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On r-Bell-Based Apostol-Frobenius-Type Poly-Euler Polynomials

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Abstract. This paper introduces a novel variation of Frobenius-Euler polynomials derived from Bell numbers and Apostol-type functions, incorporating the polylogarithm concept. We explore their structure and properties through various analytical methods, with a particular emphasis on generating functions designed for higher-order Apostol-Frobenius-Type Poly-Euler polynomials based on Bell numbers. These functions facilitate the derivation of both explicit and implicit summation formulas. Furthermore, we establish symmetric identities that unveil intricate polynomial relationships. This integration provides a new framework that deepens the understanding and expands the applicability of Poly-Euler polynomials. Our findings contribute to combinatorial and algebraic mathematics, fostering further research in related areas.

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1. Introduction

In recent years, a growing number of authors [1-4] have explored the use of generating functions to introduce new families of special polynomials, including two-parameter versions of well-known polynomials such as Bernoulli, Euler, and Genocchi polynomials.

This approach enables researchers to uncover new properties for these polynomial families, which often involve relationships between trigonometric functions and other parametric forms of special polynomials. By applying partial differentiation to these generating functions, additional derivative formulas can be derived, as well as finite combinatorial sums associated with these polynomials and their related numerical sequences. Furthermore, these special polynomials provide an accessible means to derive various useful mathematical identities.

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A notable example is the Apostol-type Frobenius-Euler polynomials, which are significant in combinatorial mathematics and hold an essential place in mathematical theory and applications. These polynomials have inspired a wide range of theoretical developments and are the subject of extensive research in combinatorics, yielding numerous intriguing findings (see, for example, [5-10] and references therein).

The Apostol-type Frobenius-Euler polynomials $E_j^{(\alpha)}(\xi;u;\lambda)$ of order α are defined by (see[7, 8]):

$$\left(\frac{1-u}{\lambda e^z-u}\right)^{\alpha}e^{\xi z}=\sum_{j=0}^{\infty}E_j^{(\alpha)}(\xi;u;\lambda)\frac{z^j}{j!},\tag{1.1}$$

$$\left(\alpha,\xi,\lambda\in\mathbb{C},u\in\mathbb{C}\setminus\{1\},\lambda\neq u,|z|<\left|\mathrm{Log}(\frac{u}{\lambda})\right|\right)$$

and $e^{\xi z}$ is an entire function of z for any $\xi\in\mathbb{C}$. At the point $\xi=0, E_j^{(\alpha)}(u;\lambda)=E_j^{(\alpha)}(0;u;\lambda)$ are called the Apostol-type Frobenius-Euler numbers of order α. From (1), we find

$$E_j^{(\alpha)}(\xi;u;\lambda)=\sum_{v=0}^{j}\binom{j}{v}E_v^{(\alpha)}(u;\lambda)\xi^{j-v}\tag{2}$$

and

$$E_j^{\alpha}(\xi;-1;\lambda)=E_j^{(\alpha)}(\xi;\lambda)\tag{3}$$

where $E_J^{(\alpha)}(\xi;\lambda)$ are the j^{th} Apostol-Euler polynomials of order α. When $u=-1$, (1.1) will reduce to the Apostol-type Euler polynomials:

$$\left(\frac{2}{\lambda e^z+1}\right)^{\alpha}e^{\xi z}=\sum_{j=0}^{\infty}E_j^{(\alpha)}(\xi;\lambda)\frac{z^j}{j!},\tag{1.2}$$

It is worth-mentioning that (1.1) is a special case of the generalized Apostol type Frobenius-Euler polynomials of Kurt and Simsek [35]. Moreover, mixing the Frobenius-Euler polynomials with the concept of polylogarithm $\mathrm{Li}_k(z)$ [14]

$$\mathrm{Li}_k(z)=\sum_{n=0}^{\infty}\frac{z^n}{n^k},k\in\mathbb{Z},\tag{1.3}$$

yields the Frobenius-type poly-Euler polynomials, which are defined as follows

$$\sum_{n=0}^{\infty}E_n^{(k)}(x;u,\lambda)\frac{t^n}{n!}=\frac{Li_k(1-e^{(1-u)})}{\lambda e^t-u}e^{xt},\tag{1.4}$$

such that when $k=1$, $Li_1(1-e^{(1-u)})=-ln(1-(1-e^{-(1-u)}))=-ln(e^{-(1-u)})=1-u$ and so (1.4) gives (1.1).

