EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6147
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Fixed Point Theorems for a Subfamily of Non-expansive Evolution Operators

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Abstract. In this manuscript, we study the fixed points of a specific subfamily within a non-expansive evolution family of bounded linear operators on a Hilbert space H. Employing the framework of nets and a progression algorithm, we establish several theorems concerning the strong convergence of sequences to a common fixed point of the considered subfamily. To illustrate the applicability of our results, we provide a concrete example that supports the theoretical findings. The manuscript concludes with an open problem to encourage further research.

2020 Mathematics Subject Classifications: 46A32, 46E20, 47B02, 47H10

Key Words and Phrases: Fixed Point, Hilbert Space, Nonexpansive mapping, evolution family, linear operator

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1. Introduction

Our primary objective in this paper is to demonstrate the convergence of an algorithm to a shared fixed point(FP) of a subfamily within a NeE family in Hilbert spaces. Initially, we delve into the significance and concept of semigroups and evolution families of bounded linear operators. Let's consider the autonomous system

$$\begin{cases} \dot{\mho}(t) &= A\mho(t), \quad t \geq 0 \\ \mho(0) &= \mho_0, \end{cases}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6147

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where A is a matrix of order m with complex entries. The solution of such a system leads to the idea of a semigroup. We recall that a family $\mathbf{S} = \{S(a),\, a \geq 0\}$, of bounded linear operators on a Hilbert space H is said to be a semigroup if it satisfies the following two conditions, $S(a+b) = S(a) + S(b)$ and $S(0) = I$ for all $a,\, b \geq 0$, where I is the identity operator on H. Similarly the solution of the following non-autonomous system

$$\begin{cases} \dot{\mho}(t) &= A(t)\mho(t) + e^{i\nu t} I \quad t \geq 0 \\ \mho(0) &= \mho_0, \end{cases}$$

where $A(t)$ is a matrix of order m, leads to the idea of an evolution family. A family of bounded linear operators $\mathbf{E} = \{E(a,\, b),\, a \geq b \geq 0\}$, is said to be an evolution family if it satisfies $E(a,\, b)E(b,\, c) = E(a,\, c)$ and $E(a,\, a) = I$ for all $a \geq b \geq c \geq 0$.

The theory of fixed points finds myriad applications in mathematics, engineering sciences, economics, and statistics, among other fields (see, for example, [1–5]). Motivated by Baillon's work [6], various authors have explored FP approximations for non-expansive families using different progressions and algorithms. For instance, in [7], it is demonstrated that a sequence of approximations converges to a FP of a non-expansive and non-linear mapping in Hilbert spaces. Halpern, in [8], provides an approximation of a FP of non-expansive maps. Indeed, all the aforementioned references focus on the convergence of algorithms or progressions to fixed points of non-expansive maps in Hilbert spaces. These results are then utilized to prove the convergence of algorithms to fixed points of semigroups in Hilbert spaces, as seen in [9].

Baillon [6] states that if Z is a convex and closed subset of a Hilbert space H, and $G : Z \to Z$ is a non-expansive mapping such that $F_{G(z)} \neq \emptyset$, where $F_{G(z)}$ represents the set of all fixed points of G, then for every element $z \in Z$, the Cesaro mean $\left(\frac{1}{m}\right)\sum_{q=1}^{m} G^q z$ converges weakly to some $z \in F_{G(z)}$. Here, if we set $z = P_{F_{G(z)}} z$ for any element $z \in Z$, then $P_{F_{G(z)}}$ from Z onto $F_{G(z)}$ is a non-expansive retraction. In 1976, Brezis and Baillon [8] proved that if $\mathbf{G} = G(s)s \geq 0 : Z \to Z$ is a non-expansive semigroup, then $\left\{\left(\frac{1}{a}\right)\int_0^a G(s)uds\right\}_{s\geq 0}$ converges weakly to a common FP of $\mathbf{G}$. These results have been generalized by several authors, first by Halpern in [10] and then by Wittmann in [11]. They considered the following algorithm

$$\begin{aligned} \varsigma_0 &= \varsigma \in Z; \\ \varsigma_{m+1} &= \mu_{m+1}\varsigma + (1 - \mu_{m+1})Tw_m, \;\; m \geq 0. \end{aligned}$$

Where $\{\varsigma_m\}$ is a progression whose terms are between 0 and 1 and satisfying the following three conditions,

- $\lim_{m\to+\infty} \varsigma_m = 0$;
- $\sum_{m=1}^{+\infty} \varsigma_m = +\infty$;
- $\sum_{m=1}^{+\infty} |\varsigma_{m+1} - \varsigma_m| < +\infty$.

In [11], Wittmann proved that for every $\varsigma \in Z$, the progression ς_m strongly converges to a unique FP $P(\varsigma) \in F_{\mathbf{G}}$, where $P : H \to F_{\mathbf{G}}$ is a metric projection. Recently, in [12], Kimure et al. generalized Wittmann's result by proving that if $\varsigma_1 \in Z$, where Z is a convex and closed subset of a Hilbert space, then the following iterative scheme converges strongly to P_ς, where $P : Z \to F$ is a non-expansive retraction and $F = \bigcap_{k=1}^{q} F_{G_k} \neq \emptyset$.

Takahashi and Shimizu, in [13], studied the strong convergence of the progression $\mho_m$ defined by the equation

$$\mho_{m+1=\delta_m}\mho + (1-\delta)\frac{1}{\alpha_m}\int_1^{\alpha_m} G(s)\mho_m ds, \quad m \geq 0,$$

where δ_m is a progression in $(0,1)$ and α_m is a progression in $(0,+\infty)$ that diverges to $+\infty$. For other recent work in the field of fixed point theory we refer to [12, 14–19].

In this paper, we aim to generalize the results provided in [9] from a semigroup to a subfamily of a NeE equation in a Hilbert space. For same generalization we refer to [20, 21].

2. Preliminaries

In this section, we present basic definitions and results that are instrumental in proving our main results.

Let $Z \neq \emptyset$ be a closed and convex subset of a real Hilbert space H. An operator $G : Z \to Z$ is considered non-expansive if $\|Gd - Gb\| \leq \|d - b\|$ for all $d, b \in Z$.

An operator $P_Z : H \to Z$ is is said to be a projection from H onto Z if for each element $\varsigma \in H$ and $P_Z\varsigma \in Z$ the following hold,

$$\|\varsigma - P_Z\varsigma\| = \inf_{\mho \in Z} \|\varsigma - \mho\| := d(\varsigma, Z),$$

and

$$\|P_Z\varsigma - P_Z\mho\| \leq \|\varsigma - \mho\|.$$

The above inequality illustrates that P_Z is non-expansive, and furthermore $\|P_Z\varsigma - P_Z\mho\|^2 \leq \langle \varsigma - \mho, P_Z\varsigma - P_Z\mho \rangle$, for all $\varsigma, \mho \in H$.

