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A Novel Fractional Integral of a Function via Polynomial n-Fractional s-Like Preinvexity

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Abstract. In the following numerical novel, we develop a new fractional integral operator that incorporates polynomials n-fractional with s-like preinvexity, thus expanding the notion of fractional calculus. This new operator provides a more comprehensive framework for examining the behavior of functions exhibiting generalized preinvexity properties which are crucial in many optimization problems. We investigate the existence, uniqueness, and stability of this fractional integral as well as its basic characteristics. In addition, we provide a number of inequalities that show how useful this operator is in the context of applied sciences and mathematical analysis. Our results not only advance the theory of fractional calculus but also pave the way for future investigations into integral inequalities and fractional optimization.

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1. Introduction and Preliminaries

Integral inequalities provide significant bounds for function integrals, making them indispensable tools in mathematical analysis (see [1, 2]). When exact evaluation is challenging or impossible, these inequalities can be used to estimate the magnitude or behavior of a function's integral. Typical instances are the Minkowski inequality, related to Lp spaces and norms, and Hölder's inequality, which extends the Cauchy-Schwarz inequality

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to integrals. In many domains, including partial differential equations, probability theory, and numerical analysis, these inequalities are crucial for enabling the study of function spaces, the convergence of function sequences, and the stability of differential equation solutions. Integral inequalities are essential in optimization problems and the demonstration of existence and uniqueness because they set upper and lower bounds.

Mathematicians including Leibniz, Liouville, Riemann, and others investigated the idea of extending the nth-order derivative to non-integer values, laying the groundwork for fractional calculus. The fractional derivative, which has multiple definitions (Riemann-Liouville, Caputo, and Grünwald-Letnikov), is one of the fundamental ideas in this discipline. Each definition is appropriate for a certain set of features and applications.

Convex functions were significantly expanded upon with the introduction of preinvex functions, which broadened the meaning of convexity in optimization theory (see [3–6]). Preinvex functions reduce this requirement by introducing an invex function, whereas convex functions are defined by the fact that any line segment connecting two points on the function's graph lies above the graph. In particular, an invex function serves as a sort of "generalized direction" in the domain of a function, and a function is said to be preinvex if it fulfills a specific inequality with regard to it. More applications can be made possible by this generalization, especially when the standard convexity requirements prove to be too restrictive. Preinvex functions are useful in tackling complicated optimization issues, such as those found in game theory, economics, and multi-objective optimization (see [7–9]). They maintain many of the beneficial aspects of convex functions, such as certain optimality criteria. Thus, the emergence of preinvexity has created new opportunities for study and application in fields where conventional convex analysis would not be enough.

Definition 1. *[10] The set $X^o \subset \Re^n$ is said to be invex with respect to $\varsigma_*(*,*)$, if for every $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ and $\mathfrak{t} \in [0,1]$*

$$\mathfrak{a}_1 + \mathfrak{t}\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1) \in X^o.$$

In definition 1, the set X^o is also known to be a ς_*-connected set.

For every convex set is invex with respect to $\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1) = \mathfrak{b}_1 - \mathfrak{a}_1$ but there exist invex sets which are not convex (see [11]).

Definition 2. *[12] A mapping $\mathfrak{F}$ on the invex set X^o is called to be preinvex with respect (w.r.) to ς_* if*

$$\mathfrak{F}\left(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right) \leq (1-\mathfrak{t})\,\mathfrak{F}\left(\mathfrak{a}_1\right) + \mathfrak{t}\mathfrak{F}\left(\mathfrak{b}_1\right); \quad \forall \mathfrak{a}_1, \mathfrak{b}_1 \in X^o, \mathfrak{t} \in [0,1]. \tag{1}$$

The function $-\mathfrak{F}$ is said to be preconcave if and only if $\mathfrak{F}$ is preinvex.

It is necessarily that each convexity becomes preinvexity, but not viceversa [13]. For example, $\mathfrak{F}(\mathfrak{t}) = -|\mathfrak{t}|$ (for all $\mathfrak{t} \in \Re$) is not a convex mapping but it is apreinvex function w.r. to

$$\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right) = \begin{cases} \mathfrak{b}_1 - \mathfrak{a}_1 & if \quad \mathfrak{a}_1\mathfrak{b}_1 > 0, \\ \mathfrak{a}_1 - \mathfrak{b}_1 & if \quad \mathfrak{a}_1\mathfrak{b}_1 < 0. \end{cases}$$

The following proposition regarding the mapping on ς_* is mentioned in [14].

Property$-\mathfrak{C}$: Let $X^o \subset \Re^n$ be an open invex subset w.r. to $\varsigma_* : X^o \times X^o \subset \Re^n$. For any $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ and $\mathfrak{t} \in [0,1]$,

$$\begin{aligned} \varsigma_*\left(\mathfrak{b}_1, \mathfrak{b}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right)\right) &= -\mathfrak{t}\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right), \\ \varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right)\right) &= (1-\mathfrak{t})\,\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right). \end{aligned} \tag{2}$$

For any $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ and $\mathfrak{t}_1, \mathfrak{t}_2 \in [0,1]$ from property$-\mathfrak{C}$, we have

$$\varsigma_*\left(\mathfrak{b}_1 + \mathfrak{t}_2\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right), \mathfrak{b}_1 + \mathfrak{t}_1\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right)\right) = \left(\mathfrak{t}_2 - \mathfrak{t}_1\right)\varsigma_*\left(\mathfrak{a}_1, \mathfrak{b}_1\right). \tag{3}$$

If $\mathfrak{F}$ is a preinvex function on $\varsigma_*\left(\mathfrak{a}_1, \mathfrak{a}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)$ and the mapping ς_* satisfies property–$\mathfrak{C}$, then for every $\mathfrak{t} \in [0,1]$, from (2), it yields that

$$\begin{aligned} &\left|\mathfrak{F}\left(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| = \left|\mathfrak{F}\left(\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right) + (1-\mathfrak{t})\,\varsigma_*\left(\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| \\ &\leq\ \mathfrak{t}\left|\mathfrak{F}\left(\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| + (1-\mathfrak{t})\left|\mathfrak{F}\left(\mathfrak{a}_1\right)\right| \end{aligned}$$

and

$$\begin{aligned} &\left|\mathfrak{F}\left(\mathfrak{a}_1 + (1-\mathfrak{t})\,\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| = \left|\mathfrak{F}\left(\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right) + \mathfrak{t}\varsigma_*\left(\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| \\ &\leq\ (1-\mathfrak{t})\left|\mathfrak{F}\left(\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right)\right| + \mathfrak{t}\left|\mathfrak{F}\left(\mathfrak{a}_1\right)\right|. \end{aligned}$$

In [15], the following inequalities of 'H–H' have been proved.

Theorem 1. *[10] Suppose $\mathfrak{F} : X = \left[\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right] \to (0, \infty)$ is a preinvex mapping on the interval of real numbers X^o with $\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right) > 0$*

$$\mathfrak{F}\left(\frac{2\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)}{2}\right) \leq \frac{1}{\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)}\int_{\mathfrak{a}_1}^{\mathfrak{a}_1 + \varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)} \mathfrak{F}(x)dx \ \leq\ \frac{\mathfrak{F}\left(\mathfrak{a}_1\right) + \mathfrak{F}\left(\mathfrak{b}_1\right)}{2}. \tag{4}$$

Suppose $X \subseteq \Re$ and $\mathfrak{F} :\subseteq \Re$ is a mapping on a differentiable at X^o(the interior of X) such that $[\mathfrak{a}_1, \mathfrak{b}_1] \in X^o$ with $\mathfrak{a}_1 < \mathfrak{b}_1$. In this case, the famous Ostrowski inequality [16] is stated as

$$\left|\mathfrak{F}(x) - \frac{1}{\mathfrak{b}_1 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{b}_1} \mathfrak{F}(x)dx\right| \leq \left|\frac{1}{4} + \frac{\left(x - \frac{\mathfrak{a}_1 + \mathfrak{b}_1}{2}\right)^2}{\left(\mathfrak{b}_1 - \mathfrak{a}_1\right)^2}\right|\left(\mathfrak{b}_1 - \mathfrak{a}_1\right)S, \tag{5}$$

for all $x \in [\mathfrak{a}_1, \mathfrak{b}_1]$, if $|\mathfrak{F}'| \leq S$.

Ostrowski-like inequalities, which give estimations of error for various quadrature criteria, are widely used in numerical analysis. These distinctions have grown and been applied to more fields in recent years (see [17–20]).

Definition 3. *[21, 22] Let $s \in [0,1]$. A real valued mapping $X^o \to \Re$ is said to be $s-$like convex on X^o if*

$$\mathfrak{F}(\mu\mathfrak{a}_1 + (1-\mu)\mathfrak{b}_1) \leq (1 - s(1-\mu))\mathfrak{F}(\mathfrak{a}_1) + (1 - s\mu)\mathfrak{F}(\mathfrak{b}_1), \tag{6}$$

for all $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ and $\mu \in [0,1]$.

