theta_k = \frac{\beta_k d_k^T y_k + \frac{t g_{k+1}^T v_k}{1 + M_1 M_2}}{g_{k+1}^T y_k}. \tag{19}$$

In particular, if we take $\beta_k = \beta_k^{HS}$ (Hestenes-Stiefel formula), then:

$$\theta_k = 1 + \frac{t g_{k+1}^T v_k}{(1 + M_1 M_2) g_{k+1}^T y_k}. \tag{20}$$

Remark 1. *When $t = 0$, the proposed spectral parameter in Eq.* (20) *reduces to $\theta_k = 1$, which implies that the search direction becomes:*

$$d_{k+1}^{New\text{-}SCG} = -g_{k+1} + \beta_k^{HS} d_k, \tag{21}$$

i.e., the method reduces to the classical HS conjugate gradient method. Therefore, the proposed algorithm can be seen as a spectral extension of the HS method.

Based on the above analysis, we propose the following algorithm:

Algorithm 1 New Spectral Conjugate Gradient Algorithm

Input: Initial guess x_0, tolerance ϵ, parameters $t \geq 0$, $0.2 < \rho_k < 1$
Compute $g_0 = \nabla f(x_0)$, set $d_0 = -g_0$, $k = 0$
while $\|g_k\| > \epsilon$ **do**
    If $\|g_k\| = 0$, then stop; otherwise, proceed to Step 5.
    Compute the step size α_k by minimizing $f(x_k + \alpha_k d_k)$.
    $x_{k+1} = x_k + \alpha_k d_k$
    $g_{k+1} = \nabla f(x_{k+1})$
    $y_k = g_{k+1} - g_k$, $v_k = x_{k+1} - x_k$
    $M_1 = \frac{(0.2-\rho_k)}{(1-\rho_k)}$
    $M_2 = \frac{\|v_k\|-2\sqrt{\epsilon_m}(1+\|x_{k+1}\|)}{2\sqrt{\epsilon_m}(1+\|x_{k+1}\|)}$
    $\beta_k = \frac{g_{k+1}^T y_k}{y_k^T d_k}$
    $\theta_k = 1 + \frac{t g_{k+1}^T v_k}{(1+M_1 M_2) g_{k+1}^T y_k}$
    $d_{k+1} = -\theta_k g_{k+1} + \beta_k d_k$
    If $\|g_{k+1}\|^2 \leq |g_k^T g_{k+1}|/0.2$, go to Step 4; otherwise, set $k = k + 1$ and repeat Step 5.
end while
Output: x_k

3. Algorithm Characteristics

This section discusses the descent properties, sufficient descent conditions, and global convergence of the proposed algorithm.

Theorem 1: If the search direction d_{k+1} is generated by equation (15), where β_k is defined by equation (6) and the spectral parameter θ_k is given by equation (20), then:

$$g_{k+1}^T d_{k+1} \leq 0. \tag{22}$$

Proof: By multiplying both sides of equation (15) by g_{k+1}, we obtain:

$$g_{k+1}^T d_{k+1} = -\left(1 + \frac{t g_{k+1}^T v_k}{(1 + M_1 M_2) g_{k+1}^T y_k}\right) g_{k+1}^T g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} g_{k+1}^T d_k. \tag{23}$$

Simplifying this expression, we get:

$$g_{k+1}^T d_{k+1} = -g_{k+1}^T g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} g_{k+1}^T d_k - \frac{t g_{k+1}^T v_k g_{k+1}^T g_{k+1}}{(1 + M_1 M_2) g_{k+1}^T y_k}. \tag{24}$$

In the case of an exact search direction, the descent condition is satisfied:

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2 \leq 0. \tag{25}$$

For an inexact search direction ($g_{k+1}^T d_k \neq 0$), we use the following inequalities:

$$g_{k+1}^T v_k \leq v_k^T y_k, \quad g_{k+1}^T y_k \leq \|g_{k+1}\|\|y_k\|. \tag{26}$$

