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The Type 2 Degenerate Hermite-Based Apostol-Frobenius-type Poly-Genocchi Polynomials with Parameters a, b and c

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Abstract. This paper introduces a novel variation of poly-Genocchi polynomials by combining elements from the modified degenerate polyexponential function, Hermite-Based Apostol-Genocchi polynomials, and Frobenius polynomials. These newly formulated polynomials, termed as the degenerate Hermite-Based Apostol-Frobenius-type poly-Genocchi polynomials with parameters a, b, and c, are explored to derive various identities and formulas. These include recurrence relations, explicit formulas, and specific differential identities. Furthermore, the paper establishes connections between these polynomials and degenerate Stirling numbers of the first and second kind, higher-order degenerate Bernoulli polynomials, and higher-order degenerate Frobenius-Euler polynomials.

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1. Introduction

The theory of special polynomials is vibrant area of modern mathematics, owing to its fundamental role in number theory, combinatorics, mathematical analysis, and mathematical physics. Among these, the Bernoulli, Euler, and Genocchi polynomials occupy a central place due to their deep connections with zeta functions, generating functions, and summation formulas [1–6]. Classical Genocchi numbers G_n are defined by the generating function:

$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t + 1}, \quad |t| < \pi. \tag{1}$$

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Several results involving these numbers, such as recurrence formulas and identities, can be found in [7–9].

Through multiplication with exponential functions, various generalizations of the Genocchi numbers have been constructed, including:

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, \tag{2}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1}e^{xt}, \tag{4}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{5}$$

where the case $\lambda = 1$ retrieves the classical forms. The first two of these have been extensively studied, with asymptotic approximations and Fourier expansions discussed in [10–14].

Further generalization has been achieved by introducing the Frobenius-Genocchi polynomials, defined as:

$$\sum_{n=0}^{\infty} G_n^F(x;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u}e^{xt}, \tag{6}$$

as considered in [15–23].

Another prominent line of generalization involves the use of the polylogarithm function

$$Li_k(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^k}.$$

When applied to Genocchi polynomials, this yields the poly-Genocchi polynomials:

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1}e^{xt}, \tag{7}$$

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt}, \tag{8}$$

as studied in [24–26]. When $k = 1$, these reduce to the classical Genocchi polynomials in (2).

Generalized versions with parameters a, b, and c were introduced by Kurt in [27], yielding:

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{x^n}{n!}, \tag{9}$$

$$\frac{2Li_k(1-(ab)^{-2t})}{a^{-t}+b^t}c^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x;a,b,c)\frac{x^n}{n!}. \tag{10}$$

These forms provide a unifying framework for further generalization. When $x = 0$, (7) simplifies to:

$$\frac{2Li_k(1-e^t)}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)}\frac{x^n}{n!}, \tag{11}$$

where $G_n^{(k)}$ are the poly-Genocchi numbers.

By extending these forms further through multi-polylogarithms and Hermite structures, Corcino et al. [28, 29] introduced the polynomials:

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!} = \left(\frac{Li_{k,\rho}(1-(ab)^{-(1-u)t})}{\lambda b^t - ua^{-t}}\right)^{\alpha} c^{xt}, \tag{12}$$

which encapsulate multiple layers of generalization, including Apostol-type, Frobenius-type, and polylogarithmic components.

Degeneracy offers yet another lens of generalization. The degenerate exponential function introduced by Carlitz [6, 30] is given by:

$$e_\lambda^x(t) = (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{13}$$

where $(x)_{n,\lambda}$ denotes the degenerate falling factorial. This function is central to constructing degenerate versions of classical polynomials. It can easily be seen that

$$\begin{aligned} e_\lambda^{x+y}(t) &= (1+\lambda t)^{(x+y)/\lambda} = (1+\lambda t)^{(x/\lambda)+(y/\lambda)} \\ &= (1+\lambda t)^{(x/\lambda)}(1+\lambda t)^{(y/\lambda)} \\ &= e_\lambda^x(t)e_\lambda^y(t), \end{aligned} \tag{14}$$

and

$$\frac{d}{dx}e_\lambda^x(t) = \log(1+\lambda t)^{1/\lambda}e_\lambda^x(t). \tag{15}$$

The degenerate Bernoulli polynomials $\mathcal{B}_{n,\lambda}(x)$ and degenerate Euler polynomials $\mathcal{E}_{n,\lambda}(x)$ were defined by Carlitz [30] by means of the following generating functions

$$\frac{t}{e_\lambda(t)+1}e_\lambda^x(t) = \frac{t}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}(x)\frac{t^n}{n!}$$

$$\frac{2}{e_\lambda(t)+1}e_\lambda^x(t) = \frac{2}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}.$$

Parallel to these, Lim [31] defined the degenerate Genocchi polynomials as follows

$$\frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \frac{2t}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{G}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{16}$$

