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Efficient Viscosity Algorithms for Solving the Split Equality Fixed-Point Problem

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Abstract. Solving the Split Equality Fixed-Point Problem (SEFPP) often requires computing the norms of bounded and linear operators, a task that can be computationally demanding. To tackle this challenge, we investigated the SEFPP for quasi-pseudocontractive mappings in Hilbert spaces and proposed innovative viscosity algorithms to solve the problem. We established the strong convergence of these algorithms under appropriate conditions. To validate our theoretical results, we conducted numerical experiments, which not only confirmed the efficacy of our results but also demonstrated their advantages over existing methods in the literature. Our work generalizes and extends several significant results from prior research, contributing to the broader understanding of fixed-point problems and related problems.

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Key Words and Phrases: fixed point problem, iterative algorithm, nonlinear mappings, weak and strong convergence

1. Introduction

For $j = 1, 2$, let $\mathcal{H}_j$ be Hilbert spaces, $\mathcal{C}_j$ be nonempty, closed, and convex subsets of $\mathcal{H}_j$, and $\mathcal{A}_j : \mathcal{H}_1 \to \mathcal{H}_2$ be linear and bounded mappings, with $\mathcal{A}_j^*$ denoting the adjoint of $\mathcal{A}_j$.

The problem of finding

$$z \in \mathcal{C}_1 \text{ such that } \mathcal{A}_1 z \in \mathcal{C}_2, \tag{1}$$

is known as the Split Feasibility Problem (SFP). This problem was introduced by Censor et al. [1]. To solve problem (1), Byrne [2] proposed the following algorithm, known as the "$\mathcal{CQ}$ algorithm":

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$$z_{n+1} = P_{\mathcal{C}_1}\left(I - \lambda \mathcal{A}_1^*(I - P_{\mathcal{C}_2})\mathcal{A}_1\right) z_n, \tag{2}$$

where $\lambda \in \left(0, \frac{2}{\mathcal{A}_1\mathcal{A}_1^*}\right)$. This algorithm requires the computation of $P_{\mathcal{C}_j}$ onto $\mathcal{C}_j$, which is feasible when these projections have closed-form expressions. More results on the SFP and its applications can be found in [3–7].

The Split Equality Problem (SEP), which is related to the SFP, was introduced by Moudafi and Al-Shemas [8]. The SEP involves finding

$$y \in \mathcal{C}_1 \text{ and } z \in \mathcal{C}_2 \text{ such that } \mathcal{A}_1 y = \mathcal{A}_2 z. \tag{3}$$

By setting $\mathcal{A}_2 = I$ (identity mapping), the SEP reduces to the SFP. To solve problem (3), Moudafi and Al-Shemas [8] proposed the following algorithm:

$$\begin{cases} y_{n+1} = P_{\mathcal{C}_2}\left(y_n + \lambda_n \mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\right); \\ z_{n+1} = P_{\mathcal{C}_1}\left(z_n - \lambda_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\right), \quad n \geq 0; \end{cases} \tag{4}$$

where $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ are chosen arbitrarily, and $P_{\mathcal{C}_j}$, are metric projections onto $\mathcal{C}_j$. Under certain conditions imposed on $\{\lambda_n\}$, a weak convergence result was obtained.

Since any nonempty, closed, and convex subset of a Hilbert space can be represented as the fixed point set of its corresponding projector, see [7], therefore, equation (3) can be simplified to finding

$$y \in \text{Fix}(\mathcal{T}_1) \text{ and } z \in \text{Fix}(\mathcal{T}_2) \text{ such that } \mathcal{A}_1 y = \mathcal{A}_2 z, \tag{5}$$

where $\mathcal{T}_j : \mathcal{H}_j \to \mathcal{H}_j$, $j = 1, 2$, are nonlinear operators with $\text{Fix}(\mathcal{T}_j) \neq \emptyset$. Problem (5) is known as the Split Equality Fixed Point Problem (SEFPP).

Motivated by the results in [8], Moudafi [5] proposed the following algorithm:

$$\begin{cases} y_{n+1} = \mathcal{T}_2\left(y_n + \lambda_n \mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\right); \\ z_{n+1} = \mathcal{T}_1\left(z_n - \lambda_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\right), \quad n \geq 0. \end{cases} \tag{6}$$

By imposing certain conditions on the parameters and operators involved, a weak convergence result for algorithm (6) was obtained. However, implementing this algorithm requires computing the inverse of a bounded linear operator, which is generally a difficult task. To address this challenge, Byrne [2] introduced an alternative algorithm for solving the SFP that eliminates the need for such an inverse.

Moudafi's algorithm in [5] involved firmly quasi-nonexpansive mapping, a class that includes quasi-nonexpansive mapping. Since quasi-pseudocontractive mapping includes firmly quasi-nonexpansive, directed, and demicontractive mappings, this motivated Chang et al., [9] to introduce the following algorithm for solving the SEFPP involving quasi-pseudocontractive mappings and proved the weak convergence result of the algorithm:

$$\begin{cases} y_{n+1} = \beta_n y_n + (1 - \beta_n)\Big((1-\eta)I + \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_2)\Big)v_n; \\ v_n = y_n + \lambda_n \mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n); \\ \\ z_{n+1} = \beta_n z_n + (1 - \beta_n)\Big((1-\eta)I + \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)\Big)u_n; \\ u_n = z_n - \lambda_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n), n \geq 0. \end{cases} \tag{7}$$

This algorithm relies on prior knowledge of operator norms. Recently, Mohammed and Kilicman [10] investigated the SEFPP involving quasi-pseudocontractive mappings in Hilbert spaces. They developed innovative algorithms and demonstrated their convergences, both with and without prior knowledge of the operator norm for bounded and linear mappings.

Recently, Wang et al., [11], studied the SEFPP for the class of demicontractive operators in Hilbert spaces and proved the strong convergence results by proposing the following algorithm:

$$\begin{cases} y_{n+1} = \beta_n g_1(y_n) + (1-\beta_n)v_n, \\ v_n = y_n - \lambda_n \left(y_n - T_1 y_n + \mathcal{A}_1^* \left(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\right)\right), \\ \\ z_{n+1} = \beta_n g_2(y_n) + (1-\beta_n)w_n, \\ w_n = z_n - \lambda_n \left(z_n - T_2 z_n + \lambda_n \mathcal{A}_2^* \left(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n\right)\right), \quad n \geq 0. \end{cases} \tag{8}$$

The results from Chang et al. [9] and Mohammed and Kilicman [10], on the other hand, only show strong convergence if the operators are thought to be semi-compact. This compactness condition can be limiting, as many nonlinear mappings do not satisfy the compactness condition (see Example 1 for more details). Consequently, it was proposed that future studies could be focused on establishing strong convergent results without relying on the compactness assumption. In this paper, we aimed to obtain the strong convergence result of the proposed algorithm without imposing the semi-compactness condition on the operators involved, which is a critical consideration in infinite-dimensional spaces.

The paper is organized as follows: The introduction offers an overview of the study's background and context. This is followed by the preliminary section, where key definitions and lemmas are introduced. Section 3 presents the main results of the research, and Section 4 discusses the numerical results.