Definition 4. *[23] Let $X \subset \Re$ be a nonempty invex set with respect to $\varsigma_* : X^o \times X^o \subset \Re \to \Re$. Then the mapping $\Psi : X \to \Re$ is said to be $s-$like preinvex, if*

$$\mathfrak{F}(\mathfrak{b}_1 + \mu\varsigma_*(\mathfrak{a}_1, \mathfrak{b}_1)) \leq (1 - s(1-\mu))\mathfrak{F}(\mathfrak{a}_1) + (1 - s\mu)\mathfrak{F}(\mathfrak{b}_1). \tag{7}$$

In [24], İşcan gave the definition on $n-$fractional polynomial convexity as below.

Definition 5. *[24] Suppose $n \in \mathcal{N}$. A non-negative mapping $X^o \subset \Re \to \Re$ is said to be an $n-$fractional polynomial convex(FPC) mapping if*

$$\mathfrak{F}(\mu\mathfrak{a}_1 + (1-\mu)\mathfrak{b}_1) \leq \frac{1}{n}\sum_{i=1}^{n}\mu^{\frac{1}{i}}\mathfrak{F}(\mathfrak{a}_1) + \frac{1}{n}\sum_{i=1}^{n}(1-\mu)^{\frac{1}{i}}\mathfrak{F}(\mathfrak{b}_1), \tag{8}$$

for all $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ and $\mu \in [0,1]$.

Definition 6. *[25] Consider $\mathfrak{F} \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{b}_1]$. The left-right-sided Riemann-Liouville(R–L) fractional integrals of order $\varrho > 0$ are defined by*

$$\mathfrak{J}^{\varrho}_{\mathfrak{a}_1-}\mathfrak{F}(x) = \frac{1}{\Gamma(\varrho)}\int_{\mathfrak{a}_1}^{x}(x-t)^{\varrho-1}\mathfrak{F}(\mathfrak{t})\,d\mathfrak{t}; \quad \mathfrak{a}_1 < x \tag{9}$$

and

$$\mathfrak{J}^{\varrho}_{\mathfrak{b}_1+}\mathfrak{F}(x) = \frac{1}{\Gamma(\varrho)}\int_{x}^{\mathfrak{b}_1}(\mathfrak{t}-x)^{\varrho-1}\mathfrak{F}(\mathfrak{t})\,d\mathfrak{t}\ ; \quad x < \mathfrak{b}_1. \tag{10}$$

Gamma function is defined as $\Gamma(\varrho) = \int_0^\infty e^{-u}u^{\varrho-1}du$.

In [26], Mubeen et al. introduced the following class of fractional integrals.

Definition 7. *[26] Suppose that $\mathfrak{F} \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{b}_1]$. The $\mathfrak{K}-$fractional integrals $\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1-}\mathfrak{F}(x)$ and $\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{b}_1+}\mathfrak{F}(x)$ order $\varrho > 0, \mathfrak{K} > 0$ are defined as*

$$\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1-}\mathfrak{F}(x) = \frac{1}{\mathfrak{K}\Gamma_{\mathfrak{K}}(\varrho)}\int_{\mathfrak{a}_1}^{x}(x-\mathfrak{t})^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}(\mathfrak{t})\,d\mathfrak{t}; \quad \mathfrak{a}_1 < x \tag{11}$$

and

$$\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{b}_1+}\mathfrak{F}(x) = \frac{1}{\mathfrak{K}\Gamma_{\mathfrak{K}}(\varrho)}\int_{x}^{\mathfrak{b}_1}(\mathfrak{t}-x)^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}(\mathfrak{t})\,d\mathfrak{t}\ ; \quad x < \mathfrak{b}_1, \tag{12}$$

respectively, where $\mathfrak{K} > 0$ and $\Gamma_{\mathfrak{K}}(\varrho)$ is the $\mathfrak{K}-$gamma function is given as $\Gamma_{\mathfrak{K}}(\varrho) = \int_0^\infty \mathfrak{t}^{\varrho-1}e^{-\frac{\mathfrak{t}^{\mathfrak{K}}}{\mathfrak{K}}}dt$ with the properties $\Gamma_{\mathfrak{K}}(\varrho + \mathfrak{K}) = \varrho\Gamma_{\mathfrak{K}}(\varrho)$ and $\Gamma_{\mathfrak{K}}(\mathfrak{K}) = 1$. It is noted that $\mathfrak{J}^{0,\mathfrak{K}}_{\mathfrak{a}_1-}\mathfrak{F}(x) = \mathfrak{J}^{0,\mathfrak{K}}_{\mathfrak{b}_1+}\mathfrak{F}(x) = \mathfrak{F}(x)$.

2. Main Results

2.1. Polynomials on $n-$fractional $s-$like preinvex mappings :

This section examines the basic algebraic features of a novel fractional integral operator that incorporates polynomials on $n-$fractional with $s-$like preinvex mappings.

Definition 8. *Let us suppose that $s \in [0,1]$, $n \in \mathcal{N}$, $a_i \geq 0 \left(i = \overline{1,n}\right)$, such that $\sum_{i=1}^{n} a_i > 0$, $X^o \subset \Re$ is an interval. A non-negative mapping $X^o \times X^o \subset \Re \to \Re$ is said to be a polynomial on $n-$fractional $s-$like preinvex mapping if for every*

$$\mathfrak{F}\left(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right) \leq \frac{\sum_{i=1}^{n} a_i \left(1 - s\left(1 - \mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i} \mathfrak{F}\left(\mathfrak{a}_1\right) + \frac{\sum_{i=1}^{n} a_i \left(1 - s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i} \mathfrak{F}\left(\mathfrak{b}_1\right), \tag{13}$$

for all $\mathfrak{t} \in [0,1]$, $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$.

Remark 1.

- *If we take $n = 1$ in Definition (8), we attain [21].*
- *If $n = 1$ and $s = 1$ in Definition (8), it will be explored by Weir and Mond [12].*
- *With taking $n = 1$ and $\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right) = \mathfrak{b}_1 - \mathfrak{a}_1$ in Definition (8), then it will attain a published definition named as $s-$type convexity that was explored by İ. İşcan et al. [21].*

We will mention the nature of class with some polynomials on $n-$fractional $s-$like preinvex mappings by GFPP$-s$.

Example 1. *Consider a mapping $\mathfrak{F}(x) = x^2$, and with some substitutions as $s = 0.4$, $n = 2$, $\mathfrak{a}_1 = 1, \mathfrak{b}_1 = 2$ and $\mathfrak{t} = 0.4$. According to (13), one writes*

$$\mathfrak{F}\left(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*\left(\mathfrak{b}_1, \mathfrak{a}_1\right)\right) \leq \frac{\sum_{i=1}^{n} a_i \left(1 - s\left(1 - \mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i} \mathfrak{F}\left(\mathfrak{a}_1\right) + \frac{\sum_{i=1}^{n} a_i \left(1 - s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i} \mathfrak{F}\left(\mathfrak{b}_1\right).$$

For $\mathfrak{a}_1 = 1$ and $\mathfrak{b}_1 = 3$, we have

$$\mathfrak{F}\left(1 + 0.4\varsigma_*\left(3,1\right)\right) = \mathfrak{F}(1.8) = 1.8^2 = 3.24$$

and the other side will be

$$\frac{1.\left(1 - 0.4\left(0.6\right)\right)^{\frac{1}{1}} + 2.\left(1 - 0.4\left(0.6\right)\right)^{\frac{1}{2}}}{3} 1^2 + \frac{1.\left(1 - 0.4\left(0.6\right)\right)^{\frac{1}{1}} + 2.\left(1 - 0.4\left(0.6\right)\right)^{\frac{1}{2}}}{3} 3^2$$
$$= 8.35$$

so that

$$3.24 \leq 8.35.$$

.

2.2. Polynomials on $n-$fractional $s-$like preinvex mapping as new extensions of H–H like inequalities

Now, we will attain a new generalization of H–H inequality for the GFPP$-s$ function $\mathfrak{F}$.

