EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6146
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Deducing Trigonometric Functions from Differential Equations: An Educational Perspective

Sameen Ahmed Khan[1], Manisha M. Kankarej[2,*],
Mohammad Nazrul Islam Khan[3]

[1] *Department of Mathematics and Sciences, College of Arts and Applied Sciences, Dhofar University, Salalah, Sultanate of Oman*
[2] *Rochester Institute of Technology, Dubai, UAE*
[3] *Department of Computer Engineering, College of Computer, Qassim University, Saudi Arabia*

Abstract. We derive the two trigonometric functions, sine and cosine using the qualitative properties of differential equations and further conclude that the two functions are periodic. Some properties such as the sine/cosine of sums and differences of two angles are also derived from the defining differential equations. The oscillations of the solutions of second-order differential equations are covered along with the related theorems and techniques. We also derive the oscillatory behaviour of the Bessel functions from its defining differential equation.

2020 Mathematics Subject Classifications: 53D10, 53C25, 53C15, 58A30

Key Words and Phrases: Trigonometric functions, differential equations, qualitative theory of differential equations, oscillation phenomenon, sturm separation theorem, sturm comparison theorem, orthogonal polynomials

1. Introduction

Trigonometry is a branch of mathematics that studies the relationships between the angles and sides of triangles. It is widely used in geometry, physics, engineering, astronomy, and many other fields. The origins of trigonometry can be traced back to ancient civilizations such as the Babylonians, Egyptians and Indians, who used basic trigonometric concepts for practical purposes [1]. The trigonometric functions are traditionally introduced using the geometric approach of right-angled triangles. A more formal approach is done using the unit circle [2–5]. We know that the trigonometric functions are solutions of differential equations [6–9]. Trigonometric arise as solutions of linear as well partial differential equations [10–15]. Individual trigonometric ratios can be calculated using geometric

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6146

Email addresses: rohelakhan@yahoo.com (S. A. Khan),
manisha.kankarej@gmail.com (M. M. Kankarej), m.nazrul@qu.edu.sa (M. N. I. Khan)

techniques [16] and algebraic techniques [17]. Trigonometric ratios can also be used as a tool to teach irrational numbers [18].

As trigonometric functions are solutions of differential equations, it is natural to explore them starting with a differential equation [6, 7, 9]. In this article, we use the qualitative properties of differential equations to deduce the periodicity and other properties of the two trigonometric functions, sine and cosine respectively. First, we deduce the sine and cosine functions along with their basic properties from the differential equation, $y''+y=0$. Then, we see that such deductions, particularly the oscillation phenomenon can be extended to the second-order linear differential equations under certain conditions. The related theorems and techniques are done in detail. We also consider the classical orthogonal polynomial, which are governed by second-order linear differential equations.

Teaching trigonometry from its defining differential equations is a valuable but advanced approach that offers a deeper, more rigorous understanding of the functions than the traditional geometrical methods. It connects trigonometry directly to calculus.

2. The Sine and Cosine Functions from $y'' + y = 0$

In this article, we will be dealing with the second-order linear differential equations, which in the nonhomogeneous case can be written as

$$y'' + P(x)y' + Q(x)y = R(x)\,, \tag{1}$$

where $P(x)$, $Q(x)$ and $R(x)$ are continuous. If $R(x) = 0$, the equation is said to be homogeneous. In this article, we will be dealing only with the second-order linear homogeneous differential equations.

A second-order linear homogeneous differential equation has two linearly independent solutions, say $y_1(x)$ and $y_2(x)$. Two functions are said to be *linearly independent*, if one is not a multiple of the other. The linear independence of $y_1(x)$ and $y_2(x)$ is established from their Wronskian

$$\begin{aligned} W(x) &= W[y_1(x)\,, y_2(x)] \\ &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= y_1(x)y_2'(x) - y_1'(x)y_2(x)\,. \end{aligned} \tag{2}$$

If the Wronskian is *not* zero, the two solutions $y_1(x)$ and $y_2(x)$ are linearly independent. The Wronskian is named in honour of the Polish mathematician, Józef Maria Hoëné-Wroński (1776-1853). A *general solution* is obtained by taking a linear combination, $y = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants. The *unique solution* is determined by fixing the values of the arbitrary constants, c_1 and c_2 by an additional input, which is the set of initial conditions. The initial conditions are the values of the solutions, $y(x_0) = y_0$ and their derivatives, $y_0'(x_0) = y_0'$ respectively. Equation (1) has a unique solution, which is summarised in the following theorem.

Theorem 1. *Existence and Uniqueness Theorem: Let $P(x)$, $Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a\,,b]$. If x_0 is any point in $[a\,,b]$, and if y_0 and y_0' are any numbers whatever, then*

$$y'' + P(x)y' + Q(x)y = R(x)$$

has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y_0'(x_0) = y_0'$.