From these, we derive:

$$g_{k+1}^T d_{k+1} \leq -\left(1 - \frac{\|y_k\|}{\|g_{k+1}\|}\right)\|g_{k+1}\|^2 - \left(\frac{t g_{k+1}^T v_k}{(1+M_1M_2) g_{k+1}^T y_k}\right) g_{k+1}^T g_{k+1}. \tag{27}$$

Since $0 \leq \frac{\|y_k\|}{\|g_{k+1}\|} \leq 1$ and using the inequalities in (26), we can rewrite equation (27) as:

$$g_{k+1}^T d_{k+1} \leq -\frac{t v_k^T y_k \|g_{k+1}\|}{(1+M_1M_2)\|y_k\|}. \tag{28}$$

Since t, $v_k^T y_k$, $\|g_{k+1}\|$, and $\|y_k\|$ are non-negative, and $M_1 < 0$, $M_2 < 0$ imply that $M_1 M_2 > 0$, it follows that:

$$g_{k+1}^T d_{k+1} \leq 0. \tag{29}$$

This completes the proof.

Theorem 2: The search direction d_{k+1}, defined by equation (15), where β_k is from equation (6) and the spectral parameter θ_k is given by equation (20), satisfies:

$$g_{k+1}^T d_{k+1} \leq -\mu\|g_{k+1}\|^2, \tag{30}$$

where $\mu > 0$ is a positive constant.

Proof: From Theorem 1 and equation (28), we have:

$$g_{k+1}^T d_{k+1} \leq -\left[\frac{t v_k^T y_k}{(1+M_1M_2)\|y_k\|\|g_{k+1}\|}\right]\|g_{k+1}\|^2. \tag{31}$$

Let:

$$\mu = \frac{t v_k^T y_k}{(1+M_1M_2)\|y_k\|\|g_{k+1}\|} > 0. \tag{32}$$

This completes the proof.

The global convergence of the algorithm is guaranteed under the following assumptions:

Assumptions I

- The level set $S = \{x \mid f(x) \leq f(x_0)\}$ is bounded.
- The function f is continuously differentiable in some neighborhood N and its gradient satisfies the Lipschitz condition:

$$\|g(x) - g(y)\| \leq L\|x - y\|, \quad \forall x, y \in S. \tag{33}$$

- There exists a constant $b > 0$ such that:

$$\|g(x)\| \leq b, \quad \forall x \in S. \tag{34}$$

Using these assumptions, we derive the following result:

Lemma 1: [46] Let Assumptions (I) hold. Consider the methods (1) and (2), where d_{k+1} is a descent direction and α_k satisfies the standard Wolfe line search. If

$$\sum_{k\geq 1} \frac{1}{\|d_{k+1}\|^2} = \infty, \tag{35}$$

then it follows that

$$\lim_{k\to\infty} \inf \|g_{k+1}\| = 0. \tag{36}$$

Theorem 3: Assuming that Assumptions (I) hold, and the sequences $\{x_k\}$, $\{d_k\}$, $\{g_k\}$, and $\{\alpha_k\}$ are generated by Algorithm 1, we conclude that

$$\lim_{k\to\infty} \inf \|g_{k+1}\| = 0. \tag{37}$$

Proof: Starting from equation (15) and using the parameters β_k from equation (6) and the spectral parameter θ_k from equation (20), we obtain the following expression:

$$\|d_{k+1}\| = \| - \left(1 + \frac{t g_{k+1}^T v_k}{(1 + M_1 M_2) g_{k+1}^T y_k}\right) g_{k+1} + \frac{g_{k+1}^T y_k}{d_k^T y_k} d_k \|. \tag{38}$$

This leads to the following bound on $\|d_{k+1}\|$:

$$\|d_{k+1}\| \leq \left|1 + \frac{t g_{k+1}^T v_k}{(1 + M_1 M_2) g_{k+1}^T y_k}\right| \|g_{k+1}\| + \left|\frac{g_{k+1}^T y_k}{d_k^T y_k}\right| \|d_k\|. \tag{39}$$

From equation (26), we have:

$$\|d_{k+1}\| \leq \left|1 + \frac{t}{(1 + M_1 M_2)}\right| \|v_k\| + \left|\frac{\|g_{k+1}\|\|y_k\|}{d_k^T y_k}\right| \|d_k\|. \tag{40}$$