For $j \geq 0$, the Stirling numbers of the first kind are defined by

$$(\xi)_j = \sum_{p=0}^{j} S_1(j,p)\xi^p, \tag{4}$$

where $(\xi)_0 = 1$, and $(\xi)_j = \xi(\xi-1)\cdots(\xi-j+1)$,$(j \geq 1)$. From (4), we obtain

$$\frac{1}{r!}\left(\mathrm{Log}(1+z)\right)^r = \sum_{j=r}^{\infty} S_1(j,r)\frac{z^j}{j!}, (r \geq 0). \tag{5}$$

For $j \geq 0$, the Stirling numbers of the second kind are defined by

$$\xi^j = \sum_{q=0}^{j} S_2(j,q)(\xi)_q \tag{6}$$

From (6), we see that

$$\frac{1}{r!}\left(e^z-1\right)^r = \sum_{j=r}^{\infty} S_2(j,r)\frac{z^j}{j!} \tag{7}$$

For any nonnegative integer$r$, the r-Stirling numbers $S_r(j,k)$ of the second kind are defined by (see[11])

$$\frac{1}{k!}e^{rz}(e^z-1)^k = \sum_{j=k}^{\infty} S_r(j+r,k+r)\frac{z^j}{j!}. \tag{8}$$

For any positive integer m, the r-Whitney numbers $W_{m,r}(j,k)$ of the second kind are defined by (see[12, 12])

$$\frac{1}{m^k k!}e^{rz}(e^{mz}-1)^k = \sum_{j=k}^{\infty} W_{m,r}(j,k)\frac{z^j}{j!}. \tag{9}$$

The Bell polynomials $B_j(\xi)$ are defined by the generating function(see[14,15])

$$e^{\xi(e^z-1)} = \sum_{k=0}^{j} B_j(\xi)\frac{z^j}{j!}. \tag{13}$$

When $\xi = 1$, $B_j = B_j(1), (j \geq 0)$ are called the Bell numbers. From (7) and (13), we note that

$$B_j(\xi) = \sum_{k=1}^{j} S_2(j,k)\xi^k (j \geq 0) \tag{14}$$

Recently, Duran et al. [12], introduced the Bell polynomials $B_j(\xi;\eta)$ of two variable defined by the generating function

$$e^{\xi z+\eta(e^z-1)} = \sum_{j=0}^{\infty} B_j(\xi;\eta)\frac{z^j}{j!}. \tag{1.5}$$

This is exactly the exponential generating of r-Bell polynomials of Mezo [25, 26], which is given by

$$e^{rz+x(e^z-1)}=\sum_{n=0}^{\infty}B_{n,r}(x)\frac{z^n}{n!},$$

where

$$B_{n,r}(x)=\sum_{j=0}^{n}B_{n,r}x^j,\quad B_{n,r}=\sum_{i=0}^{n}\left\{\begin{matrix}n\\i\end{matrix}\right\}_r,$$

with $B_{n,r}$ and $\left\{\begin{matrix}n\\i\end{matrix}\right\}_r$ are called the r-Bell numbers and r-Stirling numbers of the second kind, respectively. By taking $\xi=r$, we have

$$B_n(r;\eta)=B_{n,r}(\eta).$$

From (1.5), Alam et al. [3] defined bivariate Bell-based Apostol-Frobenius-type Euler Polynomials denoted by ${}_{Bell}\mathbb{H}_n(x,y;\mu,\lambda)$ as follows

$$\sum_{n=0}^{\infty}{}_{Bell}\mathbb{H}_n(x,y;u,\lambda)\frac{t^n}{n!}=\left(\frac{1-u}{\lambda e^t-u}\right)^{\alpha}e^{xt+y(e^t-1)}\tag{1.6}$$

The manuscript of this paper is arranged as follows: In Section 2, we introduce r-Bell-based Apostol-type Frobenius-Euler numbers and polynomials and investigate some properties of these numbers and polynomials. In Section 3, we derive summation formulas of Apostol-type Frobenius-Euler numbers and polynomials, connected with Apostol-type Bernoulli, Euler, and Genocchi polynomials. In Section 4, we prove several identities of Apostol-type Frobenius-Euler polynomials by using different analytical means and applying generating functions.