This theorem along with the Wronskian is sufficient to deduce the properties of the two trigonometric functions, sine and cosine. Let us consider the second-order linear homogeneous differential equation

$$y'' + y = 0\,, \tag{3}$$

with the initial conditions $y_1(0) = 0$, $y_1'(0) = 1$ and $y_2(0) = 1$, $y_2'(0) = 0$. We use only the qualitative properties of the differential equations to deduce the trigonometric functions as well as some of their properties. Theorem (1) ensures that, this equation has a unique solution. Let the two linearly independent solutions be $y_1 = s(x)$ and $y_2 = c(x)$ with the initial conditions $s(0) = 0$, $s'(0) = 1$ and $c(0) = 1$, $c'(0) = 0$. From the initial conditions, the graph of $s(x)$ starts at origin with slope of unity. From the defining equation Eq. (3), $s''(x) = -s(x)$. So, $s''(x)$ is a negative number whenever $s(x)$ is above the x-axis. Furthermore, $s''(x)$ is a negative number, which increases in magnitude as the curve $s(x)$ rises. Note, that $s''(x)$ is the rate of change of the slope $s'(x)$. This slope decreases at an increasing rate and it must reach zero at some point $x = x_0$. With further increase in x, the curve falls towards the x-axis, $s'(x)$ decreases at a decreasing rate. The curve crosses the x-axis at the point $x = p$. The value of p will be later determined to be π. As $s''(x)$ depends only on $s(x)$, the graph between $x = 0$ and $x = p$ is symmetric about the line $x = x_0$. So, $x_0 = p/2$ and $s'(p) = -1$. The maximum of $s(x)$ is at $x = p/2$. A similar line of arguments leads to the conclusion that the portion of the curve $x \in (p\,, 2p)$ is an inverted replica of the first arch, $x \in (0\,, p)$. The pair of arches (one above and one below the x-axis) are found to repeat indefinitely after every $2p$. Thus, $s(x)$ is periodic, $s(x) = s(x + 2p)$. With the initial conditions, $c(0) = 1$, $c'(0) = 0$, we note that the graph of $c(x)$ starts at the point $(0, 1)$ with slope zero. Using the same reasoning, we conclude that the graph of $c(x)$ also oscillates about the x-axis with a periodicity $2\tilde{p}$. Later, we shall see that $\tilde{p} = p = \pi$, the transcendental number from the circle.

In order to interconnect $s(x)$ and $c(x)$ (so also p and $\tilde{p}$), we differentiate Eq. (3) and obtain $y''' + y' = 0$ or $(y')'' + (y') = 0$. Consequently, the derivative of any solution of Eq. (3) is also a solution. Thus $s'(x)$ and $c(x)$ are both solutions of Eq. (3). So also $c'(x)$ and $s(x)$ are both solutions of Eq. (3). From the initial conditions, $s(0) = 0$, $s'(0) = 1$ and $c(0) = 1$, $c'(0) = 0$, along with Theorem (1), we conclude

$$\begin{aligned} s'(x) &= c(x) \\ c'(x) &= -s(x)\,. \end{aligned} \tag{4}$$

If $f(x)$ is a solution of $y'' + y = 0$, then $f(-x)$ is also a solution. From this, we conclude that

$$\begin{aligned} s(-x) &= -s(x) \\ c(-x) &= c(x)\,. \end{aligned} \tag{5}$$

Thus, we have established that $s(x)$ is an *odd* function and $c(x)$ is an *even* function.

The Taylor series or expansion of a real function $f(x)$ that is infinitely differentiable about a point $x = a$ is given by

$$\begin{aligned} f(x) &= f(a) + \frac{1}{1!} f^{(1)}(a)(x-a) + \frac{1}{2!} f^{(2)}(a)(x-a)^2 \\ &\quad + \frac{1}{3!} f^{(3)}(a)(x-a)^3 \\ &\quad + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \cdots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)(x-a)^k\,, \end{aligned} \tag{6}$$

where $f^{(n)}(a) = \frac{d^n}{dx^n} f(x)|_{x=a}$. The name is in honour of Brook Taylor (1685-1731). Colin Maclaurin (1698-1746) also made significant contributions to the series expansion of functions. In Maclaurin series, the derivatives are evaluated at zero ($a = 0$ in Eq. (6)) and hence, it is a special case of the Taylor series. From Eq. (4), it is seen that the higher derivatives of $s(x)$ and $c(x)$ have a simple pattern. We note, that $s^{(2n)}(x) = (-1)^n s(x)$ and $s^{(2n+1)}(x) = (-1)^n c(x)$, and $c^{(2n)}(x) = (-1)^n c(x)$ and $c^{(2n+1)}(x) = -(-1)^n s(x)$, for $n = 0, 1, 2, 3, \ldots$. Using the initial conditions, we note that $s^{(2n)}(0) = 0$ and $s^{(2n+1)}(0) = (-1)^n$, $c^{(2n)}(0) = (-1)^n$ and $c^{(2n+1)}(0) = 0$. The Taylor series for $s(x)$ and $c(x)$ are