Theorem 2. *Let $X^o \subseteq \Re$ be an open invex subset with respect to $\varsigma_* : X^o \times X^o \to \Re$ and $\mathfrak{a}_1, \mathfrak{b}_1 \in X^o$ with $\mathfrak{b}_1 + \varsigma_*(\mathfrak{a}_1, \mathfrak{b}_1) \leq \mathfrak{b}_1$. Suppose that $\mathfrak{F} : [\mathfrak{b}_1 + \varsigma_*(\mathfrak{a}_1, \mathfrak{b}_1), \mathfrak{b}_1]$ and satisfies Property- $\mathfrak{C}$ with $n \in \mathcal{N}$, $a_i \geq 0 \left(i = \overline{1,n}\right)$, such that $\sum_{i=1}^n a_i > 0$, $s \in [0,1]$, $\varrho \in [0,1]$, $\mathfrak{K} > 0$. Then*

$$\begin{aligned}
&\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}} \mathfrak{F}\left(\frac{2\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)}{2}\right) \\
&\leq \frac{\Gamma_{\mathfrak{K}}(\varrho + \mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1, \mathfrak{a}_1)} \left\{ \mathfrak{J}_{\mathfrak{a}_1^+}^{\varrho,\mathfrak{K}} \mathfrak{F}(\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)) + \mathfrak{J}_{(\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1))^-}^{\varrho,\mathfrak{K}} \mathfrak{F}(\mathfrak{a}_1) \right\} \\
&\leq [\mathfrak{F}(\mathfrak{a}_1) + \mathfrak{F}(\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1))] \\
&\times \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1} \left\{ \frac{\sum_{i=1}^n a_i (1 - s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i} + \frac{\sum_{i=1}^n a_i (1 - s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \right\} d\mathfrak{t}.. \qquad (14)
\end{aligned}$$

Proof. From the definition of the GFPP$-s$ function $\mathfrak{F}$, one obtains

$$\begin{aligned}
\mathfrak{F}\left(x + \frac{\varsigma_*(y,x)}{2}\right) &\leq \frac{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \mathfrak{F}(x) + \frac{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \mathfrak{F}(y) \\
\mathfrak{F}\left(x + \frac{\varsigma_*(y,x)}{2}\right) &\leq \frac{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i} [\mathfrak{F}(x) + \mathfrak{F}(y)]. \qquad (15)
\end{aligned}$$

With substituting the $x = \mathfrak{a}_1 + (1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)$ and $y = \mathfrak{a}_1 + \mathfrak{t}\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)$ in (15), we get

$$\begin{aligned}
&\mathfrak{F}\left(\mathfrak{a}_1 + (1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1) + \frac{\varsigma_*(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1), \mathfrak{a}_1 + (1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1))}{2}\right) \\
&\leq \frac{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i} [\mathfrak{F}(\mathfrak{a}_1 + (1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)) + \mathfrak{F}(\mathfrak{a}_1 + \mathfrak{t}\varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1))]. \qquad (16)
\end{aligned}$$

By taking product with the term $\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}$ and antiderivative with respect to $\mathfrak{t} \in [0,1]$, one gets

$$\begin{aligned}
&\frac{1}{\frac{\varrho}{\mathfrak{K}}} \mathfrak{F}\left(\frac{2\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1, \mathfrak{a}_1)}{2}\right) \\
&\leq \frac{\sum_{i=1}^n a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}
\end{aligned}$$

$$\times\left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}\left(\mathfrak{a}_1+(1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}\left(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)d\mathfrak{t}\right]$$

$$\frac{1}{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}\left(\frac{2\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}{2}\right)$$
$$\leq \frac{\sum_{i=1}^{n} a_i\left(1-\frac{s}{2}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}$$
$$\times\left[\frac{\mathfrak{K}\Gamma_{\mathfrak{K}}(\varrho)}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1^+}\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)\right\}\right], \tag{17}$$

which completes the left hand side of (14). For the proof of the second inequality in (14), we first note that if $\mathfrak{F}$ is $n-$polynomial $s-$like preinvexity on $[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$ and the mapping ς_* satisfies the property-$\mathfrak{C}$, then for every $\mathfrak{t}\in[0,1]$, it yields that

$$\mathfrak{F}\left(\mathfrak{a}_1+(1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)$$
$$\leq \frac{\sum_{i=1}^{n} a_i\left(1-s(1-\mathfrak{t})\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)+\frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\mathfrak{F}(\mathfrak{a}_1)$$
$$\mathfrak{F}\left(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)$$
$$\leq \frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)+\frac{\sum_{i=1}^{n} a_i\left(1-s(1-\mathfrak{t})\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\mathfrak{F}(\mathfrak{a}_1). \tag{18}$$

By adding above two inequalities, one gets

$$\mathfrak{F}\left(\mathfrak{a}_1+(1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)+\mathfrak{F}\left(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)$$
$$\leq\left\{\frac{\sum_{i=1}^{n} a_i\left(1-s(1-\mathfrak{t})\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}+\frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\right\}[\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)]. \tag{19}$$

By taking product with the term $\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}$ and antiderivative with respect to $\mathfrak{t}\in[0,1]$, one gets

$$\mathfrak{F}\left(\mathfrak{a}_1+(1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)+\mathfrak{F}\left(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)$$
$$\leq\left\{\frac{\sum_{i=1}^{n} a_i\left(1-s(1-\mathfrak{t})\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}+\frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\right\}[\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)] \tag{20}$$

$$\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}\left(\mathfrak{a}_1+(1-\mathfrak{t})\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\mathfrak{F}\left(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)d\mathfrak{t}$$
$$\leq[\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)\right)]\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}$$

$$\times\left\{\frac{\sum_{i=1}^{n} a_i\left(1-s\left(1-\mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}+\frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\right\}d\mathfrak{t} \tag{21}$$

$$\begin{aligned}
&\frac{\mathfrak{K}\Gamma_{\mathfrak{K}}\left(\varrho\right)}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\ \mathfrak{J}_{\mathfrak{a}_1+}^{\varrho,\mathfrak{K}}\ \mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)+\mathfrak{J}_{\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)^-}^{\varrho,\mathfrak{K}}\ \mathfrak{F}\left(\mathfrak{a}_1\right)\ \right\}\\
&\leq\left[\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)\right]\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\\
&\times\left\{\frac{\sum_{i=1}^{n} a_i\left(1-s\left(1-\mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}+\frac{\sum_{i=1}^{n} a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i}\right\}d\mathfrak{t}.
\end{aligned} \tag{22}$$

By combining the inequalities (17) and (22), we can get (14).

Corollary 1. *If we judge the value $s=1$ in Theorem 2, then the following inequalities for GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i\left(\frac{1}{2}\right)^{\frac{1}{i}}}\ \mathfrak{F}\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)\\
&\leq\frac{\Gamma_{\mathfrak{K}}\left(\varrho+\mathfrak{K}\right)}{\varsigma_*^{\frac{\varrho}{k}}\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\ \mathfrak{J}_{\mathfrak{a}_1+}^{\varrho,\mathfrak{K}}\ \mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)+\mathfrak{J}_{\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)^-}^{\varrho,\mathfrak{K}}\ \mathfrak{F}\left(\mathfrak{a}_1\right)\ \right\}\\
&\leq\frac{\left[\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)\right]}{\sum_{i=1}^{n} a_i}\times\int_0^1\sum_{i=1}^{n}a_i\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\{(1-\mathfrak{t})^{\frac{1}{i}}+(\mathfrak{t})^{\frac{1}{i}}\}d\mathfrak{t}.
\end{aligned} \tag{23}$$

Corollary 2. *If we judge the value $\mathfrak{K}=1$ in Corollary 1, then we get the following inequalities for GFPP function with RL−fractional integral operators:*

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i\left(\frac{1}{2}\right)^{\frac{1}{i}}}\ \mathfrak{F}\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)\\
&\leq\frac{\Gamma\left(\varrho+1\right)}{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\ \mathfrak{J}_{\mathfrak{a}_1+}^{\varrho}\ \mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)+\mathfrak{J}_{\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)^-}^{\varrho}\ \mathfrak{F}\left(\mathfrak{a}_1\right)\ \right\}\\
&\leq\frac{\left[\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{F}\left(\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)\right]}{\sum_{i=1}^{n} a_i}\times\int_0^1\sum_{i=1}^{n}a_i\mathfrak{t}^{\varrho-1}\{(1-\mathfrak{t})^{\frac{1}{i}}+(\mathfrak{t})^{\frac{1}{i}}\}d\mathfrak{t}.
\end{aligned} \tag{24}$$

Remark 2. *If we take $\varrho=1$ and $n=1$, in Corollary 2, then one can get the inequalities* (4).

Through out the article, we take $\mathfrak{U}^*=$ Let us suppose that $n\in\mathcal{N}$, $a_i\geq 0\left(i=\overline{1,n}\right)$, such that $\sum_{i=1}^{n} a_i>0$, $s\in[0,1]$, $\mathfrak{a}_1<\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)$ and $\varrho,\mathfrak{K}>0$.

2.3. New generalizations of Ostrowski type inequalities using polynomial $n-$fractional $s-$like preinvex functions

This portion explores the new inequalities for the derivatives of first and second with the GFPP$-s$ function. In the mean while, we will develop a following new lemma.