2. Preliminaries

This section offers a few fundamental findings that support the paper's primary findings.

Definition 1. *A mapping $\mathcal{T}_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be;*

- *(i) Fixed point of $\mathcal{T}_1$ $\big(Fix(\mathcal{T}_1)\big)$ if $\mathcal{T}_1 z = z$, for all $z \in \mathcal{H}_1$, and we denote the set of $Fix(\mathcal{T}_1)$ by $\{z \in Fix(\mathcal{T}_1) : \mathcal{T}_1 z = z\}$.*
- *(ii) Nonexpansive if $\|\mathcal{T}_1 y - \mathcal{T}_1 z\| \leq \|y - z\|, \forall y, z \in \mathcal{H}_1$.*
- *(iii) Quasi-nonexpansive if* $\|\mathcal{T}_1 y - z\| \leq \|y - z\|, \forall y \in \mathcal{H}_1$ and $z \in \text{Fix}(\mathcal{T}_1)$.
- *(iv) Directed if* $\|\mathcal{T}_1 y - z\|^2 \leq \|y - z\|^2 - \|\mathcal{T}_1 y - y\|^2, \forall y \in \mathcal{H}_1$ and $z \in \text{Fix}(\mathcal{T}_1)$.
- *(v) Demicontractive if* $\|z - \mathcal{T}_1 y\|^2 \leq \|z - y\|^2 + k\|y - \mathcal{T}_1 y\|^2$, $\forall y \in \mathcal{H}_1, z \in Fix(\mathcal{T}_1)$ and $k \in [0, 1)$, and it is called a quasi-pseudocontractive if $k = 1$.

(vi) Semi-compact if for any bounded sequence $\{z_n\} \subseteq \mathcal{H}_1$ with $\|z_n - \mathcal{T}_1 z_n\| \to 0$, then there exist $\{z_{n_i}\} \subseteq \{z_n\}$ such that $z_{n_i} \to z \in \mathcal{H}_1$.

Remark 1. *From the definitions provided above, we observe that the class of quasi pseudocontractive mapping is fundamental. This class encompasses various types of nonlinear mappings, including demicontractive mapping, directed mapping, quasi-nonexpansive mapping, and strictly pseudocontractive mapping, all of which serve as special cases. For further details and examples, see [9, 10].*

Remark 2. *It is obvious that if $\mathcal{T}_1$ is quasi-nonexpansive then*

$$\|\mathcal{T}_1 y - y\| \leq 2\langle y - \mathcal{T}_1 y, y - z\rangle .$$

Lemma 1. *(Chang et al., [9]) Suppose $\mathcal{T}_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is Lipschitz with $\mathcal{L} > 0$, and $\mathcal{U}_1 := (1-\eta)I + \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)$, then*

(i) $Fix(\mathcal{T}_1) = Fix((1-\eta)I + \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)) = Fix(\mathcal{U}_1)$;

(ii) $\mathcal{U}_1$ is demiclosed at zero only if $\mathcal{T}$ is demiclosed at zero;

(iii) $\mathcal{U}_1$ is $\mathcal{L}^2-$ Lipschitzian;

(iv) $\mathcal{U}_1$ is quasi-nonexpansive only if $\mathcal{T}_1$ is quasi-pseudocontractive.

Lemma 2. *(Xu, [12]) Let $\{a_n\}, \{b_n\} \subseteq \mathbb{R}^+$ such that $\sum_{n=0}^{\infty} b_n < \infty$. If*

$$a_{n+1} \leq (1+b_n)a_n \quad \text{or} \quad a_{n+1} \leq a_n + b_n, \quad \forall n \geq 0,$$

then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 3. *(Xu, [12]) Let $b_n > 0$ for $n \in \mathbb{N}$, and suppose the following recurrence holds:*

$$b_{n+1} \leq (1-\alpha_n)b_n + \alpha_n\Theta_n + \varepsilon_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\Theta_n\} \subset \mathbb{R}$ satisfy the conditions:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\varepsilon_n \geq 0$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \varepsilon_n < \infty$;

(iii) $\limsup\limits_{n\to\infty} \Theta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \alpha_n|\Theta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} b_n = 0$.

3. Main Results

In what follows, $\mathcal{S}$ will denote the solution set of equation (1), that is,

$$\mathcal{S} := \{\mathrm{y} \in \mathrm{Fix}(\mathcal{T}_1) \text{ and } \mathrm{z} \in \mathrm{Fix}(\mathcal{T}_2) \text{ such that } \mathcal{A}_1\mathrm{y} = \mathcal{A}_2\mathrm{z}\}. \tag{9}$$

Suppose that for $j = 1, 2$, the following assumptions hold:

(K1) $\mathcal{T}_j : \mathcal{H}_j \to \mathcal{H}_j$, are two quasi-pseudocontractive operators with $Fix(\mathcal{T}_j) \neq \emptyset$, in addition, $\mathcal{T}_1$ is also L- Lipschitz.

(K2) $\mathcal{A}_j : \mathcal{H}_j \to \mathcal{H}_j$ are linear and bounded operators with their adjoints $\mathcal{A}_j^*$.

(K3) $(\mathcal{T}_j - I)$ are demiclosed at origin.

(K4) $\mathcal{F}_j : \mathcal{H}_j \to \mathcal{H}_j$ are contraction mappings with contraction constant $\rho \in (0, 1]$;

(K5) Let $\mathcal{U}_j = (1 - \eta)I + \eta\mathcal{T}_j((1 - \zeta)I + \zeta\mathcal{T}_j)$, and define $(y_n, z_n) \subseteq \mathcal{H}_1 \times \mathcal{H}_2$ by

$$\begin{cases} y_{n+1} = \alpha_n\mathcal{F}_1(v_n) + (1 - \alpha_n)\mathcal{U}_1 v_n; \\ v_n = (1 - \tau_n)y_n + \tau_n\mathcal{U}_1 y_n + \tau_n\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n); \\ \\ z_{n+1} = \alpha_n\mathcal{F}_2(w_n) + (1 - \alpha_n)\mathcal{U}_2 w_n; \\ w_n = (1 - \tau_n)z_n + \tau_n\mathcal{U}_2 z_n + \tau_n\mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n), \forall n \geq 0; \end{cases} \tag{10}$$

where $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ are chosen arbitrary, $0 < \alpha_n < 1$ such that $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1} (1 - \alpha_n)\alpha_n < \infty$, and $\lim_{n\to\infty} \alpha_n = 0$, and $0 < \tau_n < 1$ such that

$$\inf_{n\geq 1} \tau_n \left((1 - \alpha_n(1 - 2\rho^2))\lambda - \tau_n\right) \geq \beta > 0,$$

where $\lambda = \frac{1}{2\max\{1, \|\mathcal{A}_1\|^2, \|\mathcal{A}_2\|^2\}}$, and $0 < \eta < \zeta < \frac{1}{1+\sqrt{1+L^2}}$.