$$\begin{aligned} s(x) &= x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\,, \qquad \forall x\,, \end{aligned} \tag{7}$$

$$\begin{aligned} c(x) &= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\,, \qquad \forall x\,. \end{aligned} \tag{8}$$

The odd and even properties of $s(x)$ and $c(x)$ respectively can also be seen from their Taylor series.

The series expansion of $s(x)$ and $c(x)$ can also be derived from a matrix form of the differential equation. Equation (4), $s'(x) = c(x)$ and $c'(x) = -s(x)$ has the following matrix form

$$\frac{d}{dx} \begin{bmatrix} s(x) \\ c(x) \end{bmatrix} = M \begin{bmatrix} s(x) \\ c(x) \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \quad (9)$$

Note, that $M^2 = -I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 unit matrix. The formal solution incorporating the initial conditions is

$$\begin{bmatrix} s(x) \\ c(x) \end{bmatrix} = e^{Mx} \begin{bmatrix} s(0) \\ c(0) \end{bmatrix} . \quad (10)$$

On expanding the exponential

$$\begin{aligned} \begin{bmatrix} s(x) \\ c(x) \end{bmatrix} &= \left\{ I + Mx + \frac{1}{2!}M^2x^2 + \frac{1}{3!}M^3x^3 + \cdots \right\} \\ &= \left\{ \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right) M \right. \\ &\quad \left. + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \right) I \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &\quad + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \end{aligned} \quad (11)$$

This leads to the series expansion

$$\begin{aligned} s(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\ c(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots . \end{aligned} \quad (12)$$

To establish more connections between $s(x)$ and $c(x)$, we multiply Eq. (3) by y'. Then, we obtain $y'y'' + yy' = 0$. Rewriting it, we have $\frac{1}{2}\left[(y')^2 + (y^2)\right]' = 0$. So, $(y')^2 + (y^2) =$ Constant $= C$. If $y = s(x)$, then $y' = c(x)$ and $c^2(x) + s^2(x) = C$. Using the initial conditions, $s(0) = 0$ and $c(0) = 1$, we have $C = 1$. This leads to the identity

$$s^2(x) + c^2(x) = 1 . \quad (13)$$

We will be using this identity to connect the two functions to the circle geometry, but after deriving some more properties. For further discussion, it is interesting to note that this identity connecting $s(x)$ and $c(x)$ is free of their derivatives. Moreover, the two functions $s(x)$ and $c(x)$ occur on an equal footing in the identity. From Eq. (13), it follows that the height of the first arch of $s(x)$ is 1 at $x = p/2$ and the first zero of $c(x)$ is at $x = p/2$. So, we conclude that $\tilde{p} = p$. From this we conclude that the successive zeros of $s(x)$ are at p, $2p$, $3p$ ... and the successive zeros of $c(x)$ are at $p/2$, $3p/2$, $5p/2$, ... It is worth noting, that between any two successive zeros of $s(x)$ there is exactly one zero of $c(x)$ and between any two successive zeros of $c(x)$ there is exactly one zero of $s(x)$.

To establish the linear independence of $s(x)$ and $c(x)$, we calculate their Wronskian

$$\begin{aligned} W(x) &= W[s(x)\,, c(x)] \\ &= \begin{vmatrix} s(x) & c(x) \\ s'(x) & c'(x) \end{vmatrix} \\ &= s(x)c'(x) - s'(x)c(x) \\ &= -s^2(x) - c^2(x) \\ &= -1\,. \end{aligned} \tag{14}$$

As the Wronskian is not zero, the two solutions $s(x)$ and $c(x)$ are linearly independent.