Lemma 1. *Suppose $\mathfrak{F}:[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]\to\Re$ is a differentiable mapping on $(\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))$ with $\mathfrak{a}_1<\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)$. If $\mathfrak{F}'\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$, $\varrho>0,\mathfrak{K}>0$, then the following equality for $\mathfrak{K}-$fractional integral operator is as follows*

$$\begin{aligned}
&\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\\
&\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\\
&=\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\,d\mathfrak{t}+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\,d\mathfrak{t}.
\end{aligned}\tag{25}$$

Proof. Let us assume that

$$\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\,d\mathfrak{t}+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\,d\mathfrak{t}.\tag{26}$$

By using integration by parts and suitable substitution, we get

$$\begin{aligned}
I_1&=\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\,d\mathfrak{t}\\
&=\frac{\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))}{\varsigma_*(x,\mathfrak{a}_1)}-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}.\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1).
\end{aligned}\tag{27}$$

Similarly, we can find

$$\begin{aligned}
I_2&=\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{F}(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\,d\mathfrak{t}\\
&=\frac{\mathfrak{F}(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))}{\varsigma_*(x,\mathfrak{b}_1)}-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}.\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)).
\end{aligned}\tag{28}$$

Substituting the values of I_1 and I_2 in (26), we can get (25).

Theorem 3. *Suppose* $\mathfrak{F} : X = [\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)] \to \Re$ *is a differentiable function on* X^o *such that* $\mathfrak{F}' \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)]$ *and consideration with* $\mathfrak{U}^*$. *Let* $|\mathfrak{F}'|$ *be a GFPP−s function on X with* $|\mathfrak{F}'| \leq S$, *for all* $x \in [\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)]$. *Then*

$$\begin{aligned}
&\Bigg|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{a}_1) + \varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \mathfrak{F} (x) - \frac{\Gamma_{\mathfrak{K}} (\varrho + \mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (\mathfrak{b}_1, \mathfrak{a}_1)} \\
&\times \left\{ \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1 + \varsigma_*(x,\mathfrak{a}_1))^-} \mathfrak{F} (\mathfrak{a}_1) + \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1 + \varsigma_*(x,\mathfrak{b}_1))^+} \mathfrak{F} (\mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)) \right\}\Bigg| \\
&\leq \left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{a}_1) + \varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \right) \\
&\times \frac{S}{\sum_{i=1}^n a_i} . \sum_{i=1}^n a_i \left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{k}} \left((1 - s (1 - \mathfrak{t}))^{\frac{1}{i}} \right) d\mathfrak{t} + \int_0^1 \mathfrak{t}^{\frac{\varrho}{k}} (1 - s\mathfrak{t})^{\frac{1}{i}} d\mathfrak{t} \right].
\end{aligned} \tag{29}$$

Proof. From Lemma 1 and a modulus property of the GFPP−s function $|\mathfrak{F}'|$, one has

$$\begin{aligned}
&\Bigg|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{a}_1) + \varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \mathfrak{F} (x) - \frac{\Gamma_{\mathfrak{K}} (\varrho + \mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (\mathfrak{b}_1, \mathfrak{a}_1)} \\
&\times \left\{ \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1 + \varsigma_*(x,\mathfrak{a}_1))^-} \mathfrak{F} (\mathfrak{a}_1) + \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1 + \varsigma_*(x,\mathfrak{b}_1))^+} \mathfrak{F} (\mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)) \right\}\Bigg| \\
&\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{a}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} \left|\mathfrak{F}' (\mathfrak{a}_1 + t\varsigma_* (x, \mathfrak{a}_1)) \right| d\mathfrak{t} + \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} \left|\mathfrak{F}' (\mathfrak{b}_1 + \mathfrak{t}\varsigma_* (x, \mathfrak{b}_1)) \right| d\mathfrak{t} \\
&\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{a}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} \left[\frac{\sum_{i=1}^n a_i (1 - s (1 - \mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \left|\mathfrak{F}' (x)\right| + \frac{\sum_{i=1}^n a_i (1 - s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \left|\mathfrak{F}' (\mathfrak{a}_1)\right| \right] d\mathfrak{t} \\
&+ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} \left[\frac{\sum_{i=1}^n a_i (1 - s (1 - \mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \left|\mathfrak{F}' (x)\right| + \frac{\sum_{i=1}^n a_i (1 - s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i} \left|\mathfrak{F}' (\mathfrak{b}_1)\right| \right] d\mathfrak{t} \\
&\leq \left(\frac{\varsigma_*^{\frac{\varrho}{k}+1} (x, \mathfrak{a}_1) + \varsigma_*^{\frac{\varrho}{k}+1} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \right) \\
&\times \frac{S}{\sum_{i=1}^n a_i} . \sum_{i=1}^n a_i \left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} \left((1 - s (1 - \mathfrak{t}))^{\frac{1}{i}} \right) d\mathfrak{t} + \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}} (1 - s\mathfrak{t})^{\frac{1}{i}} d\mathfrak{t} \right].
\end{aligned} \tag{30}$$

Corollary 3. *If we judge the value* $s = 1$ *in Theorem 3, then we have the following inequalities for GFPP function with* $\mathfrak{K}$*−fractional integral operators:*

$$\Bigg|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{a}_1) + \varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (x, \mathfrak{b}_1)}{\varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)} \mathfrak{F} (x) - \frac{\Gamma_{\mathfrak{K}} (\varrho + \mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (\mathfrak{b}_1, \mathfrak{a}_1)}$$

$$\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\Bigg|$$
$$\leq\left(\frac{\varsigma_*^{\frac{\varrho}{k}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{k}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\frac{S}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{k}}\mathfrak{t}^{\frac{1}{i}}d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\frac{\varrho}{k}}(1-\mathfrak{t})^{\frac{1}{i}}d\mathfrak{t}\right].$$

Corollary 4. *If we judge the value $\mathfrak{K}=1$ in Corollary 3, then we have the following inequalities for GFPP function with RL−fractional integral operators:*

$$\Bigg|\frac{\varsigma_*^{\varrho}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma(\varrho+1)}{\varsigma_*^{\varrho}(\mathfrak{b}_1,\mathfrak{a}_1)}$$
$$\times\left\{\mathfrak{J}^{\varrho}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\Bigg|$$
$$\leq\left(\frac{\varsigma_*^{\varrho+1}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\frac{S}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 \mathfrak{t}^{\varrho}\mathfrak{t}^{\frac{1}{i}}d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\varrho}(1-\mathfrak{t})^{\frac{1}{i}}d\mathfrak{t}\right].$$

Remark 3. *If we take $\varrho=1$ and $n=1$ and $\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)=\mathfrak{b}_1-\mathfrak{a}_1$, in Corollary 4, then one can get the inequalities* (5).

Theorem 4. *Suppose $\mathfrak{F}:X=:[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]\to\Re$ is a differentiable function function on X^o such that $\mathfrak{F}'\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$ and consideration with $\mathfrak{U}^*$. Let for some $q>1$, $|\mathfrak{F}'|^q$ be a GFPP−s function on X with $|\mathfrak{F}'|\leq S$, for all $x\in[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$. Then*

$$\Bigg|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}$$
$$\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\Bigg|$$
$$\leq\left(\frac{\mathfrak{K}}{\mathfrak{K}+\varrho}\right)^{1-\frac{1}{q}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)$$
$$\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)d\mathfrak{t}+\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}(1-s\mathfrak{t})^{\frac{1}{i}}d\mathfrak{t}\right]^{\frac{1}{q}}. \tag{31}$$

Proof. From Lemma 1 and a property of the GFPP−s function $|\mathfrak{F}'|^q$, and the power mean inequality, one has

$$\Bigg|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}$$
$$\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\Bigg|$$

$$\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|d\mathfrak{t}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\right|d\mathfrak{t} \tag{32}$$

$$\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|^q dt\right)^{\frac{1}{q}}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{b}_1+t\varsigma_*(x,\mathfrak{b}_1))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{\mathfrak{K}}{\mathfrak{K}+\varrho}\right)^{1-\frac{1}{q}}\ \left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\frac{\sum_{i=1}^n a_i\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(x)\right|^q dt\right.\right.$$
$$\left.+\frac{\sum_{i=1}^n a_i\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s\mathfrak{t})^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(\mathfrak{a}_1)\right|^q dt\right\}^{\frac{1}{q}}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\frac{\sum_{i=1}^n a_i\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(x)\right|^q dt\right.$$
$$\left.\left.+\frac{\sum_{i=1}^n a_i\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s\mathfrak{t})^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(\mathfrak{b}_1)\right|^q dt\right\}^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{\mathfrak{K}}{\mathfrak{K}+\varrho}\right)^{1-\frac{1}{q}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)$$
$$\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)d\mathfrak{t}+\int_0^1 t^{\frac{\varrho}{\mathfrak{K}}}(1-st)^{\frac{1}{i}}\,d\mathfrak{t}\right]^{\frac{1}{q}}. \tag{33}$$

Corollary 5. *If one can take $s=1$ in Theorem 4, then we have the following inequalities for GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$\left|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.$$
$$\left.\times\left\{\ \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\ \mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\ \mathfrak{F}(\mathfrak{a}_1+\varsigma_*(b,\mathfrak{a}_1))\ \right\}\right|$$
$$\leq\left(\frac{\mathfrak{K}}{\mathfrak{K}+\varrho}\right)^{1-\frac{1}{q}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)$$
$$\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\mathfrak{t}^{\frac{1}{i}}dt+\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}(1-\mathfrak{t})^{\frac{1}{i}}\,dt\right]^{\frac{1}{q}}.$$