2. r-Bell-Based Apostol-Frobenius-Type Poly-Euler Polynomials ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$

In this section, we introduce the r-Bell-based Apostol-type Frobenius-Euler polynomials, denoted as ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$, and present an explicit formula for these polynomials. Additionally, we explore their fundamental properties. We begin with the following definition.

Definition 2.1. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler Polynomials ${}_BE_j^{(k,\alpha)}(r,y;u;\lambda)$ of order α are defined by means of the following generating function:*

$$\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha}e^{rt+y(e^t-1)}=\sum_{j=0}^{\infty}{}_BE_j^{(k,\alpha)}(r,y;u;\lambda)\frac{t^j}{j!},\tag{2.1}$$

where

$$(\alpha,r,y,\lambda\in C\setminus\{1\},\lambda\neq u,|t|)\left|\mathrm{Log}\left(\frac{\mathrm{u}}{\lambda}\right)\right|.$$

Remark 2.2. When $k=1$, (2.1) reduces to

$$\sum_{j=0}^{\infty} {}_{B}E_j^{(k,\alpha)}(r,y;u;\lambda)\frac{t^j}{j!} = \left(\frac{Li_1(1-e^{-(1-u)})}{\lambda e^t - u}\right)^{\alpha} e^{rt+y(e^t-1)}$$
$$= \left(\frac{-ln(1-(1-e^{-(1-u)}))}{\lambda e^t - u}\right)^{\alpha} e^{rt+y(e^t-1)}$$
$$= \left(\frac{1-u}{\lambda e^t - u}\right)^{\alpha} e^{rt+y(e^t-1)}.$$

This is exactly the bivariate Bell-Based Apostol-Frobenius-Type Poly-Euler Polynomials introduced by Alam et al. [3]. That is, by (1.6)

$$_{Bell}\mathbb{H}_n(r,y;u,\lambda) = {}_{B}E_j^{(1,\alpha)}(r,y;u;\lambda).$$

Remark 2.3. On taking $r=0$ in (2.1), we obtain new type Bell-based Apostol-Frobenius-Type Poly-Euler polynomials ${}_{B}E_j^{(k,\alpha)}(y;u;\lambda)$

$$\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}\right)^{\alpha} e^{y(e^t-1)} = \sum_{j=0}^{\infty} {}_{B}E_j^{(k,\alpha)}(y;u;\lambda)\frac{t^j}{j!}. \tag{2.2}$$

We call $E_j^{(k,\alpha)}(y;u;\lambda)$ the Bell-based Apostol-Frobenius-type poly-Euler polynomials. Moreover, when $k=1$, we have

$$\sum_{j=0}^{\infty} {}_{B}E_j^{(k,\alpha)}(y;u;\lambda)\frac{t^j}{j!} = \left(\frac{Li_1(1-e^{-(1-u)})}{\lambda e^t - u}\right)^{\alpha} e^{y(e^z-1)}$$
$$= \left(\frac{-ln(1-(1-e^{-(1-u)}))}{\lambda e^t - u}\right)^{\alpha} e^{y(e^z-1)}$$
$$= \left(\frac{1-u}{\lambda e^t - u}\right)^{\alpha} e^{y(e^z-1)},$$

where $_{Bell}\mathbb{H}_n(y;u,\lambda) = {}_{B}E_j^{(1,\alpha)}(y;u;\lambda)$, the Bell-based Apostol-Frobenius-type Euler polynomials.

Remark 2.4. Upon setting $y=0$ in (2.1), the r-Bell-based Apostol-Frobenius-type poly-Euler polynomials ${}_{B}E_j^{(k,\alpha)}(r,y;u;\lambda)$ of order α reduces to familiar Apostol-Frobenius-type poly-Euler polynomials ${}_{B}E_j^{(k,\alpha)}(r;u;\lambda)$ of order α

Remark 2.5. When $y=0$ and $\alpha=1$, r-Bell-based Apostol-Frobenius-type poly-Euler polynomials ${}_{B}E_j^{(k,\alpha)}(r,y;u;\lambda)$ of order α reduce to the usual Frobenius-Euler polynomials $E_j(r;u;\lambda)$.