The addition/subtraction formulae are obtained as follows. Let us write the general solutions of Eq. (3) as $C_1 s(x) + C_2 c(x) = Rs(x + a)$ respectively, where C_1, C_2, R and a are arbitrary constants. On differentiating, $C_1 c(x) - C_2 s(x) = Rc(x + a)$. Using the initial condition at $x = 0$, we obtain $C_1 = Rc(a)$ and $C_2 = Rs(a)$. Substituting back the values of C_1 and C_2 and cancelling the constant R, we obtain

$$\begin{aligned} s(x + a) &= s(x)c(a) + c(x)s(a) \\ c(x + a) &= c(x)c(a) - s(x)s(a)\,. \end{aligned} \tag{15}$$

In Eq. (15), with $a = x$, we obtain the double angle identities

$$\begin{aligned} s(2x) &= 2s(x)c(x) \\ c(2x) &= c^2(x) - s^2(x)\,. \end{aligned} \tag{16}$$

Equation (13), $c^2(x)+s^2(x) = 1$ is recognized as the equation of the unit circle with centre at the origin and the point $(c(x), s(x))$ on it. This connects the two functions $s(x)$ and $c(x)$ to the circle geometry, establishing their geometric origins. From this, we conclude that $p = \pi$. Having derived the various significant properties of the two solutions of Eq. (3), we can now emphatically state that

$$\begin{aligned} s(x) &\equiv \sin x \\ c(x) &\equiv \cos x\,. \end{aligned} \tag{17}$$

From the graphs in Figure-1 we can see the oscillations of the two solutions namely sine and cosine functions respectively. Significantly, we see that the roots of the two solutions are interlaced.

In this Section, we deduced the basic properties of the trigonometric functions, $\sin x$ and $\cos x$ starting with the second-order linear differential equation $y'' + y = 0$, using the qualitative properties of differential equations (essentially the uniqueness of the solutions). The derivations were straightforward based on the convexity arguments (signs of derivatives and increasing/decreasing) usually done in the introductory courses of calculus. It

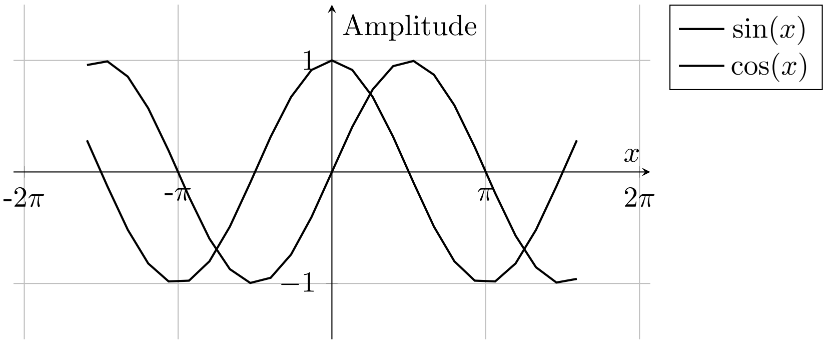

Figure 1: Graphs of $y = \sin(x)$ and $y = \cos(x)$.

is to be noted that most of the properties we derived, are unique to the sine and cosine functions. The two solutions were found to oscillate about the x-axis. These oscillations are interrelated in such a manner that the zeros of the two solutions are distinct and occur alternately. In other words, between any two successive zeros of one solution there is exactly one zero of the other solution. In Section-3, we shall see that the oscillation and this special pattern of the zeros is a common feature of many second-order linear differential equations satisfying certain conditions. In Section-4, we shall discuss the non-oscillation of the hyperbolic functions from the equation $y'' - y = 0$. In Section-5, we shall discuss the oscillation of the Bessel functions from the equation $x^2 y'' + xy' + \left(x^2 - p^2\right) y = 0$.

3. Qualitative Theory of Differential Equations

Differential equations are a fundamental tool to understand the physical phenomena. They provide mathematical models. The area of differential equations began with different perspectives in the works of Isaac Newton (1642-1727) and Gottfried Wilhelm von Leibniz (1646-1716). It was soon realized that many of the differential equations cannot be solved exactly. In fact, Joseph Liouville (1809-1882) showed that it is impossible to obtain solutions of a certain class of differential equations by a finite combination of *elementary functions*. Examples of elementary functions are the power function (including polynomials), exponential function, (including the six hyperbolic functions), logarithm (including the six inverse hyperbolic functions), and the six trigonometric functions along with their inverses. Any function built up by taking sums, products, and compositions of finitely many aforementioned functions also results in an elementary function. Hence, new approaches had to be developed to solve differential equations such as the approximate analytical solutions and numerical methods (to a desired degree of accuracy).

Here, we shall focus on another method used for deducing the properties of solutions of differential equations without actually finding their solutions [6–9]. The work in this direction can be traced back to early 19th century, but a significant growth took place in the works of Jules Henri Poincaré (1854-1912) and Aleksandr Mikhailovich Lyapunov (1857-1918). Poincaré was then analyzing the stability of the solar system [19]. This approach of deducing properties and behaviour of the solutions is known as *qualitative theory of differential equations*. We have used this approach to deduce the trigonometric functions.

Let us consider the second-order linear homogenous differential equation

$$y'' + P(x)y' + Q(x)y = 0\,, \tag{18}$$

where $P(x)$ and $Q(x)$ are continuous. The behaviour of the zeros (roots) of the two linearly independent solutions is summarised in the following theorem due to Jacques Charles François Sturm (1803-1855).