Moreover, we have

$$\langle \varsigma - \mho, P_Z\varsigma - P_Z\mho \rangle \leq 0, \tag{1}$$

and $\|\varsigma - P_Z\varsigma\|^2 + \|\mho - P_Z\mho\|^2 \leq \|\varsigma - \mho\|^2$, for all $\varsigma \in H$ and $\mho \in Z$.

The evolution family $\mathbf{E} = \{E(d,b)\}_{d \geq b \geq 0}$ is called non-expansive, continuous and periodic with period $q \geq 0$ if it satisfy the following conditions:

- $\|E(d,b)y - E(d,b)z\| \leq \|y - z\| \ \forall \ y, z \in H \, and \, d \geq b \geq 0$;
- for each $z \in H$, $(d,b) \to E(d,b)z$ is continuous.
- $E(a+q, b+q) = E(a,b)$ for all $a \geq b \geq 0$.

Evolution family is a generalized form of a semigroup, see the following remarks.

Remark 1. *[22] Every semigroup can be viewed as a special case of an evolution family, not every evolution family is a semigroup, particularly if it doesn't satisfy the semigroup property.*

Remark 2. *[22] If an evolution family repeats periodically for every positive number, then it transforms into a semigroup.*

In this paper, we aim to establish a convergence theorem to a FP of a subfamily $\mathbf{L}_s$ of a NeE family on a Hilbert space H. It's important to note that such a family need not be a semigroup. The following example will illustrate this fact.

Example 1. *The family defined by $\boldsymbol{L} = L(\mathfrak{s}, \mathfrak{r}) = \frac{\mathfrak{s}+1}{\mathfrak{r}+1} : \mathfrak{s} \geq \mathfrak{r} \geq 0$ is evidently an evolution family acting on $\mathbb{R}_+$. Since $L(\mathfrak{s}, \mathfrak{s}) = 1$ (the identity on $\mathbb{R}_+$) and*

$$L(\mathfrak{s}, \mathfrak{t})L(\mathfrak{t}, \mathfrak{r}) = (\frac{\mathfrak{s}+1}{\mathfrak{t}+1})(\frac{\mathfrak{t}+1}{\mathfrak{r}+1}) = \frac{\mathfrak{s}+1}{\mathfrak{r}+1} = L(\mathfrak{s}, \mathfrak{r}).$$

Setting $\mathfrak{r} = 0$, we obtain $\boldsymbol{L}_s = L(\mathfrak{s}, 0) = \mathfrak{s} + 1$, which is indeed a sub-family of $\boldsymbol{L}$. However, it is not a semigroup.

Definition 1. *Let $Z \neq \emptyset$ and $G : Z \to Z$, be an operator, then the set of all fixed points of G is denoted by $F_{G(z)}$ and is defined as $F_{G(z)} = \{z \in Z : G(z) = z\}$.*

Definition 2. *A mapping that is continuous from a topological space into one of its subspaces and maintains the positions of all points in that subspace is referred to as a retraction.*

Definition 3. ***Open Ball and Closed Ball:*** *An open ball is denoted by $B(\Upsilon; K)$ and define as $B(\Upsilon; K) = \{\zeta \in H : \|\zeta - \Upsilon\| < K\}$, while closed ball is denoted by $\overline{B}(\Upsilon; K)$ and define as $\overline{B}(\Upsilon; K) = \{\zeta \in H : \|\zeta - \Upsilon\| \leq K\}$.*

Throughout this paper we will denote by $F_{E(d,b)}$ and $F_{\mathbf{E}}$ the set of all fixed points of the operator $E(d, b)$ and the family $\mathbf{E}$ respectively, i.e. $F_{\mathbf{E}} = \cap_{d \geq b \geq 0} F_{E(d,b)}$.

The following lemmas will be useful in the proof of our main results.

Lemma 1. *[7] Consider $\mathbb{Z}$ as a closed and convex subset of H, and let $G : \mathbb{Z} \to \mathbb{Z}$ be a non-expansive map. Then, $I - G$ is demiclosed. Specifically, if ς_m is a sequence in $\mathbb{Z}$ such that $\varsigma_m \rightharpoonup \varsigma$ and $(I - G)\varsigma_m \to \mho$, then $(I - G)\varsigma = \mho$.*

Lemma 2. *[23] Let ς_n and $\mho_n$ be bounded sequences in a Banach space $\mathbb{Y}$, and λ_n be a sequence in the closed interval $[0, 1]$ satisfying the condition*

$$0 < \liminf_{n \to +\infty} \lambda_n \leq \limsup_{n \to +\infty} \lambda_n < 1.$$

If the sequence

$$\varsigma_{n+1} = (1 - \lambda_n)\varsigma_n + \lambda_n \mho_n \qquad \forall n \geq 0,$$

satisfy

$$\limsup_{n\to+\infty}(\|\varsigma_n - \mho_{n-1}\| - \|\varsigma_n - \varsigma_{n-1}\|) \leq 0.$$

Then

$$\lim_{n\to+\infty}\|\varsigma_n - \mho_n\| = 0.$$

Lemma 3. *[24] Consider that $\{b_n\}$ is a sequence in $[0,+\infty)$ such that $b_{n+1} \leq (1-\lambda_n)b_n + \delta_n\lambda_n$ for all n, where $\{\lambda_n\} \in]0,1[$ and $\{\delta_n\}$ are sequences satisfiyng*
(a) $\sum_{n=1}^{+\infty}\lambda_n = +\infty$, and (b) $\limsup_{n\to\infty}\lambda_n \leq 0$ or $\sum_{n=1}^{+\infty}|\delta_n\lambda_n| < +\infty$.
Then $\lim_{n\to+\infty} b_n = 0$.

3. Main Results

This section is devoted to our main results. Throughout the paper Z will be a non-empty, closed, bounded and convex subset of Hilbert space H.