Lemma 4. *Let $\{(y_n, z_n)\}$ be the sequence generated by algorithm (10), then*

(i) $\{(y_n, z_n)\}$ *is bounded;*

(ii) $\limsup_{n\to\infty} \|\mathcal{U}_1 y_n - y_n + \mathcal{A}_1^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 = 0$, *and*
$\limsup_{n\to\infty} \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 = 0$;

(iii) $\limsup_{n\to\infty} \|\mathcal{U}_1 y_n - y_n\| = 0$, $\limsup_{n\to\infty} \|\mathcal{U}_2 z_n - z_n\| = 0$, *and* $\limsup_{n\to\infty} \|\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\| = 0$.

Proof. Let $(y, z) \in \mathcal{S}$. By Lemma 1, it follows that $\mathcal{U}_1$ is quasi-nonexpansive. By algorithm (10) we have

$$\|z_{n+1} - z\|^2 = \|\alpha_n\mathcal{F}_2(w_n) + (1 - \alpha_n)\mathcal{U}_2 w_n - z\|^2$$

$$
\begin{aligned}
&= \|\alpha_n(\mathcal{F}_2(w_n) - z) + (1-\alpha_n)(\mathcal{U}_2 w_n - z)\|^2 \\
&= \alpha_n \|\mathcal{F}_2(w_n) - \mathcal{F}_2(z) + \mathcal{F}_2(z) - z\|^2 + (1-\alpha_n)\|\mathcal{U}_2 w_n - z\|^2 \\
&\quad - (1-\alpha_n)\alpha_n \|\mathcal{U}_2 w_n - \mathcal{F}_2(w_n)\|^2 \\
&\leq (1 - \alpha_n(1-2\rho^2)) \|w_n - z\|^2 + 2\alpha_n \|\mathcal{F}_2(z) - z\|^2, \text{ and} \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\|w_n - z\|^2 &= \left\|(1-\tau_n)(z_n - z) + \tau_n\Big(\mathcal{U}_2 z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n) - z\Big)\right\|^2 \\
&= (1-\tau_n)\|z_n - z\|^2 + \tau_n \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n) + z_n - z\|^2 \\
&\quad - (1-\tau_n)\tau_n \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 \\
&\leq \|z_n - z\|^2 + \tau_n^2 \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 \\
&\quad - 2\tau_n \langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle. \qquad (12)
\end{aligned}
$$

Thus, by equations (11) and (12), we have

$$
\begin{aligned}
\|z_{n+1} - z\|^2 \leq & (1 - \alpha_n(1-2\rho^2)) \|z_n - z\|^2 + 2\alpha_n \|\mathcal{F}_2(z) - z\|^2 \\
& + \tau_n^2 \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 \\
& - 2(1 - \alpha_n(1-2\rho^2))\tau_n \langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle. \qquad (13)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|y_{n+1} - y\|^2 \leq & (1 - \alpha_n(1-2\rho^2)) \|y_n - y\|^2 + 2\alpha_n \|\mathcal{F}_1(y) - y\|^2 \\
& + \tau_n^2 \|\mathcal{U}_1 y_n - y_n + \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \\
& - 2(1 - \alpha_n(1-2\rho^2))\tau_n \langle y_n - y, y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\rangle. \qquad (14)
\end{aligned}
$$

By equations (13) and (14), we deduce that

$$
\begin{aligned}
\Gamma_{n+1} \leq & (1 - \alpha_n(1-2\rho^2))\Gamma_n + 2\alpha_n\Big(\|\mathcal{F}_2(z) - z\|^2 + \|\mathcal{F}_1(y) - y\|^2\Big) + \tau_n^2 k_n \\
& - 2\tau_n(1 - \alpha_n(1-2\rho^2))\Big(\langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle \\
& + \langle y_n - y, y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\rangle\Big), \qquad (15)
\end{aligned}
$$

where $\Gamma_n := \|y_n - y\|^2 + \|z_n - z\|^2$, and

$$
k_n := \|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 + \|\mathcal{U}_1 y_n - y_n + \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2.
$$

On the other hand,

$$
\begin{aligned}
\langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle &= \langle z_n - z, z_n - \mathcal{U}_2 z_n\rangle \\
&\quad - \langle \mathcal{A}_2 z_n - \mathcal{A}_2 z, \mathcal{A}_2 z_n - \mathcal{A}_1 y_n\rangle \\
&\geq \frac{1}{2}\|z_n - \mathcal{U}_2 z_n\|^2
\end{aligned}
$$

$$- \langle \mathcal{A}_2 z_n - \mathcal{A}_2 z, \mathcal{A}_2 z_n - \mathcal{A}_1 y_n \rangle . \tag{16}$$

Similarly,

$$\begin{aligned}\langle y_n - y, y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\rangle \geq & \frac{1}{2}\|y_n - \mathcal{U}_1 y_n\|^2 \\ & - \langle \mathcal{A}_1 y_n - \mathcal{A}_1 y, \mathcal{A}_1 y_n - \mathcal{A}_2 z_n \rangle . \end{aligned}\tag{17}$$

By (16) and (17), and the fact that $\mathcal{A}_1 y = \mathcal{A}_2 z$, we have

$$\begin{aligned}
\langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle + \langle y_n - y, y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\rangle \\
\geq & \frac{1}{2}\|z_n - \mathcal{U}_2 z_n\|^2 - \langle \mathcal{A}_2 z_n - \mathcal{A}_2 z, \mathcal{A}_2 z_n - \mathcal{A}_1 y_n\rangle \\
& + \frac{1}{2}\|y_n - \mathcal{U}_1 y_n\|^2 - \langle \mathcal{A}_1 y_n - \mathcal{A}_1 y, \mathcal{A}_1 y_n - \mathcal{A}_2 z_n\rangle \\
= & \frac{1}{2}\Big(\|z_n - \mathcal{U}_2 z_n\|^2 + \|\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\|^2 \Big) \\
& + \frac{1}{2}\Big(\|y_n - \mathcal{U}_1 y_n\|^2 + \|\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\|^2 \Big) \\
\geq & \frac{1}{2}\Big(\|z_n - \mathcal{U}_2 z_n\|^2 + \frac{1}{\|\mathcal{A}_2\|^2}\|\mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \Big) \\
& + \frac{1}{2}\Big(\|y_n - \mathcal{U}_1 y_n\|^2 + \frac{1}{\|\mathcal{A}_1\|^2}\|\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \Big) \\
\geq & \frac{1}{2\max\{1, \|\mathcal{A}_2\|^2\}}\Big(\|z_n - \mathcal{U}_2 z_n\|^2 \\
& + \|\mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \Big) \\
& + \frac{1}{2\max\{1, \|\mathcal{A}_1\|^2\}}\Big(\|y_n - \mathcal{U}_1 y_n\|^2 \\
& + \|\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \Big) \\
\geq & \frac{1}{4\max\{1, \|\mathcal{A}_1\|^2, \|\mathcal{A}_2\|^2\}}\Big(\Big(\|y_n - \mathcal{U}_1 y_n\| \\
& + \|\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\| \Big)^2 \\
& + \Big(\|z_n - \mathcal{U}_2 z_n\|^2 \\
& + \|\mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\| \Big)^2\Big) \\
\geq & \frac{\lambda}{2}\Big(\|z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 \\
& + \|y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2\Big).
\end{aligned}\tag{18}$$