Theorem 2. *Sturm Separation Theorem: If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of*

$$y'' + P(x)y' + Q(x)y = 0\,,$$

then, the zeros of these functions are distinct and occur alternately—in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$, and conversely.

Proof. Let x_1 and x_2 be the two successive zeros of $y_2(x)$. That is, $y_2(x_1) = y_2(x_2) = 0$ and $y_2(x) \neq 0$ on $(x_1\,, x_2)$. Since, $y_1(x)$ and $y_2(x)$ are linearly independent, their Wronskian

$$\begin{aligned} W(x) &= W[y_1(x)\,, y_2(x)] \\ &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \end{aligned} \tag{19}$$

is never zero. Let us assume that $W(x) > 0$ for $x \in (x_1\,, x_2)$. Then

$$W(x_j) = y_1(x_j)y_2'(x_j) > 0\,, \quad j = 1\,, 2\,. \tag{20}$$

Since, x_1 and x_2 are successive zeros of $y_2(x)$, it must hold that $y_2'(x_1)$ and $y_2'(x_2)$ have opposite signs. If $y_2(x)$ increases at x_1 then it must decrease at x_2 and vice versa. This implies that $y_1(x_1)$ and $y_1(x_2)$ must have opposite signs. From the *intermediate value theorem*, $y_1(x)$ must vanish between x_1 and x_2. Same arguments work for the case, $W(x) < 0$ for $x \in (x_1\,, x_2)$. This completes the proof. The roles of $y_1(x)$ and $y_2(x)$ can be interchanged. Between any two consecutive zeros of $y_1(x)$ there is only one zero of $y_2(x)$ and vice versa. Analogous theorems exist for the eigenvalues of matrices, which we illustrate below. So, also for the orthogonal functions, which we shall see in some detail in Section-6.

A symmetric matrix is a square matrix having the property, $A^T = A$. In other words the entries satisfy the relation $a_{ij} = a_{ji}$. Let $A_r = a_{ij}$ be a sequence of N symmetric matrices of increasing order with $i\,, j = 1, 2, 3, \ldots, r$ and $r = 1, 2, 3, \ldots N$. Let $\lambda_k(A_r)$ be the k-th eigenvalue of A_r for $k = 1, 2, 3, \ldots, r$ with the ordering

$$\lambda_1(A_r) \geq \lambda_2(A_r) \geq \lambda_3(A_r) \geq \ldots \geq \lambda_r(A_r)\,.$$

Then

$$\lambda_{k+1}(A_{i+1}) \leq \lambda_k(A_i) \leq \lambda_k(A_{i+1})\,.$$

Let us examine the equation $y'' + ay = 0$, where a is any real number. For $a > 0$, the two solutions, $y_1 = \sin(\sqrt{a}x)$ and $y_2 \cos(\sqrt{a}x)$ are oscillatory and we have seen them in detail with $a = 1$, in Section-2. We note, that both the solutions have infinite zeros. For $a < 0$, the two solutions, $y_1 = e^{\sqrt{a}x}$ and $y_2 = e^{-\sqrt{a}x}$ or equivalently $y_1 = \sinh(\sqrt{a}x)$ and $y_2 = \cosh(\sqrt{a}x)$ are non-oscillatory. The solution $y_1 = \sinh(\sqrt{a}x)$ has one zero at $x = 0$ and the other solution has no zeros. For $a = 0$, the two solutions are $y_1 = x$ and $y_2 = c$ and the general solution is a straight line, $y = mx + c$ and it is called as the *trivial* solution. The trivial solution has one zero at $x = -c/m$. The other two solutions are called as *nontrivial* solutions. In the rest of this Section, we shall study the case, when the real number a is replaced by a real and continuous function $q(x)$. For $q(x) < 0$, the solutions do not oscillate at all. For $q(x) > 0$, the solutions do oscillate under additional conditions, which we shall cover in Theorem (4).

Equation (18) is in the *standard form.* For the purpose of using the convexity arguments, we shall transform it to the following form known as the *normal form*

$$\begin{aligned} & u'' + q(x)u = 0 \\ & q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x)\,. \end{aligned} \tag{21}$$

This enables us to relate the general equation in (18) to the familiar equation, $y'' + ay = 0$. The change of the dependent variable is done by choosing $y(x) = u(x)v(x)$. This substitution transforms Eq. (18) to $vu'' + (2v' + Pv)u' + (v'' + Pv' + Qv)u = 0$. Choosing the coefficient of the u' to be zero, we obtain the equation, $2v' + Pv = 0$. The solution of this equation is $v = e^{-\frac{1}{2}\int P\,dx}$. Using this solution, we obtain the normal form of Eq. (18) in Eq. (21). As the $v(x)$ in this form is never zero, the change of dependent variable from $y(x)$ to $u(x)$ does not have any effect on the zeros of Eq. (18). So, the oscillation (or non-oscillation) of the solutions is preserved.