Lemma 4. *Let $\mathbb{E} = \{E(z,0)_{z\geq 0}\}$ be a subfamily of a NeE family on Z with condition $\|\mho_r - E((\frac{s}{r})j_r,0)\mho_r\|^2 \leq \frac{j_r}{r}diam(Z)$, where $\mho_r = (\frac{1}{r})\sum_{j=1}^{r}\varsigma_{j,r} \in Z$ and $j_r \geq 0$ is an integer. For $\varsigma \in D$ and $s > 0$, set $F_t(\varsigma) = \frac{1}{s}\int_0^s E(z,0)wdz$. Then, for each $t \geq 0$,*

$$\lim_{s\to+\infty}\sup_{\varsigma\in Z}\left\|E(t,0)F_s(\varsigma) - F_s(\varsigma)\right\| = 0.$$

Proof. Assume that $s > t$ where $t \in R_+$ is fix. Then, for $r \in N$, there exist $j_r \in N$ such that

$$(\frac{s}{r})j_r \leq s \leq (\frac{s}{r})j_r + 1.$$

Therefore, we have $\lim_{r\to+\infty}(\frac{s}{r})j_r = t$. By using the idea in [3] for $\{\varsigma_{j,r}\}_{j,r=1}^{\infty} \subseteq Z$ and $\mho_r = (\frac{1}{r})\sum_{j=1}^{r}\varsigma_{j,r} \in Z$, we have

$$\|\mho_r - \eta\|^2 = \frac{1}{r}\sum_{j=1}^{r}\|\varsigma_{j,r} - \eta\|^2 - \frac{1}{r}\sum_{j=1}^{r}\|\varsigma_{j,r} - \mho_r\|^2 \quad \forall\eta \in H.$$

Set $\eta = E((\frac{t}{r})j_r,0)\mho_r$ and $\varsigma_{j,r} = E((\frac{t}{r})j,0)\varsigma$, for $\varsigma \in Z$. Therefore from the given condition we have

$$\left\|\mho_r - E((\frac{s}{r})j_r,0)\mho_r\right\|^2 \leq \frac{j_r}{r}diam(Z).$$

For $\epsilon > 0$, and from $\frac{j_r}{r} \leq \frac{t}{s}$, there exists a number $s_1 > 0$ such that

$$0 < \frac{j_r}{r}diam(Z) \leq \frac{t}{s}diam(Z) < \epsilon^2, \ \ \forall s \geq s_1.$$

From the above estimation we have

$$\left\|\mho_r - E((\frac{s}{r})j_r,0)\mho_r\right\| < \epsilon, \ \ \forall r \in N, \varsigma \in Z.$$

One can write

$$\begin{aligned}\lim_{r\to+\infty}\mho_r &= \lim_{r\to+\infty}\frac{1}{r}\sum_{j=1}^{r}E((\frac{s}{r}j),0)\varsigma\\ &= \lim_{r\to\infty}\frac{1}{s}\sum_{j=1}^{r}\frac{s}{r}E((\frac{s}{r}j),0)\varsigma\\ &= \frac{1}{s}\int_0^s E(z,0)wdz\\ &= F_s(\varsigma),\quad \forall\ \varsigma\in Z.\end{aligned}$$

Therefore, for each $\varsigma \in Z$,

$$\lim_{r\to+\infty}\|\mho_r - F_s(\varsigma)\| = 0. \tag{2}$$

Also we have

$$\begin{aligned}\|F_s(\varsigma)-E(t,0)F_s(\varsigma)\| &\leq \|F_s(\varsigma)-\mho_r\| + \left\|\mho_r - E((\frac{s}{r}j_r),0)\mho_r\right\|\\ &\quad + \left\|E((\frac{s}{r}j_r),0)\mho_r - E((\frac{s}{r}j_r),0)F_s(\varsigma)\right\|\\ &\quad + \left\|E((\frac{s}{r}j_r),0)F_s(\varsigma) - E(t,0)F_s(\varsigma)\right\|\\ &\leq 2\|\mho_r - F_s(\varsigma)\| + \left\|\mho_r - E((\frac{s}{r}j_r),0)\mho_r\right\|\\ &\quad + \left\|E((\frac{s}{r}j_r),0)F_s(\varsigma) - E(t,0)F_s(\varsigma)\right\|\\ &< 2\|\mho_r - F_s(\varsigma)\|\\ &\quad + \left\|E((\frac{s}{r}j_r),0)F_s(\varsigma) - E(t,0)F_s(\varsigma)\right\| + \epsilon \quad \forall s\geq s_1,\ r\in N,\ \varsigma\in Z.\end{aligned}$$

From $\lim_{r\to+\infty}(\frac{s}{r})j_r = t$, and (2) we have

$$\|F_s(\varsigma)-E(t,0)F_s(\varsigma)\| < \epsilon \quad \forall s\geq s_1,\ r\in N,\ \varsigma\in Z.$$

Therefore, for each $t\in R_+$ we have

$$\lim_{s\to+\infty}\sup_{\varsigma\in Z}\left\|F_s(\varsigma)-E(t,0)F_s(\varsigma)\right\| = 0.$$

Theorem 1. *Let* $\mathbb{E} = \{E(s,0)\}_{s\geq 0} : Z \to Z$ *be a subfamily of a NeE family such that* $\|\mho_m - E((\frac{t}{m})i_m,0)\mho_m\|^2 \leq \frac{i_m}{m}diam(Z)$ *where* $\varsigma \in Z$, $t > 0$ *and* $F_{\mathbb{E}} \neq \emptyset$. *Let* $\{\lambda_t\}$ *and* $\{\gamma_t\}_{0<t<1}$ *be nets of positive real numbers such that* $\lambda_t \in (0,1)$, $\lim_{t\to 0}\lambda_t = 1$, $\lim_{t\to 0}\gamma_t = +\infty$ *and* $\{\varsigma_t\}$ *be the net*

$$\varsigma_t = P_Z\bigg[t(\lambda_t\varsigma_t) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t}E(s,0)\varsigma_t ds\bigg],\quad \forall\, t\in(0,1). \tag{3}$$

Then $\varsigma_t \to \varsigma^* \in F_{\mathbb{E}}$, *strongly as* $t\to 0^+$.

Proof. First of all we will show that ς_t is well defined. For this let

$$W_\varsigma = P_Z\left[t(\lambda_t\varsigma) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)wds\right], \quad \forall\, t \in (0,1).$$

It gives

$$\begin{aligned}
\|W_\varsigma - W_\mho\| &\leq \left\|P_Z\left(t(\lambda_t\varsigma) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)wds\right)\right. \\
&\quad \left. - P_Z\left(t(\lambda_t\mho) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)uds\right)\right\| \\
&\leq t\lambda_t\|\varsigma - \mho\| + (1-t)\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t}\big(E(s,0)\varsigma - E(s,0)\mho\big)ds\right\| \\
&\leq t\lambda_t\|\varsigma - \mho\| + (1-t)\|\varsigma - \mho\| \\
&= [1-(1-\lambda_t)t]\|\varsigma - \mho\|.
\end{aligned}$$