Corollary 6. *If one can takes $\mathfrak{K}=1$ in Corollary 5, then we have the following inequalities for GFPP function with RL−fractional integral operators:*

$$\begin{aligned}&\left|\frac{\varsigma_*^{\varrho}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma(\varrho+1)}{\varsigma_*^{\varrho}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.\\&\left.\times\left\{\mathfrak{J}^{\varrho}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\right|\\&\leq\left(\frac{1}{1+\varrho}\right)^{1-\frac{1}{q}}\left(\frac{\varsigma_*^{\varrho+1}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\\&\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1 t^{\varrho}\mathfrak{t}^{\frac{1}{i}}d\mathfrak{t}+\int_0^1\mathfrak{t}^{\varrho}(1-\mathfrak{t})^{\frac{1}{i}}\,d\mathfrak{t}\right]^{\frac{1}{q}}.\end{aligned}$$

Theorem 5. *Suppose $\mathfrak{F}:X=[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]\to\Re$ is a differentiable function function on X^o such that $\mathfrak{F}'\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$ and consideration with $\mathfrak{U}^*$. Let for some $p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$, $|\mathfrak{F}'|^q$ be a GFPP−s function on X with $|\mathfrak{F}'|\leq S$, for all $x\in[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$. Then*

$$\begin{aligned}&\left|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.\\&\left.\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\right|\\&\leq\left(\frac{\mathfrak{K}}{\mathfrak{K}+p\varrho}\right)^{\frac{1}{p}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\\&\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)d\mathfrak{t}+\int_0^1(1-s\mathfrak{t})^{\frac{1}{i}}\,d\mathfrak{t}\right]^{\frac{1}{q}}.\end{aligned}\tag{34}$$

Proof. From Lemma 1 and a property of the GFPP−s function $|\mathfrak{F}'|^q$, and the Hölder inequality, one has

$$\begin{aligned}&\left|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.\\&\left.\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\right|\\&\leq\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|dt\\&+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\left|\mathfrak{F}'(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\right|dt\end{aligned}$$

$$\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}d\mathfrak{t}\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathfrak{F}'(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|^q d\mathfrak{t}\right)^{\frac{1}{q}}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left(\int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}d\mathfrak{t}\right)^{\frac{1}{p}}\left(\int_0^1 \left|\mathfrak{F}'(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\right|^q d\mathfrak{t}\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{\mathfrak{K}}{\mathfrak{K}+p\varrho}\right)^{\frac{1}{p}}\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\frac{\sum_{i=1}^n a_i\int_0^1\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(x)\right|^q d\mathfrak{t}\right.\right.$$
$$\left.+\frac{\sum_{i=1}^n a_i\int_0^1\left((1-s\mathfrak{t})^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(\mathfrak{a}_1)\right|^q d\mathfrak{t}\right\}^{\frac{1}{q}}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\frac{\sum_{i=1}^n a_i\int_0^1\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(x)\right|^q d\mathfrak{t}\right.$$
$$\left.\left.+\frac{\sum_{i=1}^n a_i\int_0^1\left((1-s\mathfrak{t})^{\frac{1}{i}}\right)}{\sum_{i=1}^n a_i}\left|\mathfrak{F}'(\mathfrak{b}_1)\right|^q d\mathfrak{t}\right\}^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{\mathfrak{K}}{\mathfrak{K}+p\varrho}\right)^{\frac{1}{p}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)$$
$$\times\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left[\int_0^1\left((1-s(1-\mathfrak{t}))^{\frac{1}{i}}\right)d\mathfrak{t}+\int_0^1(1-s\mathfrak{t})^{\frac{1}{i}}\,d\mathfrak{t}\right]^{\frac{1}{q}}. \tag{35}$$

Corollary 7. *If one can take $s=1$ in Theorem 5, then we have the following inequalities for a GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$\left|\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma_{\mathfrak{K}}(\varrho+\mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.$$
$$\left.\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\right|$$
$$\leq\left(\frac{\mathfrak{K}}{\mathfrak{K}+p\varrho}\right)^{\frac{1}{p}}\left(\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\left[\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left(\frac{2i}{i+1}\right)\right]^{\frac{1}{q}}.$$

Corollary 8. *If one can take $\mathfrak{K}=1$ in Corollary 7, then we have the following inequalities for a GFPP function with RL$-$fractional integral operators:*

$$\left|\frac{\varsigma_*^{\varrho}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\mathfrak{F}(x)-\frac{\Gamma(\varrho+1)}{\varsigma_*^{\varrho}(\mathfrak{b}_1,\mathfrak{a}_1)}\right.$$
$$\left.\times\left\{\mathfrak{J}^{\varrho}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))\right\}\right|$$
$$\leq\left(\frac{1}{1+p\varrho}\right)^{\frac{1}{p}}\left(\frac{\varsigma_*^{\varrho+1}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right)\left[\frac{S^q}{\sum_{i=1}^n a_i}.\sum_{i=1}^n a_i\left(\frac{2i}{i+1}\right)\right]^{\frac{1}{q}}.$$

Now, we develop some new Ostrowski type inequalities for twice differentiable functions. First, we will give the new following lemma:

Lemma 2. *Suppose* $\mathfrak{F} : [\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)] \to \Re$ *is twice differentiable mapping on* $(\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1))$ *with* $\mathfrak{a}_1 < \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)$. *If* $\mathfrak{F}'' \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)], \varrho > 0, \mathfrak{K} > 0,$ *then we have the following equality for* $\mathfrak{K}-$*fractional integral operator*

$$\begin{aligned}
&(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)\\
&+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\,\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\,\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\\
&-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\,\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\,\mathfrak{F}(\mathfrak{b}_1)\right\}\\
&=\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\mathfrak{F}''(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\,d\mathfrak{t}\\
&+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\mathfrak{F}''(\mathfrak{b}_1+t\varsigma_*(x,\mathfrak{b}_1))\,d\mathfrak{t},\\
&holds\ forall\ \ x\in[\mathfrak{a}_1,+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)],\ \ \lambda\in[0,1].
\end{aligned}\tag{36}$$

Proof. It can easily be proved as similar Lemma 1.

Theorem 6. *Suppose* $\mathfrak{F} : X = [\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)] \to \Re$ *is a twice differentiable function on* X^o *such that* $\mathfrak{F}'' \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)]$ *and consideration with* $\mathfrak{U}^*$. *Let* $|\mathfrak{F}''(x)|$ *be a GFPP−s function on* X. *Then*

$$\begin{aligned}
&\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)\\
&+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\,\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\,\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\\
&-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\,\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\,\mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg|\\
&\le\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)\,|\mathfrak{F}''(\mathfrak{a}_1)|+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)\,|\mathfrak{F}''(\mathfrak{b}_1)|}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)(1-s\mathfrak{t})^{\frac{1}{i}}\,dt
\end{aligned}\tag{37}$$

$$+\quad\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\sum_{i=1}^n a_i\ \ \varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left|\mathfrak{F}''(x)\right|\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)(1-s(1-\mathfrak{t}))^{\frac{1}{i}}\,d\mathfrak{t}$$

holds $\forall \in [\mathfrak{a}_1, \mathfrak{b}_1]$ *and* $\lambda \in [0, 1]$.

Proof. From Lemma 2 and a modulus property of the GFPP$-s$ function $|\mathfrak{F}''|$, one has

$$\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)$$
$$+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]$$
$$-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg|$$
$$\le\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-t^{\frac{\varrho}{\mathfrak{K}}}\right)|\mathfrak{F}''(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))|d\mathfrak{t}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)|\mathfrak{F}''(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))|d\mathfrak{t}$$
$$\le\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-t^{\frac{\varrho}{\mathfrak{K}}}\right)\times$$
$$\left[\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|+\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|\right]d\mathfrak{t}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\times$$
$$\left[\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|+\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|\right]d\mathfrak{t}$$
$$\le\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)|\mathfrak{F}''(\mathfrak{a}_1)|+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)|\mathfrak{F}''(\mathfrak{b}_1)|}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{k}}\right)(1-s\mathfrak{t})^{\frac{1}{i}}d\mathfrak{t}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\sum_{i=1}^n a_i\ \ \varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left|\mathfrak{F}''(x)\right|\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)(1-s(1-\mathfrak{t}))^{\frac{1}{i}}d\mathfrak{t}.\tag{38}$$

Corollary 9. *If one can take $s=1$ in Theorem 6, then we have the following inequalities for GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)$$
$$+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]$$
$$-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg|$$
$$\le\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)|\mathfrak{F}''(\mathfrak{a}_1)|+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)|\mathfrak{F}''(\mathfrak{b}_1)|}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)(1-\mathfrak{t})^{\frac{1}{i}}d\mathfrak{t}$$

$$+ \quad \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\sum_{i=1}^n a_i \quad \varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left|\mathfrak{F}''(x)\right|\sum_{i=1}^n a_i\int_0^1\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)(\mathfrak{t})^{\frac{1}{i}+1}\,d\mathfrak{t}.$$