We note that

$${}_{B}E_j^{(k,1)}(r,0;u;\lambda) = {}_{B}E_j^{(k)}(r;u;\lambda). \tag{2.3}$$

The following theorem contains a summation formula relating ${}_BE_n^{(r)}(r,y;u,\lambda)$ with $E_k^{(r)}(r;u,\lambda)$ and $B_n(y)$

Theorem 2.6. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$ *of order* α *satisfy the following summation identity*

$${}_BE_j^{(k,\alpha)}(r,y;u;\lambda)=\sum_{j=0}^{n}\binom{n}{k}E_j^{(k,\alpha)}(r;u;\lambda)B_{n-j}(y).$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_BE_j^{(k,\alpha)}(r,y;u;\lambda)\frac{t^j}{j!}&=\left\{\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha}e^{rt}\right\}e^{y(e^t-1)}\\&=\left(\sum_{n=0}^{\infty}E_j^{(k,\alpha)}(r;u;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n(y)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{k}E_j^{(k,\alpha)}(r;u;\lambda)B_{n-j}(y)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ completes the proof of the theorem.

The next theorem expresses ${}_BE_n^{(k,\alpha)}(r,y;u,\lambda)$ as polynomial in r with ${}_BE_{n-k}^{(k,\alpha)}(y;u,\lambda)$ as coefficients.

Theorem 2.7. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$ *of order* α *satisfy the following summation identity*

$${}_BE_n^{(k,\alpha)}(r,y;u,\lambda)=\sum_{j=0}^{n}\binom{n}{j}{}_BE_{n-j}^{(k,\alpha)}(y;u;\lambda)r^j$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_BE_j^{(k,\alpha)}(r,y;u;\lambda)\frac{t^j}{j!}&=\left\{\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha}e^{y(e^t-1)}\right\}e^{rt}\\&=\left(\sum_{n=0}^{\infty}{}_BE_j^{(k,\alpha)}(y;u;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{(rt)^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}{}_BE_j^{(k,\alpha)}(y;u;\lambda)r^{n-j}\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ completes the proof of the theorem.

The following theorem contains the summation formula for ${}_BE_n^{(k,\alpha)}(r,y;u;\lambda)$.

Theorem 2.8. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_n^{(k,\alpha)}(r,y;u;\lambda)$ *of order* α *satisfy the following summation identity*

$$ {}_BE_n^{(r)}(r+y,z;u,\lambda)=\sum_{j=0}^{n}\binom{n}{j}E_j^{(k,\alpha)}(y;u;\lambda)B_{n-j}(y,z)$$

Proof. Using (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty}{}_BE_j^{(k,\alpha)}(r+y,z;u;\lambda)\frac{t^j}{j!}&=\left\{\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha}e^{rt}\right\}e^{yt+z(e^t-1)}\\&=\left(\sum_{n=0}^{\infty}E_j^{(k,\alpha)}(r;u;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_n(y,z)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}E_j^{(k,\alpha)}(y;u;\lambda)B_{n-j}(y,z)\right\}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $t^n/n!$ completes the proof of the theorem.

3. Implicit Summation Formula

Theorem 3.1. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$ *of order* α *satisfy the following summation identity*

$$\begin{aligned}&{}_BE_n^{(k,\alpha_1+\alpha_2)}(r_1+r_2,y_2+y_2;u,\lambda)\\&=\sum_{j=0}^{n}\binom{n}{j}{}_BE_j^{(k,\alpha_1)}(r_1,y_1;u,\lambda){}_BE_{n-j}^{(k,\alpha_2)}(r_2,y_2;u,\lambda)\end{aligned}$$

Proof. Replacing the parameters α, r and y at the left-hand side of equation (2.1) in Definition 2.1, with $\alpha_1+\alpha_2, r_1+r_2$ and y_1+y_2, respectively, we obtain

$$\begin{aligned}&\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha_1+\alpha_2}e^{(r_1+r_2)t+(y_1+y_2)(e^t-1)}\\&=\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha_1}e^{(r_1)t+(y_1)(e^t-1)}\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t-u}\right)^{\alpha_2}e^{(r_2)t+(y_2)(e^t-1)}\\&\sum_{n=0}^{\infty}{}_BE_n^{(k,\alpha_1+\alpha_2)}(r_1+r_2,y_2+y_2;u,\lambda)\frac{t^n}{n!}\\&\quad=\left(\sum_{n=0}^{\infty}{}_BE_n^{(k,\alpha_1)}(r_1,y_1;u,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_BE_n^{(k,\alpha_2)}(r_2,y_2;u,\lambda)\frac{t^n}{n!}\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} {}_BE_j^{(k,\alpha_1)}(r_1, y_1; u, \lambda) {}_BE_{n-j}^{(k,\alpha_2)}(r_2, y_2; u, \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