Which implies that

$$\|W_\varsigma - W_\mho\| \leq [1-(1-\lambda_t)t]\|\varsigma - \mho\|.$$

This shows that ς_t is a contraction, so by Banach contraction principle it has a unique FP, and hence the net ς_t is well defined.
Now for any $\Upsilon \in F_{\mathbb{E}}$, we have

$$\begin{aligned}
\|\varsigma_t - \Upsilon\| &= \left\|P_Z\left[t(\lambda_t\varsigma_t) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right] - P_Z(\Upsilon)\right\| \\
&= \left\|t(\lambda_t\varsigma_t) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \Upsilon\right\| \\
&= \left\|t\lambda_t\varsigma_t - t\lambda_t\Upsilon - t\Upsilon + t\lambda_t\Upsilon + t\Upsilon + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \Upsilon\right\| \\
&\leq t\lambda_t\|\varsigma_t - \Upsilon\| + t(1-\lambda_t)\|\Upsilon\| + (1-t)\|\varsigma_t - \Upsilon\| \\
&= [1-(1-\lambda_t)t]\|\varsigma_t - \Upsilon\| + t(1-\lambda_t)\|\Upsilon\|.
\end{aligned}$$

From this we have

$$\begin{aligned}
\|\varsigma_t - \Upsilon\| &\leq [1-(1-\lambda_t)t]\|\varsigma_t - \Upsilon\| + t(1-\lambda_t)\|\Upsilon\| \\
&\leq \|\varsigma_t - \Upsilon\| - (1-\lambda_t)t\|\varsigma_t - \Upsilon\| + (1-\lambda_t)t\|\Upsilon\|.
\end{aligned}$$

It follows that $\|\varsigma_t - \Upsilon\| \leq \|\Upsilon\|$, therefore the net ς_t is bounded. Let $K := \|\Upsilon\|$, then clearly $\{\varsigma_t\} \subset B(\Upsilon, K)$.
Note that

$$\begin{aligned}
\left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \Upsilon\right\| &= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \frac{1}{\gamma_t}\int_0^{\gamma_t}\Upsilon ds\right\| \\
&= \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t}\big(E(s,0)\varsigma_t - E(s,0)\Upsilon\big)ds\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\gamma_t}\int_0^{\gamma_t}\left\|\left(E(s,0)\varsigma_t - E(s,0)\Upsilon\right)\right\|ds\\
&\leq \frac{1}{\gamma_t}\int_0^{\gamma_t}\|\varsigma_t - \Upsilon\|ds\\
&= \|\varsigma_t - \Upsilon\| \leq K.
\end{aligned}
$$

Also we note that, if $\varsigma \in B(\Upsilon, K)$, then
$\|E(s,0)\varsigma - \Upsilon\| \leq \|E(s,0)\varsigma - E(s,0)\Upsilon\| \leq \|\varsigma - \Upsilon\| \leq K$,
that is $B(\Upsilon, K)$ is $E(s,0)$-invariant $\forall\ s \geq 0$.
Set $\mho_t = t(\lambda_t\varsigma_t) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds$. Then $\varsigma_t = P_Z[\mho_t]$, therefore we have

$$
\begin{aligned}
\|E(\tau,0)\varsigma_t - \varsigma_t\| &= \|P_Z[E(\tau,0)\varsigma_t] - P_Z[\mho_t]\|\\
&\leq \|E(\tau,0)\varsigma_t - \mho_t\|\\
&\leq \left\|E(\tau,0)\varsigma_t - E(\tau,0)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\quad + \left\|E(\tau,0)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right.\\
&\quad \left. - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\| + \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \mho_t\right\|\\
&\leq \left\|E(\tau,0)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\quad + \left\|\varsigma_t - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\quad + \left\|\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \left(t(\lambda_t\varsigma_t) + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right)\right\|\\
&\leq \left\|E((\tau),0)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\quad + \left\|\varsigma_t - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\| + t\left\|\lambda_t\varsigma_t - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\leq \left\|E((\tau),0)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|\\
&\quad + 2t\left\|\lambda_t\varsigma_t - \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\right\|.
\end{aligned}
$$

Using Lemma (4), we get that

$$\lim_{t\to 0}\|E(\tau,0)\varsigma_t - \varsigma_t\| = 0, \tag{4}$$

for all $\tau \in [0,\infty)$. Note that we have $\varsigma_t = P_Z[\mho_t]$. By using metric projection property (1), we have

$$\|\varsigma_t - \Upsilon\|^2 = \langle \varsigma_t - \varsigma_t + \varsigma_t - \Upsilon, \varsigma_t - \Upsilon\rangle$$

$$
\begin{aligned}
&\leq \langle \mho_t - \Upsilon, \varsigma_t - \Upsilon\rangle \\
&= \langle t(\lambda_t\varsigma_t) - t\lambda_t\Upsilon + t\lambda_t\Upsilon \\
&\quad +(1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds + (1-t)\Upsilon - (1-t)\Upsilon, \Upsilon - \varsigma_t\rangle - \langle \Upsilon, \varsigma_t - \Upsilon\rangle \\
&= t\lambda_t\langle \varsigma_t - \Upsilon, \varsigma_t - \Upsilon\rangle + (1-t)\left\langle \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \Upsilon, \varsigma_t - \Upsilon\right\rangle \\
&\quad +\langle t\lambda_t\Upsilon + \Upsilon - t\Upsilon, \varsigma_t - \Upsilon\rangle - \langle \Upsilon, \varsigma_t - \Upsilon\rangle \\
&= t\lambda_t\langle \varsigma_t - \Upsilon, \varsigma_t - \Upsilon\rangle + (1-t)\left\langle \frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds - \Upsilon, \varsigma_t - \Upsilon\right\rangle \\
&\quad +\langle t(\lambda_t - 1)\Upsilon + \Upsilon - \Upsilon, \varsigma_t - \Upsilon\rangle \\
&= t\lambda_t\langle \varsigma_t - \Upsilon, \varsigma_t - \Upsilon\rangle + (1-t)\left\langle \frac{1}{\gamma_t}\int_0^{\gamma_t} (E(s,0)\varsigma_t - \Upsilon) ds, \varsigma_t - \Upsilon\right\rangle \\
&\quad -(1-\lambda_t)t\langle \Upsilon, \varsigma_t - \Upsilon\rangle \\
&\leq t\lambda_t\langle \varsigma_t - \Upsilon, \varsigma_t - \Upsilon\rangle + (1-t)\left|\left\langle \frac{1}{\gamma_t}\int_0^{\gamma_t} (E(s,0)\varsigma_t - \Upsilon) ds, \varsigma_t - \Upsilon\right\rangle\right| \\
&\quad -(1-\lambda_t)t\langle \Upsilon, \varsigma_t - \Upsilon\rangle \\
&\leq t\lambda_t\|\varsigma_t - \Upsilon\|^2 + (1-t)\left[\frac{1}{\gamma_t}\int_0^{\gamma_t} \|E(s,0)\varsigma_t - \Upsilon\| ds\right]\|\varsigma_t - \Upsilon\| \\
&\quad -(1-\lambda_t)t\langle \Upsilon, \varsigma_t - \Upsilon\rangle \\
&= [1-(1-\lambda_t)t]\|\varsigma_t - \Upsilon\|^2 - (1-\lambda_t)t\langle \Upsilon, \varsigma_t - \Upsilon\rangle.
\end{aligned}
$$