The following four theorems summarise the results required to study the oscillation phenomenon in the second-order linear homogeneous differential equations. The proofs are straightforward based on the convexity arguments. The following theorem ensures the non-oscillation of the solutions.

Theorem 3. *Non-Oscillation Theorem: If $q(x) < 0$, and if $u(x)$ is a nontrivial solution of $u'' + q(x)u = 0$, then $u(x)$ has at most one zero.*

The proofs of Theorem (3) is straightforward based on the convexity arguments [6–9].

For the oscillations to take place, there needs to be additional conditions as given in the following theorem.

Theorem 4. *Oscillation Theorem: Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$, where $q(x) > 0$ for all $x > 0$. If*

$$\int_1^\infty q(x)\,dx = \infty\,,$$

then $u(x)$ has infinitely many zeros on the positive x-axis.

The proofs of Theorem (4) is straightforward based on the convexity arguments [6–9].

In many problems, we are dealing with finite intervals. The following theorem rules out the occurrence of infinitely many oscillations on closed intervals.

Theorem 5. *Finite Zeros in a Closed Interval: Let $u(x)$ be a nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Then $u(x)$ has at most a finite number of zeros in this interval.*

The proofs of Theorem (5) is straightforward based on the convexity arguments [6–9].

From the Sturm separation theorem (2), we know that the zeros of the two solutions of Eq. (18) alternate. So, the number of zeros of the two solutions can not differ by more than one in a given interval. In many problems, the decisive function, $q(x) > 0$ can be different. This leads to the question of the number of zeros for different choices of $q(x)$. In the case of the familiar equation, $y'' + ay = 0$ with $a > 0$, we know that larger the a, more the number of zeros in the same interval. The following theorem addresses this scenario in totality.

Theorem 6. *Sturm Comparison Theorem: Let $y(x)$ and $z(x)$ be the nontrivial solutions of*

$$y'' + q(x)y = 0$$

and

$$z'' + r(x)z = 0\,,$$

where $q(x)$ and $r(x)$ are positive functions such that $q(x) > r(x)$. Then $y(x)$ vanishes at least once between any two successive zeros of $z(x)$.

The proof is based on convexity arguments along with the Wronskian. From the theorem, we conclude that the solutions of Eq. (21) oscillate more rapidly as $q(x)$ is increased.

4. Hyperbolic Functions from $y'' - y = 0$

Following a very similar approach, it is possible to deduce the properties of the solutions of the equation

$$y'' - y = 0\,. \tag{22}$$

From the theorems, we can conclude that the hyperbolic solutions do not have any oscillation. The solution $y = \sinh x$ has only one zero at $x = 0$. The equivalent pair, e^{-x} and e^{+x} also does not have oscillatory solutions and both of them do not have any zeros. The two linearly independent solutions indeed are

$$\begin{aligned} y &= \sinh x \\ y &= \cosh x\,. \end{aligned} \tag{23}$$

5. Bessel Functions from $x^2y'' + xy' + (x^2 - p^2)\, y = 0$

Let us consider the Bessel equation

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0\,. \tag{24}$$

Next, we write the Bessel equation in Eq. (24) in normal form using the transform in Eq. (21), which leads to

$$u'' + \left(1 + \frac{1-4p^2}{4x^2}\right) u = 0\,. \tag{25}$$

Using the theorems from Section 2, we arrive at the following result stated in the form of a theorem.

Theorem 7. *Solutions of Bessel Equation Theorem: Let $y_p(x)$ be a nontrivial solution of Bessel equation on the positive x-axis. If $0 \leq p < 1/2$, then every interval of length π contains at least one zero of $y_p(x)$; if $p = 1/2$, then the distance between successive zeros of $y_p(x)$ is exactly π; if $p > 1/2$, then every interval of length π contains at most one zero of $y_p(x)$.*

We used Eq. (25) to obtain the results stated in Theorem (7) using qualitative analysis. Now, we shall revisit these qualitative results after noting the infinite series solutions of the Bessel equation in its standard form in Eq. (24). Equation (24) is a second order equation and has two linearly independent solutions. One of the two linearly independent solutions known as the Bessel function of the *first kind* is denoted by $J_p(x)$ and has the following infinite series representation

$$J_p = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(m+p+1)} \left(\frac{x}{2}\right)^{2m+p}\,, \tag{26}$$

where $\Gamma(n)$ is the gamma function. In case n is a positive integer, we obtain $\Gamma(n) = (n-1)!$ which is the factorial function. In such cases, the solutions are

$$J_n = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}\,. \tag{27}$$

In the context of Theorem (7), we note the following properties

(i) Bessel function for negative integers is

$$J_{-n}(x) = (-1)^n J_n(x)\,.$$

(ii) In Theorem 7, there is a special value $p = 1/2$. Recalling that, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we note