By equations (15) and (18), and noticing that $(1 - \alpha_n(1 - 2\rho^2)) > 0$, we have

$$\begin{aligned}\Gamma_{n+1} &\leq (1-\alpha_n(1-2\rho^2))\Gamma_n + 2\alpha_n\Big(\|\mathcal{F}_2(z)-z\|^2 + \|\mathcal{F}_1(y)-y\|^2\Big)\\ &\quad - \tau_n\Big((1-\alpha_n(1-2\rho^2))\lambda - \tau_n\Big)k_n \\ &\leq (1-\alpha_n(1-2\rho^2))\Gamma_n + 2\alpha_n\Big(\|\mathcal{F}_2(z)-z\|^2 + \|\mathcal{F}_1(y)-y\|^2\Big).\end{aligned} \tag{19}$$

Thus,

$$\begin{aligned}\Gamma_{n+1} \leq& (1-\alpha_n(1-2\rho^2))\Gamma_n + \frac{\alpha_n(1-2\rho^2)}{(1-2\rho^2)}\Big(2\|\mathcal{F}_2(z)-z\|^2 + 2\|\mathcal{F}_1(y)-y\|^2\Big).\\ \leq& \max\left\{\Gamma_n, 2\|\mathcal{F}_2(z)-z\|^2 + 2\|\mathcal{F}_1(y)-y\|^2\right\}.\end{aligned}$$

By induction, we deduce that

$$\Gamma_{n+1} \leq \max\left\{\Gamma_0, 2\|\mathcal{F}_2(z)-z\|^2 + 2\|\mathcal{F}_1(y)-y\|^2\right\}.$$

Thus $\Gamma_n := \|z_n - z\|^2 + \|y_n - y\|^2$ is bounded, which further implies that $\{(y_n, z_n)\}$ is also bounded.

By equation (19), we have

$$\begin{aligned}\tau_n\left((1-\alpha_n(1-2\rho^2))\lambda - \tau_n\right)\Big(&\|z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2\\ &+ \|y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2\Big)\\ &\leq \|z_n - z\|^2 - \|z_{n+1} - z\|^2\\ &+ \|y_n - y\|^2 - \|y_{n+1} - y\|^2\\ &+ 2\alpha_n\Big(\|\mathcal{F}_2(z)-z\|^2 + \|\mathcal{F}_1(y)-y\|^2\Big).\end{aligned}$$

Since $\inf\limits_{n\geq 1}\tau_n\left((1-\alpha_n(1-2\rho^2))\lambda - \tau_n\right) \geq \beta$, we therefore deduce that

$$\limsup_{n\to\infty}\Big(\|z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 + \|y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2\Big) = 0.$$

This turns to implies that

$$\limsup_{n\to\infty}\|y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 = 0, \quad \text{and}$$

$$\limsup_{n\to\infty}\|z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 = 0. \tag{20}$$

Since $\mathcal{U}_1$ is quasi-nonexpansive mapping, by Remark 2, we have

$$\frac{1}{2}\|y_n - \mathcal{U}_1 y_n\|^2 + \langle \mathcal{A}_1 y_n - \mathcal{A}_2 z_n, \mathcal{A}_1 y_n - \mathcal{A}_1 y\rangle \leq \langle y_n - \mathcal{U}_1 y_n, y_n - y\rangle$$

$$
\begin{aligned}
&+ \langle \mathcal{A}_1 y_n - \mathcal{A}_2 z_n, \mathcal{A}_1 y_n - \mathcal{A}_1 y \rangle \\
&= \langle y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n), y_n - y \rangle \\
&\leq \| y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\| \| y_n - y\|. \qquad (21)
\end{aligned}
$$

Since $\{\|y_n - y\|\}$ is bounded, thus, by equation (20) we deduce that

$$
\limsup_{n\to\infty} \|y_n - \mathcal{U}_1 y_n\| = 0. \qquad (22)
$$

Similarly,

$$
\limsup_{n\to\infty} \|z_n - \mathcal{U}_2 z_n\| = 0. \qquad (23)
$$

On the other hand,

$$
\begin{aligned}
\|\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\|^2 =& \langle \mathcal{A}_1 y_n - \mathcal{A}_2 z_n, \mathcal{A}_1 y_n - \mathcal{A}_2 z_n \rangle \\
=& \langle \mathcal{A}_1 y_n - \mathcal{A}_2 z_n, \mathcal{A}_1 y_n - \mathcal{A}_1 y \rangle + \langle \mathcal{A}_2 z_n - \mathcal{A}_1 y_n, \mathcal{A}_2 z_n - \mathcal{A}_2 z \rangle, \\
=& \langle \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n), y_n - y \rangle + \langle \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n), z_n - z \rangle \\
\leq& \|\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n) + (y_n - \mathcal{U}_1 y_n) - (y_n - \mathcal{U}_1 y_n)\| \|y_n - y\|\| \\
&+ \|\mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n) + (z_n - \mathcal{U}_2 z_n) - (z_n - \mathcal{U}_2 z_n)\| \|z_n - z\| \\
\leq& \|\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n) - (y_n - \mathcal{U}_1 y_n)\| \|y_n - y\| \\
&+ \|y_n - \mathcal{U}_1 y_n\| \|y_n - y\| \\
&+ \|\mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n) - (z_n - \mathcal{U}_2 z_n)\| \|z_n - z\| \\
&+ \|z_n - \mathcal{U}_2 z_n\| \|z_n - z\|.
\end{aligned}
$$

Thus, by (20) and the fact that $\{\|z_n - z\|\}$ and $\{\|y_n - y\|\}$ are bounded, we see that

$$
\limsup_{n\to\infty} \|\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\| = 0.
$$

This completes the proof of this lemma.

We are now in a position to prove that $(y_n, z_n) \to (y, z)$.

Theorem 1. *Suppose conditions $(K1)$–$(K5)$ are satisfied, and that $\mathcal{S} \neq \emptyset$. Then the sequence $\{(y_n, z_n)\}$ generated by algorithm (10) converges strongly to $(y, z) \in \mathcal{S}$.*