From above we have

$$\|\varsigma_t - \Upsilon\|^2 \leq \|\varsigma_t - \Upsilon\|^2 - (1-\lambda_t)t\|\varsigma_t - \Upsilon\|^2 - (1-\lambda_t)t\langle \Upsilon, \varsigma_t - \Upsilon\rangle.$$

Which implies that

$$\|\varsigma_t - \Upsilon\|^2 \leq \langle \Upsilon, \Upsilon - \varsigma_t\rangle, \qquad \forall\, \Upsilon \in F_{\mathbb{E}}. \tag{5}$$

The last inequality (5), shows that $\omega_s(\varsigma_t) = \omega_\varsigma(\varsigma_t)$, where $\omega_s(\varsigma_t)$ and $\omega_\varsigma(\varsigma_t)$ denoting the strong and weak limit points sets of ς_t respectively. Let $\{t_n\}$ be a progression in $(0,1)$ such that $t_n \to 0$, as $n \to \infty$. Put $\varsigma_n := \varsigma_{t_n}$, $\varsigma_n := \mho_{t_n}\, and\, \gamma_n := \beta_{t_n}$. So ς_n become bounded, hence we may assume that the sequence $\{\varsigma_n\}$ converge to a point $\varsigma^* \in Z$, in a weak sense. Also, $\mho_n \to \varsigma^*$. From equation(4) and Lemma (1) we obtain that $\varsigma^* \in F_{\mathbb{E}}$. From (5) we have

$$\|\varsigma_n - \Upsilon\|^2 \leq \langle \Upsilon, \Upsilon - \varsigma_n\rangle \qquad \forall\, \Upsilon \in F_{\mathbb{E}}. \tag{6}$$

In particular, if we replace Υ by ς^* in (6), then we have

$$\|\varsigma_n - \varsigma^*\|^2 \leq \langle \varsigma^*, \varsigma^* - \varsigma_n\rangle. \tag{7}$$

However, $\varsigma_n \to \varsigma^*$. This and (7) guarantees that $\varsigma_n \to \varsigma^*$ and so if t goes to 0, from the right side then $\{\varsigma_t\}$ become relatively compact, in the norm topology. Taking limit n goes to ∞ in (6), we have

$$\|\varsigma^* - \Upsilon\|^2 \leq \langle \Upsilon, \Upsilon - \varsigma^*\rangle \qquad \forall\, \Upsilon \in F_{\mathbb{E}}.$$

Which implies that

$$0 \leq \langle \Upsilon, \Upsilon - \varsigma^* \rangle \qquad \forall\, \Upsilon \in F_{\mathbb{E}}.$$

Hence, $\varsigma^* = P_{F_{\mathbb{E}}}(0)$, which is obviously unique and this shows that the net ς_t converge strongly to ς^*.

Remark 3. *Clearly the net*

$$\varsigma_t = P_Z\big[tw_t + (1-t)\frac{1}{\gamma_t}\int_0^{\gamma_t} E(s,0)\varsigma_t ds\big], \quad \forall\, 0 < \ t\ 1,$$

has only weak convergence. But, the similar net (3) has strong convergence (with $\lambda_t \to 1$*).*

Now we define a sequence ς_n for the non-expansive family $\mathbb{E} = \{E(s,0)\}_{s\geq 0} : Z \to Z$ and we will prove that the sequence ς_n converge strongly to $x \in F_{\mathbb{E}}$

Theorem 2. *Let* $\mathbb{E} = \{E(s,0)\}_{s\geq 0} : Z \to Z$ *is non-expansive family with conditions* $\|\mho_m - E((\frac{t}{m})i_m, 0)\mho_m\|^2 \leq \frac{i_m}{m} diam(Z)$ *and* $F_{\mathbb{E}} \neq \emptyset$*. Consider* $\{\varsigma_n\}$ *be the sequence*

$$\varsigma_{n+1} = (1-\mu_n)\varsigma_n + \mu_n P_Z\big[\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\big], \qquad \forall n \geq 0. \tag{8}$$

Where $\{\eta_n\}, \{\mu_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $[0,1]$ *and* $\{\gamma_n\}$ *is a sequence in* $(0,+\infty)$*, with the following conditions:*

- **(a)** $\lim_{n\to+\infty} \eta_n = 0, \Sigma_{n=0}^{+\infty}\eta_n = +\infty and \lim_{n\to+\infty} \lambda_n = 1;$
- **(b)** $0 < \liminf_{n\to+\infty} \mu_n \leq \limsup_{n\to+\infty} \mu_n < 1;$
- **(c)** $\lim_{n\to+\infty} \gamma_n = +\infty$ *and* $\lim_{n\to+\infty} \frac{\gamma_n - 1}{\gamma_n} = 1.$

Then $\varsigma_n \to \varsigma^* \in F_{\mathbb{E}}$*, in a strong sense.*

Proof. Let $p \in F_{\mathbb{E}}$, then we have

$$\begin{aligned}
&\|\varsigma_{n+1} - p\| \\
&\quad = \left\|(1-\mu_n)\varsigma_n + \mu_n P_Z\left[\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right] - p\right\| \\
&\quad = \left\|(1-\mu_n)\varsigma_n - p + \mu_n p + \mu_n\left(P_Z\left[\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right] - p\right)\right\| \\
&\quad \leq (1-\mu_n)\|\varsigma_n - p\| + \mu_n\left\|P_Z\left[\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds - P_Z(p)\right]\right\| \\
&\quad = (1-\mu_n)\|\varsigma_n - p\| + \mu_n\left\|\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds - p\right\| \\
&\quad = (1-\mu_n)\|\varsigma_n - p\| + \mu_n\Big\|\eta_n\lambda_n(\varsigma_n - p) - \eta_n(1-\lambda_n)p
\end{aligned}$$

$$
\begin{aligned}
&+(1-\eta_n)\left(\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds - \frac{1}{\gamma_n}\int_0^{\gamma_n} p ds\right)\Big\| \\
\leq\ & (1-\mu_n)\|\varsigma_n - p\| + \mu_n\big(\eta_n\lambda_n\|\varsigma_n - p\| + \eta_n(1-\lambda_n)\|p\| \\
&+(1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n}\|\varsigma_n - p\|ds\big) \\
=\ & [1-(1-\lambda_n)\eta_n\mu_n]\|\varsigma_n - p\| + (1-\lambda_n)\eta_n\mu_n\|p\|.
\end{aligned}
$$