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x\,.$$

6. Orthogonal Polynomials

Any sequence of polynomials $\{P_n(x)\}$ in which the degree of $P_n(x)$ is n for all n is said to be orthogonal with respect to the measure $d\alpha(x)$ if

$$\begin{aligned} \int_a^b P_n(x)P_m(x)d\alpha(x) &= 0\,, \quad n \neq m\,, \\ \int_a^b P_n(x)P_n(x)d\alpha(x) &\neq 0\,, \quad \forall\, n \geq 0\,. \end{aligned} \tag{28}$$

The limits of integration can be finite (as in the case of Laguerre polynomials and Jacobi polynomials) or infinite (as in the case of Hermite polynomials). When the measure is absolutely continuous, that is $d\alpha(x) = w(x)\,dx$, where $w(x)$ is called the *weight function*, then the relations in (28) become

$$\begin{aligned} \int_a^b P_n(x)P_m(x)w(x)dx &= 0\,, \quad n \neq m\,, \\ \int_a^b P_n(x)P_n(x)w(x)dx &\neq 0\,, \quad \forall\, n \geq 0\,. \end{aligned} \tag{29}$$

The *classical* orthogonal polynomials are Hermite polynomials, Laguerre polynomials and Jacobi polynomials (also known as hypergeometric polynomials and the special cases include, Gegenbauer, Legendre, Zernike and Chebyshev polynomials). The orthogonal polynomials occur across mathematics and sciences. All the classical orthogonal polynomials are governed by the second-order linear differential equation of the Sturm-Liouville type

$$Q(x)y'' + L(x)y' + \lambda_n y = 0\,, \tag{30}$$

where $Q(x)$ is a polynomial in x of degree ≤ 2 (i.e., at most a quadratic polynomial), $L(x)$ is a polynomial in x of degree ≤ 1 (i.e., at most a linear polynomial), and λ_n is independent of x (i.e., a number). Both $Q(x)$ and $L(x)$ are independent of n. For example, the Hermite polynomials satisfy the differential equation, $y'' - 2xy' + 2ny = 0$. For each value of $n = 0, 1, 2, 3, \ldots$, the solutions are n-th degree polynomials. The equation for the Laguerre polynomials is $xy'' + (1 - x)y' + ny = 0$.

Orthogonal polynomials have many properties. Derivatives of the orthogonal polynomials sets also form orthogonal polynomials sets. The roots of the orthogonal polynomials are all real, distinct and lie in the interval of orthogonality. In the present context, we note the property of the interlacing of zeros of the orthogonal polynomials. If $\{x_{n\,,\,k}\}_{k=1}^{n}$ and $\{x_{n+1\,,\,k}\}_{k=1}^{n+1}$ denote the consecutive zeros of $P_n(x)$ and $P_{n+1}(x)$ respectively, then we have

$$\begin{aligned} a < x_{n+1\,,\,1} < x_{n\,,\,1} < x_{n+1\,,\,2} < x_{n\,,\,2} < \ldots \\ < x_{n+1\,,\,n} < x_{n\,,\,n} < x_{n+1\,,\,n+1} < b\,. \end{aligned} \tag{31}$$

These properties acquire an extra significance as no formula is known for the roots of any of the orthogonal polynomials with the exception of the Chebyshev polynomials. The

Chebyshev polynomials are one of the special cases of the Jacobi polynomials. Chebyshev polynomials of the *first kind* occur in the expansion of $\cos(n\theta) = T_n(\cos\theta)$. Chebyshev polynomials of the *second kind* occur in the expansion of $\sin(n\theta) = \sin\theta\, U_{n-1}(\cos\theta)$. A Chebyshev polynomial of either kind of degree n has n different simple roots, called Chebyshev roots, in the interval of orthogonality $[-1, 1]$. The n roots of $T_n(x)$ are

$$x_k = \cos\left(\frac{(k+\frac{1}{2})}{n}\pi\right), \qquad k = 0, \ldots, n-1\,. \tag{32}$$

The n roots of $U_n(x)$ are

$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \qquad k = 1, \ldots, n\,. \tag{33}$$

As no such formulae are known for any of the other orthogonal polynomials, the topic is of active research interest.

In passing, we note that all these polynomials have been generalized in various ways. One possibility of the generalizations is through the *quantum algebras* [20–23].