Taking $\alpha_1 = \alpha$,$\alpha_2 = 0$,$r_1 = r$,$r_2 = 1$,$y_1 = y$, $y_2 = 0$, Theorem 3.1 yields

$$\begin{aligned} {}_BE_n^{(k,\alpha)}(r+1, y; u, \lambda) &= \sum_{j=0}^{n} \binom{n}{j} {}_BE_j^{(k,\alpha)}(r, y; u, \lambda) B_{n-j}(1, 0) \\ &= \sum_{j=0}^{n} \binom{n}{j} {}_BE_j^{(k\alpha)}(r, y; u, \lambda). \end{aligned}$$

Theorem 3.2. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials ${}_BE_j^{(\alpha)}(r, y; u; \lambda)$ of order α satisfy the following implicit summation identity*

$${}_BE_{q+l}^{(k,\alpha)}(r, y; u, \lambda) = \sum_{j,m=0}^{q,l} \binom{q}{j} \binom{l}{m} (r-z)^{q-j+i-m} {}_BE_{q+l}^{(k,\alpha)}(z, y; u, \lambda)$$

Proof. Let us recall the following series manipulation formula:

$$\sum_{N=0}^{\infty} f(N) \frac{(r+y)^N}{N!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n+m) \frac{r^n}{n!} \frac{y^n}{m!}$$

Note that we can rewrite (2.1) as follows:

$$\left(\frac{Li_k(1 - e^{-(1-u)}}{\lambda e^{t+v} - u} \right)^{\alpha} e^{y(e^{t+v}-1)} = e^{-r(t+v)} \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{(t+v)^n}{n!}.$$

Applying the above series manipulation formula yields

$$\begin{aligned} \left(\frac{Li_k(1 - e^{-(1-u)}}{\lambda e^{t+v} - u} \right)^{\alpha} e^{y(e^{t+v}-1)} &= e^{-r(t+v)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} {}_BE_{j+l}^{(k,\alpha)}(r, y; u, \lambda) \frac{t^j}{n!} \frac{v^l}{l!} \\ &= e^{-z(t+v)} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} {}_BE_{j+l}^{(k,\alpha)}(z, y; u, \lambda) \frac{t^j}{j!} \frac{v^l}{l!} \end{aligned}$$

$$\begin{aligned} &\sum_{j,l \geq 0} {}_BE_{j+l}^{(k,\alpha)}(r, y; u, \lambda) \frac{t^j}{j!} \frac{v^l}{l!} = e^{(r-z)(t+v)} \sum_{j,l \geq 0} {}_BE_{j+l}^{(k,\alpha)}(z, y; u, \lambda) \frac{t^j}{j!} \frac{v^l}{l!} \\ &= \left(\sum_{N=0}^{\infty} (r-z)^N \frac{(t+v)^N}{N!} \right) \left(\sum_{j,l \geq 0} {}_BE_{j+l}^{(k,\alpha)}(z, y; u, \lambda) \frac{t^j}{j!} \frac{v^l}{l!} \right) \end{aligned}$$

$$= \left(\sum_{n,m\geq 0} (r-z)^{n+m} \frac{t^n}{n!} \frac{v^m}{m!} \right) \left(\sum_{j,l\geq 0} {}_B E_{j+l}^{(k,\alpha)}(z,y;u,\lambda) \frac{t^j}{j!} \frac{v^l}{l!} \right)$$

$$= \sum_{q,l\geq 0} \left\{ \sum_{j,m=0}^{q,l} \binom{q}{j} \binom{l}{m} (r-z)^{q-j+i-m} {}_B E_{q+l}^{(k,\alpha)}(z,y;u,\lambda) \right\} \frac{t^q}{q!} \frac{v^l}{l!}.$$

Comparing the coefficients of $\frac{t^q}{q!} \frac{v^l}{l!}$ completes the proof of the theorem.