From the above

$$
\begin{aligned}
\|\varsigma_{n+1} - p\| &\leq [1-(1-\lambda_n)\eta_n\mu_n]\|\varsigma_n - p\| + (1-\lambda_n)\eta_n\mu_n\|p\|. \\
&\leq [1-\eta_n\mu_n + \eta_n\mu_n\lambda_n + \eta_n\mu_n - \eta_n\mu_n\lambda_n] max\{\|\varsigma_n - p\|, \|p\|\}.
\end{aligned}
$$

Which imply that

$$
\|\varsigma_{n+1} - p\| \leq max\{\|\varsigma_n - p\|, \|p\|\}.
$$

By induction, we have,

$$
\|\varsigma_n - p\| \leq max\{\|\varsigma_0 - p\|, \|p\|\}.
$$

Set $\mho_n = P_Z[\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)v_n]$ for all $n \geq 0$, where $v_n = \frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds$. So we have

$$
\begin{aligned}
\|\mho_n - \mho_{n-1}\| &\leq \|\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)v_n - \mu_{n-1}(\alpha_{n-1}\varsigma_{n-1}) - (1-\mu_{n-1})v_{n-1}\| \\
&= \|\eta_n\lambda_n(\varsigma_n - \varsigma_{n-1}) + (\eta_n\lambda_n - \mu_{n-1}\alpha_{n-1})\varsigma_{n-1} \\
&\quad +(\eta_{n-1} - \eta_n)v_{n-1} + (1-\eta_n)(v_n - v_{n-1})\| \\
&= \eta_n\lambda_n\|\varsigma_n - \mho_{n-1}\| + |\eta_n\lambda_n - \eta_{n-1}\alpha_{n-1}|\|\varsigma_{n-1}\| + |\eta_{n-1} - \eta_n|\|v_{n-1}\| \\
&\quad +(1-\eta_n)\|v_n - v_{n-1}\|.
\end{aligned}
$$

And

$$
\begin{aligned}
\|v_n - v_{n-1}\| &= \Big\|\frac{1}{\gamma_n}\int_0^{\gamma_n}[E(s,0)\varsigma_n - E(s,0)\varsigma_{n-1}]ds \\
&\quad +\left(\frac{1}{\gamma_n} - \frac{1}{\gamma_{n-1}}\right)\int_0^{\gamma_{n-1}}[E(s,0)\varsigma_{n-1} - E(s,0)p]ds \\
&\quad -\frac{1}{\gamma_{n-1}}\int_0^{\gamma_n}[E(s,0)\varsigma_{n-1} - E(s,0)p]ds\Big\| \\
&\leq \frac{1}{\gamma_n}\int_0^{\gamma_n}\|\varsigma_n - \varsigma_{n-1}\|ds + (\gamma_{n-1})\left|\frac{1}{\gamma_n} - \frac{1}{\gamma_{n-1}}\right|\frac{1}{\gamma_{n-1}}\int_0^{\gamma_{n-1}}\|\varsigma_{n-1} - p\|ds \\
&\quad +\frac{1}{\gamma_{n-1}}\int_0^{\gamma_n}\|\varsigma_{n-1} - p\|ds \\
&\leq \|\varsigma_n - \varsigma_{n-1}\| + 2\frac{|\gamma_n - \gamma_{n-1}|}{\gamma_n}\|\varsigma_{n-1} - p\|.
\end{aligned}
$$

Therefore, we have

$$
\|\mho_n - \mho_{n-1}\| \leq \eta_n\lambda_n\|\varsigma_n - \varsigma_{n-1}\| + |\eta_n\lambda_n - \eta_{n-1}\alpha_{n-1}|\|\varsigma_{n-1}\| + |\eta_{n-1} - \eta_n|\|v_{n-1}\|
$$

$$
\begin{aligned}
&+(1-\eta_n)\Big(\|\varsigma_n-\varsigma_{n-1}\|+2\frac{|\gamma_n-\gamma_{n-1}|}{\gamma_n}\|\varsigma_{n-1}-p\|\Big)\\
\leq\ & [1-(1-\lambda_n)\eta_n]\|\varsigma_n-\varsigma_{n-1}\|+|\eta_n\lambda_n-\eta_{n-1}\alpha_{n-1}|\|\varsigma_{n-1}\|\\
&+|\eta_{n-1}-\eta_n|\|\upsilon_{n-1}\|+\frac{|\gamma_n-\gamma_{n-1}|}{\gamma_n}2\|\varsigma_{n-1}-p\|.\\
\leq\ & [1-(1-\lambda_n)\eta_n]\|\varsigma_n-\varsigma_{n-1}\|\\
&+M\big(|\eta_n\lambda_n-\eta_{n-1}\alpha_{n-1}|+|\eta_{n-1}-\eta_n|+\frac{|\gamma_n-\gamma_{n-1}|}{\gamma_n}\big).
\end{aligned}
$$

Where $M>0$, and

$$\sup_{n\geq 1}\big\{\|\varsigma_{n-1}\|,\|\upsilon_{n-1}\|,2\|\varsigma_{n-1}-p\|\big\}\leq M.$$

Hence

$$\limsup_{n\to+\infty}\big(\|\mho_n-\mho_{n-1}\|-\|\varsigma_n-\varsigma_{n-1}\|\big)\leq 0.$$

Using Lemma (2) we have

$$\lim_{n\to+\infty}\|\mho_n-\varsigma_n\|=0.$$

So, it follows that

$$\lim_{n\to+\infty}\|\varsigma_{n+1}-\varsigma_n\|=\lim_{n\to+\infty}\mu_n\|\mho_n-\varsigma_n\|.$$

Note that

$$
\begin{aligned}
\|E(\tau,0)\varsigma_n-\varsigma_n\| \leq\ & \Big\|E(\tau,0)\varsigma_n-E(\tau,0)\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|\\
&+\Big\|E(\tau,0)\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds-\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|\\
&+\Big\|\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds-\varsigma_n\Big\|\\
\leq\ & \Big\|E(\tau,0)\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds-\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|\\
&+2\Big\|\varsigma_n-\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|. \qquad (9)
\end{aligned}
$$

From (8), we have

$$
\begin{aligned}
\Big\|\varsigma_n &- \frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|\\
&\leq \big\|\varsigma_n-\varsigma_{n+1}\big\|+\Big\|\varsigma_{n+1}-\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|\\
&\leq \big\|\varsigma_n-\varsigma_{n+1}\big\|+(1-\mu_n)\Big\|\varsigma_n-\frac{1}{\gamma_n}\int_0^{\gamma_n}E(s,0)\varsigma_n ds\Big\|
\end{aligned}
$$

$$+\eta_n\lambda_n\left\|\varsigma_n - \frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\| + \eta_n(1-\lambda_n)\left\|\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\|.$$