Proof. Let $(y, z) \in \mathcal{S}$, then by Lemma 1, we see that $\mathcal{U}_1$ is quasi-nonexpansive, and by algorithm (10), we have

$$
\begin{aligned}
\|y_{n+1} - y\|^2 =& \|\alpha_n(\mathcal{F}_1(v_n) - y) + (1-\alpha_n)(\mathcal{U}_1 v_n - y)\|^2 \\
=& \alpha_n^2 \|\mathcal{F}_1(v_n) - y\|^2 + (1-\alpha_n)^2 \|\mathcal{U}_1 v_n - y\|^2 \\
&+ 2(1-\alpha_n)\alpha_n \langle \mathcal{F}_1(v_n) - y, \mathcal{U}_1 v_n - y \rangle \\
=& (\alpha_n^2 \rho^2 + (1-\alpha_n)^2)\|v_n - y\|^2 + \alpha_n^2 \|\mathcal{F}_1(y) - y\|^2
\end{aligned}
$$

$$+ 2(1-\alpha_n)\alpha_n \langle \mathcal{F}_1(v_n) - y, \mathcal{U}_1 v_n - y\rangle \text{, and} \tag{24}$$

$$\begin{aligned}
2\langle \mathcal{F}_1(v_n) - y, \mathcal{U}_1 v_n - y\rangle =& 2\langle \mathcal{F}_1(v_n) - \mathcal{F}_1(y), \mathcal{U}_1 v_n - y\rangle + 2\langle \mathcal{F}_1(y) - y, \mathcal{U}_1 v_n - y\rangle \\
\leq& (1+\rho^2)\|v_n - y\|^2 + 2\langle \mathcal{F}_1(y) - y, \mathcal{U}_1 v_n - y\rangle \\
\leq& (1+\rho^2)\|v_n - y\|^2 + \|\mathcal{F}_1(y) - y\|^2 + \|\mathcal{U}_1 v_n - v_n\|^2 \\
&+ 2\langle \mathcal{F}_1(y) - y, v_n - y\rangle .
\end{aligned} \tag{25}$$

By equations (24) and (25), we have

$$\begin{aligned}
\|y_{n+1} - y\|^2 \leq& (\alpha_n^2\rho^2 + (1-\alpha_n)^2)\|v_n - y\|^2 + \alpha_n^2 \|\mathcal{F}_1(y) - y\|^2 \\
&+ (1-\alpha_n)\alpha_n\Big((1+\rho^2)\|v_n - y\|^2 + \|\mathcal{F}_1(y) - y\|^2 + \|\mathcal{U}_1 v_n - v_n\|^2 \\
&+ 2\langle \mathcal{F}_1(y) - y, v_n - y\rangle\Big) \\
\leq& (1-(1-\rho^2)\alpha_n)\|v_n - y\|^2 + \alpha_n\|\mathcal{F}_1(y) - y\|^2 \\
&+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + 2\langle \mathcal{F}_1(y) - y, v_n - y\rangle\Big).
\end{aligned} \tag{26}$$

By equations (12) and (26), we see that

$$\begin{aligned}
\|y_{n+1} - y\|^2 \leq& (1-(1-\rho^2)\alpha_n)\|y_n - y\|^2 + \alpha_n\|\mathcal{F}_1(y) - y\|^2 \\
&+ (1-(1-\rho^2)\alpha_n)\tau_n^2\|\mathcal{U}_1 y_n - y_n + \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\|^2 \\
&- 2(1-(1-\rho^2)\alpha_n)\tau_n\langle y_n - y, y_n - \mathcal{U}_1 y_n - \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n)\rangle \\
&+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + 2\langle \mathcal{F}_1(y) - y, v_n - y\rangle\Big).
\end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned}
\|z_{n+1} - z\|^2 \leq& (1-(1-\rho^2)\alpha_n)\|z_n - z\|^2 + \alpha_n\|\mathcal{F}_2(z) - z\|^2 \\
&+ (1-(1-\rho^2)\alpha_n)\tau_n^2\|\mathcal{U}_2 z_n - z_n + \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\|^2 \\
&- 2(1-(1-\rho^2)\alpha_n)\tau_n\langle z_n - z, z_n - \mathcal{U}_2 z_n - \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n)\rangle \\
&+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_2 w_n - w_n\|^2 + 2\langle \mathcal{F}_2(z) - z, w_n - z\rangle\Big).
\end{aligned} \tag{28}$$

Equations (18), (27), and (28) give

$$\begin{aligned}
\Gamma_{n+1} \leq& (1-(1-\rho^2)\alpha_n)\Gamma_n + \alpha_n\Big(\|\mathcal{F}_1(y) - y\|^2 + \|\mathcal{F}_2(z) - z\|^2\Big) \\
&- \tau_n((1-(1-\rho^2)\alpha_n)\lambda - \tau_n)k_n \\
&+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + \|\mathcal{U}_2 w_n - w_n\|^2 \\
&+ 2\langle \mathcal{F}_1(y) - y, v_n - z\rangle + 2\langle \mathcal{F}_2(z) - z, w_n - y\rangle\Big) \\
\leq& (1-(1-\rho^2)\alpha_n)\Gamma_n + \alpha_n\Big(\|\mathcal{F}_1(y) - y\|^2 + \|\mathcal{F}_2(z) - z\|^2\Big)
\end{aligned}$$

$$+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + \|\mathcal{U}_2 w_n - w_n\|^2 + 2\|\mathcal{F}_1(y)-y\|\|v_n - y\| + 2\|\mathcal{F}_2(z)-z\|\|w_n - z\|\Big). \tag{29}$$

Since (y_n, z_n) is bounded, we see that $\|w_n - z\|$ is bounded, and similarly, $\|v_n - y\|$ is also bounded.

On the other hand,

$$\begin{aligned}\|\mathcal{U}_1 v_n - v_n\| &\le \|\mathcal{U}_1 v_n - y\| + \|v_n - y\| \\ &\le 2\|v_n - y\|.\end{aligned} \tag{30}$$

Similarly,

$$\|\mathcal{U}_2 w_n - w_n\| \le 2\|w_n - z\|. \tag{31}$$

Therefore, $\|\mathcal{U}_1 v_n - v_n\|$ and $\|\mathcal{U}_2 w_n - w_n\|$ are also bounded. Since (y_n, z_n) is bounded, it follows that there exist $(y, z) \in \mathcal{S}$ for which $z_n \rightharpoonup z$ and $y_n \rightharpoonup y$. Thus $w_n = z_n - \tau_n(\mathcal{U}_2 z_n - z_n) + \tau_n \mathcal{A}_1^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n) \rightharpoonup z$ and $v_n = y_n - \tau_n(\mathcal{U}_1 y_n - y_n) + \tau_n \mathcal{A}_2^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n) \rightharpoonup y$.

On the other hand, we see that

$$\limsup_{n\to\infty}\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + \|\mathcal{U}_2 w_n - w_n\|^2 + 2\|\mathcal{F}_2(z)-z\|\|w_n - z\| + 2\|\mathcal{F}_1(y)-y\|\|v_n - y\|\Big) = 0. \tag{32}$$

By equation (29), we deduce that

$$\begin{aligned}\Gamma_{n+1} \le& (1-(1-\rho^2)\alpha_n)\Gamma_n + (1-\rho^2)\alpha_n \frac{\alpha_n(\|\mathcal{F}_1(y)-y\|^2 + \|\mathcal{F}_2(z)-z\|^2)}{(1-\rho^2)} \\ &+ (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + \|\mathcal{U}_2 w_n - w_n\|^2 + \|\mathcal{F}_1(y)-y\|^2 + \|\mathcal{F}_2(z)-z\|^2 \\ &+ 2\|\mathcal{F}_1(y)-y\|\|v_n - y\| + 2\|\mathcal{F}_2(z)-z\|\|w_n - z\|\Big).\end{aligned}$$