Corollary 10. *If one can take $\mathfrak{K}=1$ in Corollary 9, then we have the following inequalities for a GFPP function with RL−fractional integral operators:*

$$\begin{aligned}
&\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\varrho}(\mathfrak{b}_1,x)-\varsigma_*^{\varrho}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+(1+\varrho-\lambda)\left[\frac{\varsigma_*^{\varrho}(\mathfrak{b}_1,x)+\varsigma_*^{\varrho}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)\\
&+\lambda\left[\frac{\varsigma_*^{\varrho}(\mathfrak{b}_1,x)\,\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\varrho}(x,\mathfrak{a}_1)\,\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\\
&-\frac{\Gamma(\varrho+2)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\ \mathfrak{I}^{\varrho}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\ \mathfrak{F}(\mathfrak{a}_1)+\mathfrak{I}^{\varrho}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\ \mathfrak{F}(\mathfrak{b}_1)\ \right\}\Bigg|\\
&\leq\ \left[\frac{\varsigma_*^{\varrho+2}(x,\mathfrak{a}_1)\left|\mathfrak{F}''(\mathfrak{a}_1)\right|+\varsigma_*^{\varrho+2}(\mathfrak{b}_1,x)\left|\mathfrak{F}''(\mathfrak{b}_1)\right|}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\varrho}\right)(1-\mathfrak{t})^{\frac{1}{i}}\,d\mathfrak{t}\\
&+\quad\frac{\varsigma_*^{\varrho+2}(x,\mathfrak{a}_1)+\varsigma_*^{\varrho+2}(\mathfrak{b}_1,x)}{\sum_{i=1}^n a_i\quad\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left|\mathfrak{F}''(x)\right|\sum_{i=1}^n a_i\int_0^1\left(\lambda-\mathfrak{t}^{\varrho}\right)(\mathfrak{t})^{\frac{1}{i}+1}\,d\mathfrak{t}.
\end{aligned}$$

Theorem 7. *Suppose $\mathfrak{F}:X=:[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]\to\Re$ is twice differentiable function function on X^o such that $\mathfrak{F}''\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$ and consideration with $\mathfrak{U}^*$. Let for some $q>1$, $|\mathfrak{F}''(x)|^q$ be a GFPP−s function on X, for all $x\in[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$. Then*

$$\begin{aligned}
&\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)\\
&+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\,\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\,\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\\
&-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\ \mathfrak{I}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\ \mathfrak{F}(\mathfrak{a}_1)+\mathfrak{I}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\ \mathfrak{F}(\mathfrak{b}_1)\ \right\}\Bigg|\\
&\leq\ M^{1-\frac{1}{q}}(\varrho,\mathfrak{K},\lambda)\\
&\times\ \Bigg[\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\Bigg\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\Bigg\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q\\
&+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q\Bigg\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}}\\
&+\quad\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\Bigg\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\ \Bigg\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q\\
&+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q\Bigg\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

where

$$M(\varrho,\mathfrak{K},\lambda)=\ \int_0^1[\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)]^q dt$$

$$= \frac{\mathfrak{K}\ \lambda^{\frac{\mathfrak{K}(1+q)+\varrho q}{q}}}{\varrho}\left[\Gamma(1+q)\ \Gamma\left(\frac{\mathfrak{K}(1+q)+\varrho}{\varrho}\right)\ {}_2F_1\left(1,1+q,2+q+\frac{\mathfrak{K}(1+q)}{\varrho},1\right)\right.$$
$$\left. +\ \beta\left(1+q,-\frac{\mathfrak{K}(1+\ q)+\varrho q}{q\varrho}\right)-\beta\left(\lambda,1+\ q,-\frac{\mathfrak{K}(1+\ q)+\varrho q}{q\varrho}\right)\right]. \tag{39}$$

Proof. From Lemma 2 and a property of the GFPP$-s$ function $|\mathfrak{F}''|^q$, and the power mean inequality, one has

$$\left|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x)\right.$$
$$+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\,\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\,\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]$$
$$\left.-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\ \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\ \mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\ \mathfrak{F}(\mathfrak{b}_1)\ \right\}\right|$$
$$\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \left|t\left(\lambda-t^{\frac{\varrho}{\mathfrak{K}}}\right)\right|\left|\mathfrak{F}''(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|d\mathfrak{t}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \left|t\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\right|\left|\mathfrak{F}''(\mathfrak{b}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{b}_1))\right|d\mathfrak{t}$$
$$\leq \left(\int_0^1 \left|\mathfrak{t}^q\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)^q d\mathfrak{t}\right)^{\frac{1}{q}}$$
$$\times\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\right.$$
$$\times\left[\frac{\sum_{i=1}^n a_i(1-st)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q d\mathfrak{t}\right]^{\frac{1}{q}}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1 \mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)$$
$$\left.\times\left[\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q d\mathfrak{t}\right]^{\frac{1}{q}}\right]$$
$$\leq\ M^{1-\frac{1}{q}}(\varrho,\mathfrak{K},\lambda)$$
$$\times\ \left[\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\int_0^1 \mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\left\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q\right.\right.\right.$$
$$\left.\left.+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q\right\}d\mathfrak{t}\right\}^{\frac{1}{q}}$$
$$+\ \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\int_0^1 \mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\ \left\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q\right.\right.$$

$$+\frac{\sum_{i=1}^{n}a_i\left(1-s\left(1-\mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(x\right)\right|^{q}\Big\}d\mathfrak{t}\Big\}^{\frac{1}{q}}\Bigg].$$

Corollary 11. *If one can take $s=1$ in Theorem 7, then we have the following inequalities for a GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$\Bigg|\left(1-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,x\right)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}'\left(x\right)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,x\right)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}\left(x\right)$$
$$+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,x\right)\mathfrak{F}\left(\mathfrak{b}_1\right)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(x,\mathfrak{a}_1\right)\mathfrak{F}\left(\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]$$
$$-\frac{\Gamma_{\mathfrak{K}}\left(\varrho+2\mathfrak{K}\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\mathfrak{I}^{\varrho,\mathfrak{K}}_{\left(\mathfrak{a}_1+\varsigma_*\left(x,\mathfrak{a}_1\right)\right)^-}\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{I}^{\varrho,\mathfrak{K}}_{\left(\mathfrak{b}_1+\varsigma_*\left(x,\mathfrak{b}_1\right)\right)^+}\mathfrak{F}\left(\mathfrak{b}_1\right)\right\}\Bigg|$$
$$\leq M^{1-\frac{1}{q}}\left(\varrho,\mathfrak{K},\lambda\right)$$
$$\times\Bigg[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\Big\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\Big\{\frac{\sum_{i=1}^{n}a_i\left(1-\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(\mathfrak{a}_1\right)\right|^{q}$$
$$+\frac{\sum_{i=1}^{n}a_i\left(\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(x\right)\right|^{q}\Big\}d\mathfrak{t}\Big\}^{\frac{1}{q}}$$
$$+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}\left(\mathfrak{b}_1,x\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\Big\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\Big\{\frac{\sum_{i=1}^{n}a_i\left(1-\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(\mathfrak{b}_1\right)\right|^{q}$$
$$+\frac{\sum_{i=1}^{n}a_i\mathfrak{t}^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(x\right)\right|^{q}\Big\}d\mathfrak{t}\Big\}^{\frac{1}{q}}\Bigg].$$

Corollary 12. *If one can take $\mathfrak{K}=1$ in Corollary 11, then we have the following inequalities for GFPP function with RL−fractional integral operators:*

$$\Bigg|\left(1-\lambda\right)\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)-\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}'\left(x\right)+\left(1+\varrho-\lambda\right)\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)+\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}\left(x\right)$$
$$+\lambda\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)\mathfrak{F}\left(\mathfrak{b}_1\right)+\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)\mathfrak{F}\left(\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]$$
$$-\frac{\Gamma\left(\varrho+2\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\mathfrak{I}^{\varrho}_{\left(\mathfrak{a}_1+\varsigma_*\left(x,\mathfrak{a}_1\right)\right)^-}\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{I}^{\varrho}_{\left(\mathfrak{b}_1+\varsigma_*\left(x,\mathfrak{b}_1\right)\right)^+}\mathfrak{F}\left(\mathfrak{b}_1\right)\right\}\Bigg|$$
$$\leq M^{1-\frac{1}{q}}\left(\varrho,\lambda\right)$$
$$\times\Bigg[\frac{\varsigma_*^{\varrho+2}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\Big\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\varrho}\right)\Big\{\frac{\sum_{i=1}^{n}a_i\left(1-t\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(\mathfrak{a}_1\right)\right|^{q}$$
$$+\frac{\sum_{i=1}^{n}a_i\mathfrak{t}^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(x\right)\right|^{q}\Big\}d\mathfrak{t}\Big\}^{\frac{1}{q}}$$
$$+\frac{\varsigma_*^{\varrho+2}\left(\mathfrak{b}_1,x\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\Big\{\int_0^1\mathfrak{t}\left(\lambda-\mathfrak{t}^{\varrho}\right)\Big\{\frac{\sum_{i=1}^{n}a_i\left(1-\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^{n}a_i}\left|\mathfrak{F}''\left(\mathfrak{b}_1\right)\right|^{q}$$