Now

$$\begin{aligned}(1-1+\mu_n-\eta_n\lambda_n)\left\|\varsigma_n\right. &- \left.\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\| \\ &\leq \left\|\varsigma_n-\varsigma_{n+1}\right\| + \eta_n(1-\lambda_n)\left\|\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\|.\end{aligned}$$

It gives that

$$\begin{aligned}&\left\|\varsigma_n - \frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\| \\ &\leq \frac{1}{\mu_n-\eta_n\lambda_n}\left[\left\|\varsigma_n-\varsigma_{n+1}\right\| + \eta_n(1-\lambda_n)\left\|\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\right\|\right]. \quad (10)\end{aligned}$$

From (9), (10) and Lemma (4), we have

$$\lim_{n\to+\infty}\|E(\tau,0)\varsigma_n-\varsigma_n\| = 0, \qquad \forall \tau \geq 0. \quad (11)$$

Note that the sequence $\{\varsigma_n\}$ is bounded and $\{\varsigma_n\} \to \tilde{\varsigma}$ weakly. Let $\varsigma^* = P_{F_{\mathbb{E}}}(0)$, then there exists $M > 0$ such that $B(\varsigma^*, M)$ contains $\{\varsigma_n\}$. Also, $B(\varsigma^*, K)$ is $E(s)$-invariant for all $s \geq 0$ and so, we can assume that $\{E(s,0)\}_{s\geq 0}$ is a nonexpansive family on $B(\varsigma^*, K)$. By making use of (11) and Lemma 1(demiclosedness principle), we have $\tilde{\varsigma} \in F_{\mathbb{E}}$ and therefore

$$\limsup_{n\to+\infty}\langle\varsigma^*, \varsigma_{n+1}-\varsigma^*\rangle = \lim_{n\to+\infty}\langle\varsigma^*, \tilde{\varsigma}-\varsigma^*\rangle \leq 0.$$

In the end, we prove that $\varsigma_n \to \varsigma^*$. Set $\acute{\varsigma}_n = \eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds$, which gives that $\mho_n = P_Z[\acute{\varsigma}_n] \ \ \forall\, n \geq 0$. By making use of metric projection property 1, we have

$$\langle\mho_n-\acute{\varsigma}_n, \mho_n-\varsigma^*\rangle = \langle\acute{\varsigma}_n-\mho_n, \varsigma^*-\mho_n\rangle \leq 0.$$

Therefore

$$\begin{aligned}\|\mho_n-\varsigma^*\|^2 &= \langle\mho_n-\varsigma^*, \mho_n-\varsigma^*\rangle \\ &= \langle\mho_n-\acute{\varsigma}_n, \mho_n-\varsigma^*\rangle + \langle\acute{\varsigma}_n-\varsigma^*, \mho_n-\varsigma^*\rangle \\ &\leq \langle\acute{\varsigma}_n-\varsigma^*, \mho_n-\varsigma^*\rangle \\ &= \langle\eta_n(\lambda_n\varsigma_n) + (1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds - \varsigma^*, \mho_n-\varsigma^*\rangle \\ &= \langle\eta_n(\lambda_n\varsigma_n) + \eta_n\lambda_n\varsigma^* - \eta_n\lambda_n\varsigma^* - \eta_n\varsigma^* + \eta_n\varsigma^* \\ &\quad +(1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds - \varsigma^*, \mho_n-\varsigma^*\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \eta_n\lambda_n(\varsigma_n-\varsigma^*)-\eta_n(1-\lambda_n)\varsigma^*+(1-\eta_n)(v_n-\varsigma^*), \mho_n-\varsigma^*\rangle \\
&= \eta_n\lambda_n\langle \varsigma_n-\varsigma^*, \mho_n-\varsigma^*\rangle-\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle \\
&\quad +(1-\eta_n)\langle v_n-\varsigma^*, \mho_n-\varsigma^*\rangle \\
&= \eta_n\lambda_n\|\varsigma_n-\varsigma^*\|\|\mho_n-\varsigma^*\|-\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle \\
&\quad +(1-\eta_n)\|v_n-\varsigma^*\|\|\mho_n-\varsigma^*\| \\
&\leq \eta_n\lambda_n\|\varsigma_n-\varsigma^*\|\|\mho_n-\varsigma^*\|-\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle \\
&\quad +(1-\eta_n)\|\varsigma_n-\varsigma^*\|\|\mho_n-\varsigma^*\| \\
&= [1-(1-\lambda_n)\eta_n]\|\varsigma_n-\varsigma^*\|\|\mho_n-\varsigma^*\|-\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle \\
&\leq \frac{[1-(1-\lambda_n)\eta_n]}{2}\|\varsigma_n-\varsigma^*\|^2+\frac{1}{2}\|\mho_n-\varsigma^*\|-\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle,
\end{aligned}
$$

that is,

$$\|\varsigma_n-\varsigma^*\|^2 \leq [1-(1-\lambda_n)\eta_n]\|\varsigma_n-\varsigma^*\|^2-2\eta_n(1-\lambda_n)\langle \varsigma^*, \mho_n-\varsigma^*\rangle.$$

By the convexity of norm, we have

$$
\begin{aligned}
\|\varsigma_{n+1}-\varsigma^*\|^2 &\leq (1-\eta)\|\varsigma_n-\varsigma^*\|^2+\eta\|\mho_n-\varsigma^*\|^2 \\
&\leq (1-\eta)\|\varsigma_n-\varsigma^*\|^2 \\
&\quad +\eta\Big([1-(1-\lambda_n)\eta_n]\|\varsigma_n-\varsigma^*\|^2-2\eta_n(1-\lambda_n)\langle \varsigma^*, \varsigma_n-\varsigma^*\rangle\Big) \\
&= [1-(1-\lambda_n)\eta_n\eta]\|\varsigma_n-\varsigma^*\|^2-2(1-\lambda_n)\eta_n\eta\langle \varsigma^*, \mho_n-\varsigma^*\rangle.
\end{aligned}
$$

So all the conditions of Lemma 1 are satisfied. Therefore, we obtain that $\varsigma_n \to \varsigma^*$.

If we put 1 instead of μ_n, $\forall\ n \geq 0$, in Theorem 2, then we have the following result.