Using equation (34) and the Lipschitz condition $\|y_k\| \leq L\|v_k\|$, along with the fact that $y_k^T v_k \geq \vartheta \|v_k\|^2$, we obtain:

$$\|d_{k+1}\| \leq \left|1 + \frac{t}{(1 + M_1 M_2)}\right| \|v_k\| + \frac{\alpha_k b L \|v_k\|}{\vartheta \|v_k\|^2} \|d_k\|, \tag{41}$$

which simplifies to:

$$\|d_{k+1}\| \leq \left|1 + \frac{t}{(1 + M_1 M_2)}\right| \|v_k\| + \frac{bL}{\vartheta}. \tag{42}$$

Since $\|v_k\| = \|x - x_k\|$, we define $D = \max\{\|x - x_k\|\}$ for all $x, x_k \in \mathbb{R}$. Thus, equation (35) becomes:

$$\|d_{k+1}\| \leq b + \frac{tD}{(1 + M_1 M_2)} + \frac{bL}{\vartheta} = \phi. \tag{43}$$

It follows that:

$$\sum_{k\geq 1} \frac{1}{\|d_{k+1}\|^2} \geq \sum_{k\geq 1} \frac{1}{\phi^2} = \infty. \tag{44}$$

By applying Lemma 1, we conclude that:

$$\lim_{k\to\infty} \inf \|g_{k+1}\| = 0, \tag{45}$$

which completes the proof.

4. Numerical Results for Unconstrained Optimization

This section presents the results of implementing the New-SCG algorithm for solving unconstrained optimization problems. A comparative analysis is conducted between the proposed algorithm and the classical HS and B-SCG methods. The tests were carried out using well-established benchmark functions [47] with varying problem dimensions. All codes were developed in `FORTRAN 95`, and the line search routine employed cubic interpolation based on both function and gradient evaluations. The letter 'F' in the tables indicates that the method failed to locate the minimum.

Table 1 reports the performance of the algorithms in terms of the number of iterations (NOI) and the number of function evaluations (NOF). The summary in Table 2 further confirms that the New-SCG algorithm outperforms both HS and B-SCG methods with respect to NOI and NOF metrics.

All tests were initialized from standard starting points, and the comparative numerical results are visualized in Figures 1a and 1b. The relative performance of the New-SCG, HS, and B-SCG methods is assessed using performance profiles in accordance with the approach described in [48].

Let $S = 2$ represent the set of solvers being compared, and $p = 72$ denote the total number of test problems. Let $l_{p,s}$ represent the number of objective function evaluations required by solver s to solve problem p. The performance ratio is defined as

$$r_{p,s} = \frac{l_{p,s}}{l_p^*},$$

where l_p^* is the minimum number of evaluations among all solvers:

$$l_p^* = \min\{l_{p,s} : s \in S\}.$$

It is evident that $r_{p,s} \geq 1$ for all p and s. If a solver fails to solve a problem, the ratio $r_{p,s}$ is assigned a large number M.

The performance profile for each solver s is given by the cumulative distribution function of the performance ratio $r_{p,s}$:

$$p_\tau(\tau) = \frac{|\{p \in P : r_{p,s} \leq \tau\}|}{n_p}.$$

Here, $p_s(1)$ represents the percentage of problems for which solver s is the best.

Table 1: Comparison of Performance Metrics (NOI and NOF) for Different Methods Across Various Test Functions and Dimensions

Method		Classical HS		B-SCG		New-SCG	
Test Function	Dimensions	NOI	NOF	NOI	NOF	NOI	NOF
Wolfe	4	11	24	11	29	11	25
	10	32	65	32	71	32	65
	100	49	99	49	100	44	89
	500	52	105	52	107	46	93
	1000	70	141	60	122	50	101
	5000	165	348	168	338	129	270
G-Central	4	22	159	16	77	21	147
	10	22	159	16	103	22	160
	100	22	159	19	103	22	160
	500	23	171	28	153	22	160
	1000	23	171	34	186	22	160
	5000	28	248	52	287	26	214
Non-diagonal	4	24	64	30	223	24	64
	10	26	72	29	238	26	72
	100	29	79	27	285	29	79
	500	F	F	27	335	29	79
	1000	29	79	30	385	29	79
	5000	30	81	30	420	30	81
Powell	4	37	102	47	231	34	93
	10	37	102	47	231	34	93
	100	40	117	47	231	34	93
	500	44	136	47	231	34	93
	1000	44	136	51	254	34	93
	5000	44	136	55	280	34	93
Rosen	4	30	83	30	83	16	44
	10	30	83	30	83	16	44
	100	30	83	30	83	17	47
	500	30	83	30	83	17	47
	1000	30	83	30	83	17	47
	5000	30	83	30	83	17	47
Miele	4	28	85	40	138	28	89