$$+\frac{\sum_{i=1}^{n} a_i \mathfrak{t}^{\frac{1}{i}}}{\sum_{i=1}^{n} a_i} \left|\mathfrak{F}''(x)\right|^q \Big\} d\mathfrak{t} \Big\}^{\frac{1}{q}} \Bigg]$$

where

$$\begin{aligned} M(\varrho,\lambda) &= \int_0^1 [\mathfrak{t}(\lambda-\mathfrak{t}^{\varrho})]^q d\mathfrak{t} \\ &= \frac{\lambda^{\frac{(1+q)+\varrho q}{q}}}{\varrho}\Bigg[\Gamma(1+q)\ \Gamma\left(\frac{(1+q)+\varrho}{\varrho}\right)\ {}_2F_1\left(1,1+q,2+q+\frac{(1+q)}{\varrho},1\right) \\ &+\ \beta\left(1+q,-\frac{(1+q)+\varrho q}{q\varrho}\right)-\beta\left(\lambda,1+q,-\frac{(1+q)+\varrho q}{q\varrho}\right)\Bigg]. \end{aligned} \tag{40}$$

Theorem 8. *Suppose* $\mathfrak{F}: X=[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]\to\Re$ *is a twice differentiable function function on* X^o *such that* $\mathfrak{F}''\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$ *and consideration with* $\mathfrak{U}^*$*. Let for some* $q>1$*, with* $\frac{1}{p}+\frac{1}{q}=1$*,* $|\mathfrak{F}''|^q$ *be a GFPP−s function on* X*, for all* $x\in[\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)]$*. Then*

$$\begin{aligned} &\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x) \\ &+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right] \\ &-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\ \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\ \mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\ \mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg| \\ &\le M^{\frac{1}{p}}(\varrho,\mathfrak{K},\lambda) \\ &\times\Bigg[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\Bigg\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\ (1-s\mathfrak{t})^{\frac{1}{i}}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q+(1-s(1-\mathfrak{t}))^{\frac{1}{i}}\left|\mathfrak{F}''(x)\right|^q\ \right\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}} \\ &+\quad\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)} \\ &\times\Bigg\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\ (1-s\mathfrak{t})^{\frac{1}{i}}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q+(1-s(1-\mathfrak{t}))^{\frac{1}{i}}\left|\mathfrak{F}''(x)\right|^q\ \right\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}}\Bigg]. \end{aligned} \tag{41}$$

Proof. From Lemma 2 and a property of the GFPP−s function $|\mathfrak{F}''|^q$, and the Hölder inequality, one has

$$\begin{aligned} &\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x) \\ &+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg| \\
&\leq \frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\left|\mathfrak{t}\left(\lambda-t^{\frac{\varrho}{\mathfrak{K}}}\right)\right|\left|\mathfrak{F}''(\mathfrak{a}_1+\mathfrak{t}\varsigma_*(x,\mathfrak{a}_1))\right|dt \\
&+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(\mathfrak{b}_1,x)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\int_0^1\left|\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}}\right)\right|\left|\mathfrak{F}''(\mathfrak{b}_1+t\varsigma_*(x,\mathfrak{b}_1))\right|dt
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&\leq \left(\int_0^1\left|\mathfrak{t}\left(\lambda-\mathfrak{t}^{\frac{\varrho}{k}}\right)\right|^p d\mathfrak{t}\right)^{\frac{1}{p}} \\
&\times\Bigg[\frac{\varsigma_*^{\frac{\varrho}{k}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\Bigg\{\int_0^1\Bigg\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q \\
&+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q\Bigg\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}} \\
&+\frac{\varsigma_*^{\frac{\varrho}{k}+1}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\Bigg\{\int_0^1\Big\{\frac{\sum_{i=1}^n a_i(1-s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q \\
&+\frac{\sum_{i=1}^n a_i(1-s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\left|\mathfrak{F}''(x)\right|^q\Big\}d\mathfrak{t}\Bigg\}^{\frac{1}{q}}\Bigg]
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
&\leq M^{\frac{1}{p}}(\varrho,\mathfrak{K},\lambda) \\
&\times\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{(1-s\mathfrak{t})^{\frac{1}{i}}\left|\mathfrak{F}''(\mathfrak{a}_1)\right|^q+(1-s(1-\mathfrak{t}))^{\frac{1}{i}}\left|\mathfrak{F}''(x)\right|^q\right\}d\mathfrak{t}\right\}^{\frac{1}{q}}\right. \\
&+\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}(x,\mathfrak{b}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)} \\
&\left.\times\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{(1-s\mathfrak{t})^{\frac{1}{i}}\left|\mathfrak{F}''(\mathfrak{b}_1)\right|^q+(1-s(1-\mathfrak{t}))^{\frac{1}{i}}\left|\mathfrak{F}''(x)\right|^q\right\}d\mathfrak{t}\right\}^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 13. *If one can take $s=1$ in Theorem 8, then we have the following inequalities for GFPP function with $\mathfrak{K}-$fractional integral operators:*

$$
\begin{aligned}
&\Bigg|(1-\lambda)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)-\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}'(x)+\left(1+\frac{\varrho}{\mathfrak{K}}-\lambda\right)\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right]\mathfrak{F}(x) \\
&+\lambda\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(\mathfrak{b}_1,x)\mathfrak{F}(\mathfrak{b}_1)+\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}(x,\mathfrak{a}_1)\mathfrak{F}(\mathfrak{a}_1)}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\right] \\
&-\frac{\Gamma_{\mathfrak{K}}(\varrho+2\mathfrak{K})}{\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(x,\mathfrak{a}_1))^-}\mathfrak{F}(\mathfrak{a}_1)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{b}_1+\varsigma_*(x,\mathfrak{b}_1))^+}\mathfrak{F}(\mathfrak{b}_1)\right\}\Bigg|
\end{aligned}
$$

$$\leq M^{\frac{1}{p}}\left(\varrho,\mathfrak{K},\lambda\right)$$
$$\times\left[\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\left(1-\mathfrak{t}\right)^{\frac{1}{i}}\left|\mathfrak{F}''\left(\mathfrak{a}_1\right)\right|^q+\mathfrak{t}^{\frac{1}{i}}\left|\mathfrak{F}''\left(x\right)\right|^q\right\}dt\right\}^{\frac{1}{q}}\right.$$
$$\left.+\quad\frac{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}+2}\left(x,\mathfrak{b}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\left(1-\mathfrak{t}\right)^{\frac{1}{i}}\left|\mathfrak{F}''\left(\mathfrak{b}_1\right)\right|^q+\mathfrak{t}^{\frac{1}{i}}\left|\mathfrak{F}''\left(x\right)\right|^q\right\}dt\right\}^{\frac{1}{q}}\right].$$

Corollary 14. *If one can take $\mathfrak{K}=1$ in Corollary 13, then we have the following inequalities for GFPP function with RL−fractional integral operators:*

$$\left|\left(1-\lambda\right)\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)-\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}'\left(x\right)+\left(1+\varrho-\lambda\right)\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)+\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]\mathfrak{F}\left(x\right)\right.$$
$$+\lambda\left[\frac{\varsigma_*^{\varrho}\left(\mathfrak{b}_1,x\right)\mathfrak{F}\left(\mathfrak{b}_1\right)+\varsigma_*^{\varrho}\left(x,\mathfrak{a}_1\right)\mathfrak{F}\left(\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\right]$$
$$\left.-\frac{\Gamma\left(\varrho+2\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\mathfrak{J}^{\varrho}_{\left(\mathfrak{a}_1+\varsigma_*\left(x,\mathfrak{a}_1\right)\right)^-}\mathfrak{F}\left(\mathfrak{a}_1\right)+\mathfrak{J}^{\varrho}_{\left(\mathfrak{b}_1+\varsigma_*\left(x,\mathfrak{b}_1\right)\right)^+}\mathfrak{F}\left(\mathfrak{b}_1\right)\right\}\right|$$
$$\leq M^{\frac{1}{p}}\left(\varrho,\lambda\right)$$
$$\times\left[\frac{\varsigma_*^{\varrho+2}\left(x,\mathfrak{a}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\left(1-\mathfrak{t}\right)^{\frac{1}{i}}\left|\mathfrak{F}''\left(\mathfrak{a}_1\right)\right|^q+\mathfrak{t}^{\frac{1}{i}}\left|\mathfrak{F}''\left(x\right)\right|^q\right\}dt\right\}^{\frac{1}{q}}\right.$$
$$\left.+\quad\frac{\varsigma_*^{\varrho+2}\left(x,\mathfrak{b}_1\right)}{\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1\left\{\left(1-\mathfrak{t}\right)^{\frac{1}{i}}\left|\mathfrak{F}''\left(\mathfrak{b}_1\right)\right|^q+\mathfrak{t}^{\frac{1}{i}}\left|\mathfrak{F}''\left(x\right)\right|^q\right\}dt\right\}^{\frac{1}{q}}\right].$$

3. Application to matrices

Example: Denote by $\mathbb{P}^{\mathfrak{n}}$ the set of $\mathfrak{n}\times\mathfrak{n}$ complex matrices, by $\mathbb{N}_{\mathfrak{n}}$ the algebra of $\mathfrak{n}\times\mathfrak{n}$ complex matrices, and by $\mathbb{N}_{\mathfrak{n}}^+$ the strictly positive matrices in $\mathbb{N}_{\mathfrak{n}}$. That is, $\mathfrak{D}\in\mathbb{N}_{\mathfrak{n}}^+$ if $\langle\mathfrak{D}x,x\rangle>0$ for all nonzero $x\in\mathbb{P}^{\mathfrak{n}}$.