Corollary 1. *Let $\mathbb{E} = \{E(s,0)\}_{s\geq 0} : Z \to Z$ be nonexpansive family with condition $F_{\mathbb{E}} \neq \emptyset$. Let $\{\varsigma_n\}$ be the sequence*

$$\varsigma_{n+1} = (1-\mu_n)\varsigma_n+\mu_n P_Z\big[\frac{(1-\eta_n)}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds+\eta_n(\lambda_n\varsigma_n)\big], \forall n \geq 0. \tag{12}$$

Where $\{\eta_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$ and $\{\gamma_n\}$ is a sequence in $(0,\infty)$. Suppose the following axioms holds:

(i) $\lim_{n\to+\infty}\eta_n = 0, \Sigma_{n=0}^{+\infty}\eta_n = +\infty$ *and* $\lim_{n\to\infty}\lambda_n = 1$;

(ii) $\lim_{n\to+\infty}\gamma_n = \infty$ *and* $\lim_{n\to+\infty}\frac{\gamma_n-1}{\gamma_n} = 1$.

Then the sequence $\{\varsigma_n\}$ strongly converge to a point $\varsigma^* \in F_{\mathbb{E}}$.

Remark 4. *The algorithm*

$$\varsigma_{n+1} = (1-\mu_n)\varsigma_n+\mu_n P_Z\big[\eta_n(\lambda_n\varsigma_n)+(1-\eta_n)\frac{1}{\gamma_n}\int_0^{\gamma_n} E(s,0)\varsigma_n ds\big], \qquad \forall n \geq 0,$$

has just weak convergence. But, the sequence (8) (with $\lambda_n \to 1$) has strong convergence.

Using Remark 2, we have the following remark.

Remark 5. *The results given in [9], become special case of our work, if an evolution family is periodic of every positive real number.*

4. Example and Open Problem

Example 2. *Let $H := L^2([0,\pi], C)$ be the Hilbert space of all square integrable functions on $[0,\pi]$ and $\boldsymbol{S} = \{S(a) : a \geq 0\}$ be a semigroup defined by*

$$(S(a)\vartheta)(t) = \frac{2}{\pi}\sum_{m=1}^{\infty} e^{-am^2} c_m(\vartheta)\sin m\Upsilon, \quad \Upsilon \in [0,\pi],\, a \geq 0, \tag{13}$$

where $c_m(\vartheta) := \int_0^\pi y(s)\sin(ms)ds$. Obviously, $\boldsymbol{S}$ is a strongly continuous and non-expansive semigroup on H and it is generated by the linear operator A given by $\ddot{\vartheta} = A\vartheta$. Now Consider the following non-autonomous Cauchy problem

$$\begin{cases} \frac{\partial \mho(\Upsilon,\zeta)}{\partial \Upsilon} &= & k(\Upsilon)\frac{\partial^2 \mho(\Upsilon,\zeta)}{\partial^2 \zeta}, \quad \Upsilon > 0,\, \zeta \in [0,\pi], \\ \mho(\Upsilon, 0) &= & \mho(\Upsilon,\pi) = 0, \quad \Upsilon \geq 0, \\ \mho(0,\zeta) &= & q(\zeta), \end{cases}$$

where $q(\Upsilon) \in H$, and $k : R_+ \to [1,\infty)$ is a non-expansive and periodic function, i.e., $k(\Upsilon + p) = k(\Upsilon)$ for all $\Upsilon \in R_+$ for some $p \geq 1$.

Let $K(\Upsilon) = \int_0^\Upsilon k(\Upsilon)d\Upsilon$. Clearly the solution $y(.)$ of the above Cauchy problem satisfies the evolution property:

$$y(\Upsilon) = E(\Upsilon,\xi)y(\xi), \tag{14}$$

where $E(\Upsilon,\xi) = S(K(\Upsilon) - K(\xi))$. See [[25], Example 2.9b]. We can find $\vartheta \geq 0$ such that the function $\Upsilon \mapsto e^{\vartheta\Upsilon}\|\mho(\Upsilon)\|$ is bounded on R_+. In fact, we have

$$\begin{aligned} \int_0^{+\infty} \|E(\Upsilon,0)\vartheta\|^2 d\Upsilon &= \frac{2}{\pi}\int_0^{+\infty}\sum_{\vartheta=1}^{+\infty} c_\vartheta^2(\vartheta)e^{-2\vartheta^2 K(\Upsilon)}d\Upsilon \\ &= \frac{2}{\pi}\sum_{\vartheta=1}^{+\infty} c_\vartheta^2(\vartheta)\int_0^{+\infty} e^{-2\vartheta^2 K(\Upsilon)}d\Upsilon \\ &= \|\vartheta\|_2^2 \int_0^{+\infty} e^{-2\vartheta^2 K(\Upsilon)}d\Upsilon \\ &= \|\vartheta\|_2^2 \int_0^{+\infty} e^{-2K(\Upsilon)}d\Upsilon. \end{aligned} \tag{15}$$

Now consider

$$\int_0^{+\infty} e^{-2K(\Upsilon)}d\Upsilon = \sum_{j=0}^{+\infty}\int_{jz}^{(j+1)z} e^{-2K(\Upsilon)}d\Upsilon$$

$$
\begin{aligned}
&= \sum_{j=0}^{+\infty} \int_0^z e^{-2K(jz+\Upsilon)} d\Upsilon \\
&= \sum_{j=0}^{+\infty} e^{-2jK(z)} \int_0^z e^{-2K(\Upsilon)} d\Upsilon \\
&\leq z \sum_{j=0}^{+\infty} e^{-2jK(z)} \\
&= \frac{ze^{2K(z)}}{e^{2K(z)}-1} := C.
\end{aligned}
\tag{16}
$$

Hence,

$$
\int_0^{+\infty} \|E(\Upsilon, 0)\vartheta\|^2 d\Upsilon \leq C\|\vartheta\|_2^2. \tag{17}
$$

Now, by Theorem 8 in [26], for $M \geq 1$ the growth bound ω_0 of the family E satisfy $\omega_0 \leq \frac{-1}{2M}$, for further details see [26], obviously this shows that the evolution family is non-expansive on the Hilbert space H. Therefore both the net ς_t in Theorem **??** and the sequence ς_m in Theorem 2 strongly converges to a FP of a subfamily of the above NeE family.

Open problem: We leave open the question whether all the main results could be generalized for the whole periodic and then for the general non-periodic evolution families in Hilbert spaces?

5. Conclusion

In this work, we investigated the fixed point properties of a specific subfamily within a non-expansive evolution family of bounded linear operators on a Hilbert space H. By utilizing the framework of nets and a progression algorithm, we established several results concerning the strong convergence of sequences to a common fixed point of the considered subfamily. The theoretical findings were further supported by a concrete example, demonstrating the applicability of the developed framework. We conclude by posing an open problem to inspire future research in this direction.

Acknowledgements

The authors M. Sarwar, M.Y.B.Mufarrej and K. Abodayeh would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Competing interest

The authors declare that they have no competing interests concerning the publication of this article.

Author's contributions

All authors contribute equally to the writing of this manuscript. All authours reads and approved the final version.

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