Thus, we see that

(i) $\sum_{n\ge 0}(1-\rho^2)\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}\frac{\alpha_n(\|\mathcal{F}_1(y)-y\|^2+\|\mathcal{F}_2(z)-z\|^2)}{(1-\rho^2)} = 0$; and $\sum_{n\ge 0}(1-\rho^2)\frac{\alpha_n^2(\|\mathcal{F}_1(y)-y\|^2+\|\mathcal{F}_2(z)-z\|^2)}{(1-\rho^2)} < \infty$;

(iii) $\eta_n \ge 0$ and $\sum_{n\ge 0}\eta_n < \infty$;
where $\eta_n = (1-\alpha_n)\alpha_n\Big(\|\mathcal{U}_1 v_n - v_n\|^2 + \|\mathcal{U}_2 w_n - w_n\|^2 + \|\mathcal{F}_1(y)-y\|^2 + \|\mathcal{F}_2(z) - z\|^2 + 2\|\mathcal{F}_1(y)-y\|\|v_n - y\| + 2\|\mathcal{F}_2(z)-z\|\|w_n - z\|\Big)$.

therefore, by Lemma 3, we deduce that $\lim_{n\to\infty} \Gamma_n = 0$. That is

$$\lim_{n\to\infty} \left(\|y_n - y\|^2 + \|z_n - z\|^2 \right) = 0.$$

Finally, we show that $(y, z) \in \mathcal{S}$.

Since, $\mathcal{U}_1 = (1-\eta)I - \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)I$, where $0 < \eta < \zeta < \frac{1}{1+\sqrt{1+L^2}}$ and $\mathcal{T}_1$ is Lipschitz, we have

$$\begin{aligned}
\eta\|y_n - \mathcal{T}_1 y_n\| &= \|y_n - (1-\eta)y_n - \eta\mathcal{T}_1 y_n\| \\
&= \|y_n - (1-\eta)y_n - \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)y_n + \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)y_n - \eta\mathcal{T}_1 y_n\| \\
&\le \|y_n - (1-\eta)y_n - \eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)y_n\| \\
&+ \|\eta\mathcal{T}_1((1-\zeta)I + \zeta\mathcal{T}_1)y_n - \eta\mathcal{T}_1 y_n\| \\
&\le \|y_n - \mathcal{U}_1 y_n\| + \eta L\|((1-\zeta)I + \zeta\mathcal{T}_1)y_n - y_n\| \\
&= \|y_n - \mathcal{U}_1 y_n\| + \eta\zeta L\|y_n - \mathcal{T}_1 y_n\|.
\end{aligned}$$

Therefore,

$$\|y_n - \mathcal{T}_1 y_n\| \le \frac{1}{(1-\zeta L)\eta}\|y_n - \mathcal{U}_1 y_n\|. \tag{33}$$

Similarly,

$$\|z_n - \mathcal{T}_2 z_n\| \le \frac{1}{(1-\zeta L)\eta}\|z_n - \mathcal{U}_2 z_n\|. \tag{34}$$

By (22) and (23), we see that

$$\lim_{n\to\infty} \|y_n - \mathcal{T}_1 y_n\| = 0, \text{ and } \lim_{\mathrm{n}\to\infty} \|\mathrm{z_n} - \mathcal{T}_2 \mathrm{z_n}\| = 0. \tag{35}$$

Now that $y_n \to y$ and $\lim_{n\to\infty} \|\mathcal{T}_1 y_n - y_n\| = 0$ couple with the demiclosedness of $(\mathcal{T}_1 - I)$ at origin, we have $y \in Fix(\mathcal{T}_1)$.

Similarly, $z_n \to z$ and $\lim_{n\to\infty} \|\mathcal{T}_2 z_n - z_n\| = 0$ together with the demiclosedness of $(\mathcal{T}_2 - I)$ at origin, we see that $z \in Fix(\mathcal{T}_2)$.

On the other hand, since $y_n \to y$, it is not difficult to see that

$$v_n = (1-\tau_n)y_n + \tau_n\mathcal{U}_1 y_n + \tau_n\mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n) \to 0.$$

Similarly, $w_n \to z$.

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are continuous mappings, we have

$$\mathcal{A}_1 v_n \to \mathcal{A}_1 y, \quad \text{and} \quad \mathcal{A}_2 w_n \to \mathcal{A}_2 z.$$

This implies that

$$\mathcal{A}_1 v_n - \mathcal{A}_2 w_n \to \mathcal{A}_1 y - \mathcal{A}_2 z,$$

which further implies that

$$\|\mathcal{A}_1 y - \mathcal{A}_2 z\| \leq \liminf_{n\to\infty} \|\mathcal{A}_1 v_n - \mathcal{A}_2 w_n\| = 0.$$

Thus, $\mathcal{A}_1 y = \mathcal{A}_2 z$, and noticing that $(y,z) \in Fix(\mathcal{T}_1) \times Fix(\mathcal{T}_2)$, we conclude that $(y,z) \in \mathcal{S}$, which completes the proof.

Corollary 2. *Suppose conditions $(K1)-(K4)$ are satisfied, and that $\mathcal{S} \neq \emptyset$. Then the sequence $\{(y_n, z_n)\}$ generated by*

$$\begin{cases} y_{n+1} = \alpha_n v_n + (1-\alpha_n)\mathcal{U}_1 v_n; \\ v_n = (1-\tau_n)y_n + \tau_n \mathcal{U}_1 y_n + \tau_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n); \\ \\ z_{n+1} = \alpha_n w_n + (1-\alpha_n)\mathcal{U}_2 w_n; \\ w_n = (1-\tau_n)z_n + \tau_n \mathcal{U}_2 z_n + \tau_n \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n), \forall n \geq 0; \end{cases} \tag{36}$$

where $\mathcal{U}_j = (1-\eta)I + \eta\mathcal{T}_j((1-\zeta)I + \zeta\mathcal{T}_j), j = 1,2$, $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ are chosen arbitrary, $0 < \alpha_n < 1$ such that $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1}(1-\alpha_n)\alpha_n < \infty$, and $\lim_{n\to\infty} \alpha_n = 0$, $0 < \tau_n < 1$ such that $\inf_{n\geq 1} \tau_n \left((1-\alpha_n(1-2\rho^2))\lambda - \tau_n\right) \geq \beta > 0$, where $\lambda = \frac{1}{2\max\{1, \|\mathcal{A}_1\|^2, \|\mathcal{A}_2\|^2\}}$, and $0 < \eta < \zeta < \frac{1}{1+\sqrt{1+L^2}}$. Then the sequence $\{(y_n, z_n)\}$ converges to $(y,z) \in \mathcal{S}$.