	10	31	102	43	150	28	89
	100	33	114	49	186	28	89
	500	40	146	55	218	29	102
	1000	46	176	55	218	29	102
	5000	54	211	61	247	30	105
Wood	4	30	68	30	68	26	60
	10	30	68	30	68	26	60
	100	30	68	30	68	26	60
	500	30	68	30	68	26	60
	1000	30	68	30	68	26	60
	5000	30	68	30	68	27	62
Sum	4	3	11	3	11	3	11
	10	6	34	6	26	6	28
	100	14	81	17	88	15	89
	500	21	124	28	162	20	104
	1000	23	128	42	199	25	139
	5000	31	159	43	227	28	117
Edger	4	5	14	5	14	5	14
	10	5	14	5	14	5	14
	100	5	14	5	14	5	14
	500	6	16	6	16	5	14
	1000	6	16	6	16	5	14
	5000	6	16	6	16	6	17
Shallow	4	8	21	8	21	8	21
	10	8	21	8	21	8	21
	100	8	21	8	21	8	21
	500	8	21	8	21	8	21
	1000	9	24	9	24	8	21
	5000	9	24	9	24	8	21
Cubic	4	12	35	22	141	12	35
	10	13	37	22	141	13	37
	100	13	37	22	141	13	37
	500	13	37	22	141	13	37
	1000	13	37	22	141	13	37
	5000	13	37	22	141	13	37
Beale	4	11	28	11	28	11	28
	10	11	28	11	28	11	28
	100	12	30	12	30	12	30
	500	12	30	12	30	12	30
	1000	12	30	12	30	12	30
	5000	12	30	12	30	12	30
Total		1852	5927	2091	9040	1570	4967

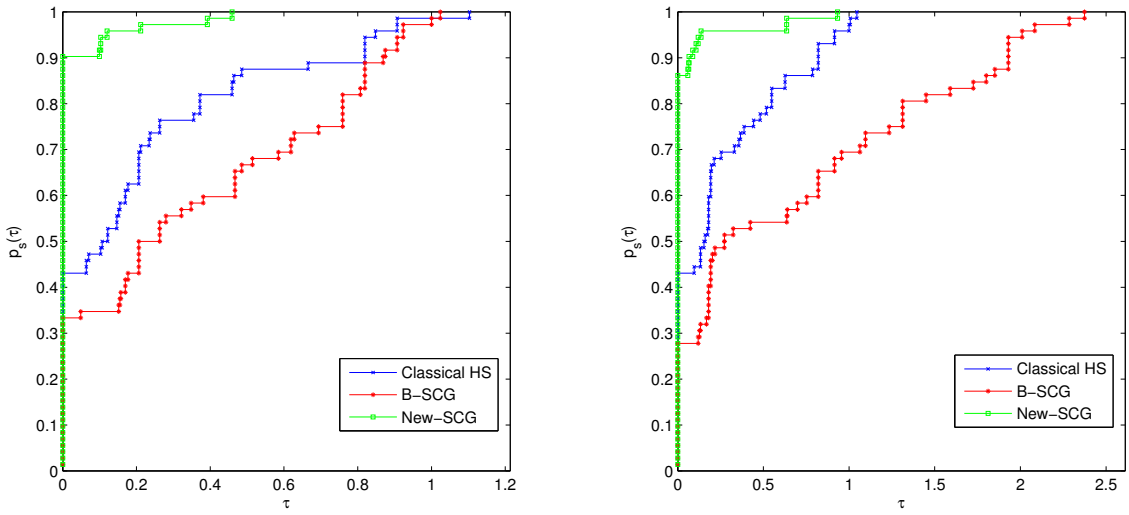
Table 2: Comparing the Rate of Improvement between the New-SCG Method and the Classical HS and B-SCG Methods.