7. Concluding Remarks

Starting with the 'simple' differential equation $y'' + y = 0$, we deduced that the solutions are $\sin x$ and $\cos x$, using qualitative arguments. We also derived some of their properties including, periodicity and the series expansions. The connection to the geometric origins of $\sin x$ and $\cos x$ was established through the unit circle. Some properties such as the sine/cosine of sums and differences of two angles are also derived from the defining differential equation. We also derived the series expansion of the sine and cosine functions. In fact, the series expansion was also derived by the alternate procedure of a matrix differential equation.

It was further shown that under certain conditions, the solutions of the second-order linear differential equations oscillate. In which a case, the zeros of the two solutions are interlaced. We also looked at the Sturm comparison theorem, which covers the amount of oscillation for different choices of the decisive function. We briefly looked at the hyperbolic and Bessel functions. We further looked at the case of classical orthogonal polynomials. The approach presented in this article will be useful to see the connection between the geometric approach to trigonometry and the calculus based approach presented here.

References

[1] V. Katter. *Historische, logische und individuelle Genese der Trigonometrie aus didaktischer Sicht.* Springer, 2023.

[2] L. Rico E. Martín-Fernández, J. F. Ruiz-Hidalgo. Meaning and Understanding of School Mathematical Concepts by Secondary Students: The Study of Sine and Cosine. *EURASIA Journal of Mathematics, Science and Technology Education*, 15(12):em1782, 2019.

[3] S. L. Loney. *Plane Trigonometry Part-1.* AITBS Publishers India, 2019.

[4] S. L. Loney. *Plane Trigonometry Part-2.* AITBS Publishers India, 2019.

[5] Eli Maor. *Trigonometric Delights.* Princeton University Press, 2013.

[6] G. F. Simmons. *Differential Equations with Applications and Historical Notes.* Taylor & Francis, 2016.

[7] P. Bedient E. Rainville and R. Bedient. *Elementary Differential Equations.* Pearson, 2014.

[8] M. D. Weir J. R. Hass, C. E. Heil and P. Bogacki. *Thomas' Calculus, 15th edition.* Pearson, 2022.

[9] H. J. Weber G. B. Arfken and F. E. Harris. *Mathematical Methods for Physicists.* Academic Press, London, UK, 2012.

[10] B. Agheli. Approximate solution of Bratu differential equations using trigonometric basic functions. *Kragujevac Journal of Mathematics*, 45(2):203–214, 2021.

[11] M. Pakdemirli. Complex Exponential Methodfor Solving Partial Differential Equations. *Engineering Transactions*, 72(4):461–474, 2024.

[12] H. K. Sevindir S. Çetinkaya, A. Demir. The analytic solution of initial boundary valueproblem including time fractional diffusion equation. *Facta Universitatis, Series: Mathematics and Informatics*, 35(1):243–252, 2020.

[13] C. Valls. Trigonometric Polynomial Solutions of Bernoulli Trigonometric Polynomial Differential Equations. *Mathematics*, 10:4022, 2022.

[14] C. O. Alakofa O. O. Enoch. A fifth order block methods for solving second-order stiff ordinary differential equations using trigonometric functions and polynomial function as the basis function. *African Scientific Reports*, 3(2):156, 2024.

[15] B. V. Rao. Fun with Differential Equations. *Resonance – Journal of Science Education*, 18(6):543–557, 2013.

[16] S. A. Khan. Trigonometric Ratios Using Geometric Methods. *Advances in Mathematics: Scientific Journal*, 10(9):8685–8702, 2020.

[17] S. A. Khan. Trigonometric Ratios Using Algebraic Methods. *Mathematics and Statistics*, 9(6):899–907, 2021.

[18] S. A. Khan. Teaching Irrational Numbers through Trigonometry. *Resonance Journal of Science Education*, 26(6):813–827, 2021.

[19] J. Barrow-Green. Oscar II's prize competition and the error in Poincaré's memoir on the three body problem. *Archive for History of Exact Sciences*, 48:107–131, 1994.

[20] R. Chakrabarti and R. Jagannathan. A (p, q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A: Math. Gen.*, 24:L7711–L718, 1991.

[21] R. Jaganathan and S. Sinha. A q-deformed nonlinear map. *Phys. Lett. A.*, 338:277–287, 2005.

[22] R. Jagannathan and S. A. Khan. On the deformed oscillator and the deformed derivative associated with the Tsallis q-exponential. *International Journal of Theoretical Physics*, 59(8):2647–2669, 2020.

[23] S. A. Khan and R. Jagannathan. On Certain Appell Polynomials and Their Generalizations Based on the Tsallis q-exponential. *Bulletin of the Malaysian Mathematical Sciences Society*, 45(4):145–1472, 2022.