Proof. *This proof is a direct consequence of Theorem 1 by setting $\mathcal{F} = I$. Algorithm 36 was studied by Mohammed and Kilicman [10] in their work.*

Corollary 3. *Suppose conditions $(K1)-(K4)$ are satisfied, and that $\mathcal{S} \neq \emptyset$. Then the sequence $\{(y_n, z_n)\}$ generated by*

$$\begin{cases} y_{n+1} = \alpha_n v_n + (1-\alpha_n)\mathcal{U}_1 v_n; \\ v_n = y_n + \lambda_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n); \\ \\ z_{n+1} = \alpha_n w_n + (1-\alpha_n)\mathcal{U}_2 w_n; \\ w_n = z_n + \lambda_n \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n), \forall n \geq 0; \end{cases} \tag{37}$$

where $\mathcal{U}_j = (1-\eta)I + \eta\mathcal{T}_j((1-\zeta)I + \zeta\mathcal{T}_j), j = 1,2$, $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ are chosen arbitrary, $0 < \alpha_n < 1$, such that $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1}(1-\alpha_n)\alpha_n < \infty$, and $\lim_{n\to\infty} \alpha_n = 0$, $0 < \lambda_n < 1$, and $0 < \eta < \zeta < \frac{1}{1+\sqrt{1+L^2}}$. Then the sequence $\{(y_n, z_n)\}$ converges to $(y,z) \in \mathcal{S}$.

Proof. *This proof follows directly from Theorem 1 by taking $\mathcal{F} = I$, and $\tau_n = 0$. Algorithm 37 was proposed by (Chang et al., [9]).*

Corollary 4. *Suppose conditions $(K1)-(K4)$ are satisfied, and that $\mathcal{S} \neq \emptyset$. Then the sequence $\{(y_n, z_n)\}$ generated by*

$$\begin{cases} y_{n+1} = \mathcal{U}_1 v_n; \\ v_n = y_n + \lambda_n \mathcal{A}_1^*(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n); \\ \\ z_{n+1} = \mathcal{U}_2 w_n; \\ w_n = z_n + \lambda_n \mathcal{A}_2^*(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n), \forall n \geq 0; \end{cases} \tag{38}$$

where $\mathcal{U}_j = (1-\eta)I + \eta\mathcal{T}_j((1-\zeta)I + \zeta\mathcal{T}_j), j = 1, 2$ $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ *are chosen arbitrary,* $0 < \alpha_n < 1$, *such that* $\sum_{n\geq 1} \alpha_n = \infty$, $\sum_{n\geq 1} (1-\alpha_n)\alpha_n < \infty$, *and* $\lim_{n\to\infty} \alpha_n = 0$, *and* $0 < \lambda_n < 1$, *and* $0 < \eta < \zeta < \frac{1}{1+\sqrt{1+L^2}}$. *Then the sequence* $\{(y_n, z_n)\}$ *converges to* $(y, z) \in \mathcal{S}$.

Proof. This proof follows directly from Theorem 1 by taking $\mathcal{F} = I$, *and* $\tau_n = \alpha_n = 0$. *Algorithm 38 was proposed and studied by (Moudafi and Al-Shemas [8]).*

Corollary 5. *Suppose conditions* $(K1) - (K4)$ *are satisfied, and that* $\mathcal{S} \neq \emptyset$. *Then the sequence* $\{(y_n, z_n)\}$ *generated by*

$$\begin{cases} y_{n+1} = \beta_n \mathcal{F}_1(y_n) + (1-\beta_n) v_n, \\ v_n = y_n - \lambda_n \left(y_n - T_1 y_n + \mathcal{A}_1^* \left(\mathcal{A}_1 y_n - \mathcal{A}_2 z_n\right)\right), \\ \\ z_{n+1} = \beta_n \mathcal{F}_2(y_n) + (1-\beta_n) w_n, \\ w_n = z_n - \lambda_n \left(z_n - T_2 z_n + \mathcal{A}_2^* \left(\mathcal{A}_2 z_n - \mathcal{A}_1 y_n\right)\right), \quad n \geq 0; \end{cases} \tag{39}$$

where $(y_0, z_0) \in \mathcal{H}_1 \times \mathcal{H}_2$ *are chosen arbitrary,* $0 < \beta_n < 1$, *such that* $\sum_{n\geq 1} \beta_n = \infty$, $\sum_{n\geq 1} (1-\beta_n)\beta_n < \infty$, *and* $\lim_{n\to\infty} \beta_n = 0$, $0 < \lambda_n < 1$ *such that* $\inf_{n\geq 1} \lambda_n (\lambda - \lambda_n) \geq \beta > 0$, *where* $\lambda = \frac{1}{2\max\{1, \|\mathcal{A}_1\|^2, \|\mathcal{A}_2\|^2\}}$. *Then the sequence* $\{(y_n, z_n)\}$ *converges to* $(y, z) \in \mathcal{S}$.

Proof. This proof is a direct consequence of Theorem 1 by setting $\mathcal{U}_j = (1-\eta)I + \eta\mathcal{T}_j((1-\zeta)I + \zeta\mathcal{T}_j) = I$. *Algorithm 39 was studies by Wang et al., [11] in their work.*

4. Numerical Examples

This section presents numerical results that demonstrate our theoretical findings and compares them with some existing results from the literature.

The following example is an example of a nonlinear mapping that is not semi-compact

Example 1. *Let* $\mathcal{H}_1 = \ell_2(\mathbb{N})$, *and define a mapping* $\mathcal{T} : \ell_2 \to \ell_2$ *by*

$$T(x) = \begin{cases} \left(1 - \frac{1}{n}\right) e_n, & \text{if } x = e_n \text{ for some } n, \\ 0, & \text{otherwise.} \end{cases}$$

then $\mathcal{T}$ *is not semi-compact.*

Proof. Clearly, $\mathcal{T}$ is not linear. L et $\{x_n\} \subseteq \ell^2$ defined by $x_n = e_n$, where e_n is the standard basis vector with 1 in the n-th position and 0 elsewhere. This sequence is bounded since $\|e_n\| = 1$ for all n.

On the other hand, $\|x_n - T(x_n)\| = \frac{1}{n} \to 0$

However, no subsequence of $\{x_n\}$ converges strongly in ℓ^2 since

$$\|x_n - x_m\| = \|e_n - e_m\| = \sqrt{2}$$

does not converge to 0. Thus, $\mathcal{T}$ is not semi-compact.

Example 2. *The mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ define by $\mathcal{F}(z) = \frac{1}{2}\sin(z)$, for all $z \in \mathbb{R}$ is a contraction mapping.*

Proof. Clearly, the function is continuous for all $z \in \mathbb{R}$. By the Mean Value Theorem (MVT), there exists $c \in \mathbb{R}$ such that

$$\begin{aligned} |\mathcal{F}(y) - \mathcal{F}(z)| &= \frac{1}{2}|\sin(y) - \sin(z)| \\ &\leq \cos(c)|y - z| \\ &\leq |y - z|. \end{aligned}$$

Thus we see that $\mathcal{F}$ is a contraction mapping with a contraction constant $\cos(c)$.

The following are examples of quasi pseudocontractive mapping.

Example 3. *The mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ define by $\mathcal{F}(z) = \frac{z}{2}$ for all $z \in \mathbb{R}$ then $\mathcal{F}$ is quasi pseudocontractive with $Fix(\mathcal{F}) = 0$.*

Example 4. *The mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ define by $\mathcal{F}(x) = x + \sin(x)$ for all $x \in \mathbb{R}$ then $\mathcal{F}$ is quasi pseudocontractive.*

Proof. It is not difficult to see that with $Fix(\mathcal{F}) = 0$, $\mathcal{F}$ is quasi pseudocontractive mapping.