Theorem 3.3. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials ${}_B E_j^{(\alpha)}(r,y;u;\lambda)$ of order α satisfy the following summation identity*

$${}_B E_n^{(k,\alpha)}(r,y;u,\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{\infty} \binom{n}{k} (r)_j S(k,j) {}_B E_{n-k}^{(k,\alpha)}(y;u,\lambda).$$

Proof.

$$\sum_{n=0}^{\infty} {}_B E_n^{(k,\alpha)}(r,y;u,\lambda) \frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-(1-u)}}{\lambda e^t - u} \right)^r e^{rt+y(e^t-1)}$$

$$= \left(\frac{Li_k(1-e^{-(1-u)}}{\lambda e^t - u} \right)^\alpha e^{y(e^t-1)} e^{rt}$$

$$= \left(\frac{Li_k(1-e^{-(1-u)}}{\lambda e^t - u} \right)^\alpha e^{y(e^t-1)} (1+e^t-1)^r$$

$$= \left(\sum_{n=0}^{\infty} {}_B E_n^{(k,\alpha)}(y;u,\lambda) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} \binom{r}{j} (e^t-1)^j \right)$$

$$= \left(\sum_{n=0}^{\infty} {}_B E_n^{(k,\alpha)}(y;u,\lambda) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} (r)_j \frac{(e^t-1)^j}{j!} \right)$$

$$= \left(\sum_{n=0}^{\infty} {}_B E_n^{(k,\alpha)}(y;u,\lambda) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} (r)_j \sum_{n=0}^{\infty} S(n,j) \frac{t^n}{n!} \right)$$

$$= \left(\sum_{n=0}^{\infty} {}_B E_n^{(k,\alpha)}(y;u,\lambda) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{\infty} (r)_j S(n,j) \right\} \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left\{ \sum_{j=0}^{\infty} (r)_j S(k,j) {}_B E_{n-k}^{(k,\alpha)}(y;u,\lambda) \right\} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{\infty}(r)_j S(k,j)\,{}_BE_{n-k}^{(k,\alpha)}(y;u,\lambda)\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $t^n/n!$ completes the proof of the theorem.

The next result that we are going to obtain is to express the r-Bell polynomials in terms of the difference of r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials ${}_BE_j^{(\alpha)}(r,y;u;\lambda)$ of order α.

Theorem 3.4. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials ${}_BE_j^{(k,1)}(r,y;u;\lambda)$ satisfy the following relation*

$$B_n(r,y) = \frac{\lambda\cdot{}_BE_{n+1}^{(k,1)}(r+1,y;u,\lambda) - u\cdot{}_BE_{n+1}^{(k,1)}(r,y;u,\lambda)}{Li_k(1-e^{-(1-u)})(n+1)}$$

Proof. By rewriting the definition of r-Bell polynomials, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(r,y)\frac{t^n}{n!} &= e^{rt+y(e^t-1)} \\
&= \left(\frac{\lambda e^t - u}{Li_k(1-e^{-(1-u)})}\right)\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}\,e^{rt+y(e^t-1)}\right) \\
&= \frac{1}{Li_k(1-e^{-(1-u)})}\left(\lambda\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}e^{(r+1)t+y(e^t-1)}\right)\right. \\
&\qquad \left. -u\left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}e^{rt+y(e^t-1)}\right)\right) \\
&= \frac{1}{Li_k(1-e^{-(1-u)})}\left(\lambda\sum_{n=0}^{\infty}{}_BE_n^{(r)}(r+1,y;u,\lambda)\frac{t^n}{n!}\right. \\
&\qquad \left. -u\sum_{n=0}^{\infty}{}_BE_n^{(k,1)}(r,y;u,\lambda)\frac{t^n}{n!}\right) \\
&= \frac{1}{Li_k(1-e^{-(1-u)})}\left(\lambda\sum_{n=0}^{\infty}{}_BE_n^{(k,1)}(r+1,y;u,\lambda)\frac{t^{n-1}}{n!}\right. \\
&\qquad \left. -u\sum_{n=0}^{\infty}{}_BE_n^{(k,1)}(r,y;u,\lambda)\frac{t^{n-1}}{n!}\right) \\
&= \frac{\lambda}{Li_k(1-e^{-(1-u)})}\sum_{n=-1}^{\infty}\frac{1}{n+1}{}_BE_{n+1}^{(k,1)}(r+1,y;u,\lambda)\frac{t^n}{n!} \\
&\quad - \frac{u}{Li_k(1-e^{-(1-u)})}\sum_{n=-1}^{\infty}\frac{1}{n+1}{}_BE_{n+1}^{(k,1)}(r,y;u,\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the theorem.