Methods	**NOI (%)**	**NOF (%)**
Classical HS	100.00	100.00
New-SCG	84.77	83.80
B-SCG	100.00	100.00
New-SCG	75.08	54.94
Rate of Improvement (%)		
Compared with	**NOI**	**NOF**
Classical HS	15.23	16.20
B-SCG	24.92	45.06

The table 2 compares the performance of the proposed New-SCG algorithm with both the classical HS and the B-SCG methods in terms of the NOI and the NOF. The analysis of the data reveals the following improvements:

- **Compared with Classical HS:**
  - **Improvement in NOI:** The New-SCG algorithm achieves a 15.23% reduction in the NOI, indicating faster convergence.
  - **Improvement in NOF:** There is a 16.20% reduction in the NOF, suggesting higher computational efficiency.
  - **Overall Average Improvement:** The mean of both improvements (NOI and NOF) is approximately 15.72%, demonstrating the New-SCG method's significant advantage over the classical HS method.
- **Compared with B-SCG:**
  - **Improvement in NOI:** The New-SCG method reduces the NOI by 24.92%, showing a more pronounced convergence speedup.
  - **Improvement in NOF:** The reduction in function evaluations reaches 45.06%, highlighting a substantial gain in computational cost.
  - **Overall Average Improvement:** The average of both metrics yields an overall improvement of approximately 34.99%, clearly indicating that the New-SCG method outperforms the B-SCG method in both convergence rate and computational efficiency.

These results confirm that the proposed New-SCG algorithm offers consistent and notable improvements over both classical HS and B-SCG methods. Its reduced number of iterations and function evaluations collectively contribute to faster and more cost-effective optimization performance.



(a) Performance profiles based on the NOI.

(b) Performance profiles based on the NOF.

Figure 1: Performance profiles of the New method.

5. Conclusion

This paper proposed a novel search method, the New-SCG algorithm, designed to enhance the efficiency of large-scale unconstrained optimization problems. The theoretical properties of the method were thoroughly analyzed, including proofs for both descent and sufficient descent conditions and global convergent, establishing its robustness. Numerical evaluations demonstrated that the New-SCG algorithm significantly outperforms classical optimization methods such as HS and B-SCG methods. Specifically, the New-SCG algorithm achieved a 15.23% reduction in the NOI and a 16.20% reduction in the NOF when compared to the HS method. In comparison to B-SCG, the New-SCG method showed even more substantial improvements, with reductions of 24.92% in NOI and 45.06% in NOF. These findings underscore the superior efficiency of the New-SCG method in reducing computational costs while enhancing convergence speed. To provide a more comprehensive assessment, future studies will expand the evaluation of the New-SCG algorithm by applying it to non-convex and non-smooth optimization problems, which are typically more complex both theoretically and computationally. Furthermore, comparisons will be made with other optimization techniques widely used in real-world applications, such as in engineering design, control systems, and machine learning. This broader scope will help evaluate the algorithm's general applicability in solving diverse, real-world optimization problems. While this study focuses primarily on the theoretical and numerical analysis of the New-SCG algorithm, future research will explore its practical implications, especially in engineering fields. For example, applications in structural optimization, process

control, and large-scale machine learning models could benefit from the improved computational efficiency and reduced cost associated with this method. In addition to these practical considerations, future work will address the limitations of the proposed algorithm. In particular, further analysis is needed on the algorithm's performance in the presence of ill-conditioned optimization problems, where traditional methods may struggle. Future versions of the New-SCG algorithm will also seek to extend its applicability to constrained optimization problems and non-smooth objective functions, ensuring that it can handle a broader class of problems encountered in engineering and industrial applications. In summary, the New-SCG algorithm presents a promising advancement in large-scale unconstrained optimization. With its superior computational efficiency and theoretical foundation, it holds significant potential for a wide range of practical applications, while offering several avenues for future enhancement and refinement.

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