Corollary 6. *In Theorem (9), let $\mathcal{H}_1 = \mathbb{R}$, $\mathcal{C}, \mathcal{Q} \subseteq (0, \infty)$, and define the operators $\mathcal{A}_1 y = y$ and $\mathcal{A}_2 y = \frac{z}{5}$. It follows that $\mathcal{A}_1 = \mathcal{A}_1^* = 1$ and $\mathcal{A}_2 = \mathcal{A}_2^* = \frac{1}{5}$, respectively. Let $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\mathcal{F}(z) = \frac{1}{2}\sin(z), \quad \textit{for all } z \in \mathbb{R},$$

which is a contraction mapping. Define the mappings $\mathcal{T}_1 : \mathcal{C} \to \mathbb{R}$ and $\mathcal{T}_2 : \mathcal{Q} \to \mathbb{R}$ as

$$\mathcal{T}_1 y = \frac{y}{2}, \quad \forall y \in \mathcal{C},$$

and

$$\mathcal{T}_2 z = z + \sin(z), \quad \forall z \in \mathcal{Q}.$$

Clearly, $\mathcal{T}_1$ and $\mathcal{T}_2$ are quasi-pseudocontractive mappings with fixed points $Fix(\mathcal{T}_1) = 0$ and $Fix(\mathcal{T}_2) = 0$. Let $\eta = \frac{1}{5}$, $\xi = \frac{1}{7}$, $\tau_n = \frac{1}{9}$, and $\alpha_n = \frac{1}{11}$. These parameters satisfy the hypotheses of Theorem 9. By setting the number of iterations to 150 and using Maple, we obtain the following results:

n	Algorithm 10	
	$\{y_n\}$	$\{z_n\}$
1	1.0000000000	1.0000000000
2	0.8958712178	0.9167912023
3	0.8037675809	0.8418283630
4	0.7219891992	0.7740135339
.	.	.
.	.	.
.	.	.
148	0.0000000258	0.0000110780
149	0.0000000233	0.0000102608
150	0.0000000210	0.0000095038

Table 1: The numerical results of algorithm (10), starting with the initial values $y_1 = 1$ and $z_1 = 1$, showed how the sequence (y_n, z_n) converges to $(0, 0)$.

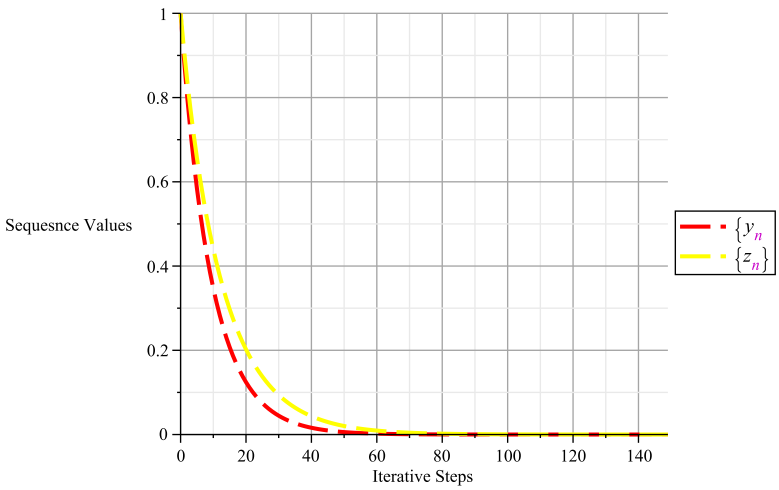

Figure 1: Graphical results presentation of algorithm (10), starting with the initial values $y_1 = 1$ and $z_1 = 1$, demonstrated how the sequence (y_n, z_n) converges to $(0, 0)$.

The following Table provides a comparison of algorithm (10) as presented in Table 1 with some of the recent results published in the literature.

n	Algorithm 36		Algorithm 37		Algorithm 39	
	$\{y_n\}$	$\{z_n\}$	$\{y_n\}$	$\{z_n\}$	$\{y_n\}$	$\{z_n\}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.0000
2	0.94774	0.96880	0.95916	0.97832	1.07480	1.0935
3	0.89822	0.93872	0.91999	0.95721	1.14970	1.1725
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
148	0.00036	0.01277	0.00209	0.05188	1.93210	2.0360
149	0.00034	0.01240	0.00200	0.05088	1.93210	2.0360
150	0.00032	0.012046	0.00192	0.04989	1.93210	2.0360

Table 2: This table presents the convergence rates of Algorithms 36, 37, and 39, as studied by Mohammed and Kilicman [10], Chang et al. [9], and Wang et al. [11], respectively. By comparing Algorithm 10 with these algorithms, it is clear that the proposed in this paper converges more rapidly. This demonstrates that the method proposed in this paper is more efficient in terms of convergence speed compared to the existing methods.

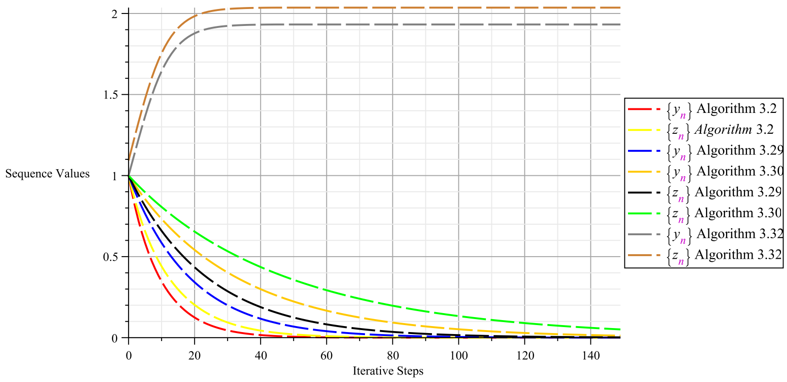

Figure 2: Graphical results presentation of algorithms (10), (36), (37) and (39) starting with the initial values $y_1 = 1$ and $z_1 = 1$, demonstrated how the sequence (y_n, z_n) converges to $(0, 0)$.

5. Conclusion

This study focused on the Split Equality Fixed-Point Problem (SEFPP) within the context of quasi-pseudocontractive mappings in Hilbert spaces. We introduced new viscosity algorithms for solving this problem, which indicated that the proposed algorithms converged strongly in an infinite-dimensional Hilbert space. Our findings not only broadened several key results from the existing literature but also addressed the computational challenges associated with calculating operator norms. To support our theoretical results, numerical experiments were performed, and a comparison with existing methods confirmed the superior efficiency and effectiveness of our proposed algorithms.

In summary, the algorithms developed in this paper provide a reliable solution to the SEFPP, achieving strong convergence without the need for the compactness assumption on the operators involved. Our research advances the field of fixed-point theory by extending prior results and offering more practical solutions to the SEFPP, especially in scenarios where computing operator norms is challenging. The algorithms' strong convergence makes them well-suited for practical applications.

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