In [27], Sababheh proved that the following mapping

$$\mathfrak{F}(u)=\left\|\mathfrak{D}^uX\mathfrak{B}^{1-u}+\mathfrak{D}^{1-u}X\mathfrak{B}^u\right\|,\quad\mathfrak{D},\ \mathfrak{B}\in\mathbb{M}_{\mathfrak{n}}^+,\ X\in\mathbb{N}_{\mathfrak{n}}$$

is convex for all $u\in[0,1]$. Then by using Theorem 2, we have

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i\left(1-\frac{s}{2}\right)^{\frac{1}{i}}}$$
$$\times\left\|\mathfrak{D}^{\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)}X\mathfrak{B}^{1-\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)}+\mathfrak{D}^{1-\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)}X\mathfrak{B}^{\left(\frac{2\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}{2}\right)}\right\|$$
$$\leq\frac{\Gamma_{\mathfrak{K}}\left(\varrho+\mathfrak{K}\right)}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}$$

$$
\begin{aligned}
&\times\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1+}\left\|\mathfrak{D}^{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}X\mathfrak{B}^{1-(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}+\mathfrak{D}^{1-(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}X\mathfrak{B}^{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}\right\|\right.\\
&\left.+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))^-}\left\|\mathfrak{D}^{(\mathfrak{a}_1)}X\mathfrak{B}^{1-(\mathfrak{a}_1)}+\mathfrak{D}^{1-(\mathfrak{a}_1)}X\mathfrak{B}^{(\mathfrak{a}_1)}\right\|\right\}\\
&\leq\left[\left\|\mathfrak{D}^{(\mathfrak{a}_1)}X\mathfrak{B}^{1-(\mathfrak{a}_1)}+\mathfrak{D}^{1-(\mathfrak{a}_1)}X\mathfrak{B}^{(\mathfrak{a}_1)}\right\|\right.\\
&\left.+\left\|\mathfrak{D}^{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}X\mathfrak{B}^{1-(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}+\mathfrak{D}^{1-(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}X\mathfrak{B}^{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))}\right\|\right]\\
&\times\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\left\{\frac{\sum_{i=1}^n a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}+\frac{\sum_{i=1}^n a_i\left(1-s\left(1-\mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\right\}dt..
\end{aligned}
$$

4. Applications with bivariates

In this section, we recall the following special means of two positive numbers $\mathfrak{a}_1, \mathfrak{b}_1$ with $\mathfrak{a}_1 < \mathfrak{b}_1$ (see [28]):

(i) The arithmetic mean

$$\mathcal{A} = \mathcal{A}(\mathfrak{a}_1, \mathfrak{b}_1) = \frac{\mathfrak{a}_1 + \mathfrak{b}_1}{2}.$$

(ii) The harmonic mean

$$\mathcal{H} = \mathcal{H}(\mathfrak{a}_1, \mathfrak{b}_1) = \frac{2\mathfrak{a}_1\mathfrak{b}_1}{\mathfrak{a}_1 + \mathfrak{b}_1}.$$

The next relationship is well-known in the literature:

$$\mathcal{H}(\mathfrak{a}_1, \mathfrak{b}_1) \leq \mathcal{G}(\mathfrak{a}_1, \mathfrak{b}_1) \leq \mathcal{A}(\mathfrak{a}_1, \mathfrak{b}_1).$$

Proposition 1. *Suppose that* $0 < \mathfrak{a}_1 < \mathfrak{b}_1$ *and* $s \in [0, 1]$, *then*

$$
\begin{aligned}
&\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i\left(1-\frac{s}{2}\right)^{\frac{1}{i}}}\mathcal{A}(2\mathfrak{a}_1,\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right))\\
&\leq\frac{\Gamma_{\mathfrak{K}}\left(\varrho+\mathfrak{K}\right)}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}}\left(\mathfrak{b}_1,\mathfrak{a}_1\right)}\left\{\mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1+}\ 2\mathcal{A}\left(\mathfrak{a}_1,\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)+\mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1+\varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))^-}\ \left(\mathfrak{a}_1\right)\right\}\\
&\leq\ 2\mathcal{A}\left(\mathfrak{a}_1,\mathfrak{a}_1+\varsigma_*\left(\mathfrak{b}_1,\mathfrak{a}_1\right)\right)\\
&\times\int_0^1\mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1}\left\{\frac{\sum_{i=1}^n a_i\left(1-s\mathfrak{t}\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}+\frac{\sum_{i=1}^n a_i\left(1-s\left(1-\mathfrak{t}\right)\right)^{\frac{1}{i}}}{\sum_{i=1}^n a_i}\right\}d\mathfrak{t}.
\end{aligned}
\tag{44}
$$

Proof. We attain the above inequality from Proposition 1 if we put $\mathfrak{F}(u) = u$ *for* $u > 0$ *in Theorem 2.*

Proposition 2. *Suppose that* $0 < \mathfrak{a}_1 < \mathfrak{b}_1$ *and* $s \in [0,1]$, *then*

$$\begin{aligned}
&\frac{\sum_{i=1}^{\mathfrak{n}} a_i}{\sum_{i=1}^{\mathfrak{n}} a_i \left(1-\frac{s}{2}\right)^{\frac{1}{i}}} \mathcal{A}^{-1}(2\mathfrak{a}_1, \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)) \\
&\leq \frac{\Gamma_{\mathfrak{K}}(\varrho + \mathfrak{K})}{\varsigma_*^{\frac{\varrho}{\mathfrak{K}}} (\mathfrak{b}_1, \mathfrak{a}_1)} \left\{ \mathfrak{J}^{\varrho,\mathfrak{K}}_{\mathfrak{a}_1+} \; 2\mathcal{A}^{-1}(\mathfrak{a}_1, \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)) + \mathfrak{J}^{\varrho,\mathfrak{K}}_{(\mathfrak{a}_1 + \varsigma_*(\mathfrak{b}_1,\mathfrak{a}_1))^-} \; \left(\mathfrak{a}_1{}^{-1}\right) \right\} \\
&\leq \; 2\mathcal{H}^{-1}(\mathfrak{a}_1, \mathfrak{a}_1 + \varsigma_* (\mathfrak{b}_1, \mathfrak{a}_1)) \\
&\times \int_0^1 \mathfrak{t}^{\frac{\varrho}{\mathfrak{K}}-1} \left\{ \frac{\sum_{i=1}^{\mathfrak{n}} a_i (1 - s\mathfrak{t})^{\frac{1}{i}}}{\sum_{i=1}^{\mathfrak{n}} a_i} + \frac{\sum_{i=1}^{\mathfrak{n}} a_i (1 - s(1-\mathfrak{t}))^{\frac{1}{i}}}{\sum_{i=1}^{\mathfrak{n}} a_i} \right\} d\mathfrak{t}.
\end{aligned} \tag{45}$$

Proof. We attain the above inequality from Proposition 2 if we put $\mathfrak{F}(u) = \frac{1}{u}$ *for* $u > 0$ *in Theorem 2.*

5. Conclusion

In this article, we introduced a novel fractional integral operator for functions, leveraging the concept of polynomial n-fractional s-like preinvexity. The proposed operator extends the classical fractional calculus framework by incorporating polynomial and preinvexity properties, offering a more versatile tool for analyzing functions with specific convexity and fractional characteristics. We established key properties of the new operator, including its convergence, boundedness, and applicability to various classes of functions. Furthermore, we demonstrated its utility in solving fractional differential equations and optimizing problems involving preinvex functions. The results presented herein not only generalize existing fractional integral operators but also open new avenues for research in fractional calculus and its applications in optimization, mathematical modeling, and applied sciences. Future work could explore the extension of this operator to higher dimensions, its application in real-world problems, and its relationship with other fractional operators.

Authors' Contributions

All authors contribute equally in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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