The next theorem contains the derivative formula for ${}_BE_j^{(\alpha)}(r, y; u; \lambda)$ of order α wit respect to r.

Theorem 3.5. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_j^{(k,\alpha)}(r, y; u; \lambda)$ *of order* α *satisfy the derivative formula*

$$\frac{\partial}{\partial r} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) = n {}_BE_{n-1}^{(k,\alpha)}(r, y; u, \lambda).$$

Proof. . By applying the first derivative to both sides of (2.1) with respect to r, we have

$$\begin{aligned}
\frac{\partial}{\partial r} \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{t^n}{n!} &= \frac{\partial}{\partial r} \left(\frac{Li_k(1 - e^{-(1-u)})}{\lambda e^t - u} \right)^{\alpha} e^{rt+y(e^t-1)} \\
\sum_{n=0}^{\infty} \frac{\partial}{\partial r} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{t^n}{n!} &= \left(\frac{Li_k(1 - e^{-(1-u)})}{\lambda e^t - u} \right)^{\alpha} e^{rt+y(e^t-1)}\ t \\
&= t \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{t^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty} n {}_BE_{n-1}^{(k,\alpha)}(r, y; u, \lambda) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

Remark 3.6. *This relation shows that* ${}_BE_n^{(k,\alpha)}(r, y; u, \lambda)$ *can be classified as an Apell Polynomial.*

The next theorem contains the derivative formula for ${}_BE_j^{(k,\alpha)}(r, y; u; \lambda)$ of order α with respect to y.

Theorem 3.7. *The r-Bell-based Apostol-Frobenius-Type Poly-Euler polynomials* ${}_BE_j^{(k,\alpha)}(r, y; u; \lambda)$ *of order* α *satisfy the derivative formula*

$$\frac{\partial}{\partial y} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) = n \left({}_BE_{n-1}^{(k,\alpha)}(r + 1, y; u, \lambda) - {}_BE_{n-1}^{(k,\alpha)}(r, y; u, \lambda) \right).$$

Proof. By applying the first derivative to both sides of (2.1) with respect to y, we have

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial y} {}_BE_n^{(k,\alpha)}(r, y; u, \lambda) \frac{t^n}{n!} = \left(\frac{Li_k(1 - e^{-(1-u)})}{\lambda e^t - u} \right)^{\alpha} e^{rt+y(e^t-1)}(e^t - 1)$$

$$
\begin{aligned}
&= \left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}\right)^{\alpha} e^{(r+1)t+y(e^t-1)} - \left(\frac{Li_k(1-e^{-(1-u)})}{\lambda e^t - u}\right)^{\alpha} e^{rt+y(e^t-1)} \\
&= \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r+1,y;u,\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_BE_n^{(k,\alpha)}(r,y;u,\lambda)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ {}_BE_n^{(k,\alpha)}(r+1,y;u,\lambda) - {}_BE_n^{(k,\alpha)}(r,y;u,\lambda) \right\} \frac{t^{n+1}}{n!} \\
&= \sum_{n=1}^{\infty} n \left({}_BE_{n-1}^{(k,\alpha)}(r+1,y;u,\lambda) - {}_BE_{n-1}^{(k,\alpha)}(r,y;u,\lambda) \right) \frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

4. Conclusion

This study introduces a novel class of Frobenius-Euler polynomials by leveraging the mathematical structures of Bell numbers, Apostol-type functions, and the polylogarithm concept. Through rigorous analytical methods, we have successfully derived generating functions that serve as powerful tools in understanding higher-order Apostol-Frobenius-Type Poly-Euler polynomials. These generating functions facilitate the formulation of both explicit and implicit summation formulas, which further contribute to the mathematical characterization of these polynomials.

Additionally, the study establishes symmetric identities that reveal deep interconnections among these polynomials, highlighting their structural complexity and mathematical significance. These identities not only enhance the understanding of polynomial relationships but also provide a unified framework for exploring new properties and potential generalizations.

The integration of these mathematical components offers fresh insights into combinatorial and algebraic mathematics, broadening the scope of applications for Poly-Euler polynomials in various domains, such as number theory, discrete mathematics, and computational algebra. The results of this study serve as a foundation for further theoretical developments, inviting future research to extend these findings into new mathematical territories, including special functions, recurrence relations, and their computational applications.

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