Kim and Kim [32] introduced the degenerate Frobenius-Euler polynomials as coefficients of the following generating function

$$\frac{1-u}{e_\lambda(t)-u}e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!} \tag{17}$$

and Kim et al. [33] derived formulas that express any polynomial in terms of $h_{n,\lambda}(x|u)$. In their separate paper, Kim and Kim [34] defined the generalized degenerate Euler-Genocchi polynomials, denoted by $A_{n,\lambda}^{(r)}(x)$, as coefficients of the following generating function

$$\frac{2t^r}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} A_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},$$

which reduce to the degenerate Genocchi polynomials in (26) when $r = 1$. That is,

$$A_{n,\lambda}^{(1)}(x) = \mathcal{G}_{n,\lambda}(x).$$

The growing literature on degenerate versions includes significant contributions from Kim and collaborators [35–42], who explored degenerate Bernoulli, Euler, and Genocchi-type polynomials using probabilistic and algebraic techniques.

The degenerate Stirling numbers of the first and second kind, denoted by $S_{1,\rho}(n,k)$ and $S_{2,\rho}(n,k)$, were defined in ([43–45])

$$\frac{(\log_\rho(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_{1,\rho}(n,k)\frac{t^n}{n!}, \quad \frac{(e_\rho(t)-1)^k}{k!} = \sum_{n=0}^{\infty} S_{2,\rho}(n,k)\frac{t^n}{n!}, \tag{18}$$

where

$$\log_\rho(e_\rho(t)) = e_\rho(\log_\rho(t)) = t. \tag{19}$$

When $\rho \to 0$,

$$\lim_{\rho\to 0} S_{1,\rho}(n,k) = S_1(n,k), \quad \lim_{\rho\to 0} S_{2,\rho}(n,k) = S_2(n,k)$$

where $S_1(n,k)$ and $S_2(n,k)$ are the classical Stirling numbers of the first and second kind. Also, the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{(1+t)^x}{\log_\rho(1+t)}t = \sum_{n=0}^{\infty} b_{n,\rho}(x)\frac{t^n}{n!}. \tag{20}$$

The degenerate Stirling numbers of the second kind appeared in the probability distribution of the random variable given as the sum of a finite number of random variables with degenerate zero-truncated Poisson distributions and a random variable with degenerate Poisson distribution, all having the same parameter (see [45]).

The polyexponential functions are defined by the following generating functions [43, 46–48]

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}, \quad k \in \mathbb{Z}. \tag{21}$$

For $k = 1$, $\mathrm{Ei}_1(x) = e^x - 1$. The modified degenerate polyexponential function are given by ([43, 46–48])

$$\mathrm{Ei}_{k,\rho}(x) = \sum_{n=1}^{\infty} \frac{(1)_{n,\rho} x^n}{n^k(n-1)!}, \quad \rho \in \mathbb{R}. \tag{22}$$

Note that

$$\mathrm{Ei}_{1,\rho}(x) = \sum_{n=1}^{\infty} (1)_{n,\rho} \frac{x^n}{n!} = e_\rho(x) - 1, \quad \rho \in \mathbb{R}. \tag{23}$$

Also, we have

$$\frac{d}{dx}\mathrm{Ei}_{k,\rho}(\log_\rho(1+x)) = \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)} \mathrm{Ei}_{k-1,\rho}(\log_\rho(1+x)), \tag{24}$$

which implies

$$\begin{aligned}
\mathrm{Ei}_{k,\rho}(\log_\rho(1+x)) &= \int_0^x \frac{(1+t)^{\rho-1}}{\log_\rho(1+t)} \int_0^y \cdots \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)} \int_0^y \frac{(1+x)^{\rho-1}}{\log_\rho(1+x)} x dx \ldots dx \\
&= \sum_{m=0}^{\infty} \sum_{m_1+m_2+\ldots+m_{k-1}=m} \binom{m}{m_1, \ldots, m_{k-1}} \\
&\quad \times \frac{b_{m_1,\rho}(\rho-1)}{m_1+1} \frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1} \cdots \frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1} \frac{x^{m+1}}{m!}.
\end{aligned} \tag{25}$$

The degenerate poly-Euler polynomials were defined in [49] by means of the following generating function

$$2\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+t))}{te_\rho(t)+1} e_\rho^x(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,\rho}^{(k)}(x) \frac{t^n}{n!}, \tag{26}$$

where $k \in \mathbb{Z}$.

Building upon these foundations, this paper introduces a novel variant of poly-Genocchi polynomials, created by integrating concepts from the modified degenerate polyexponential function, Apostol-Genocchi polynomials, and Frobenius polynomials. Termed as the type 2 degenerate hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b, special cases of these polynomials are outlined, along with several identities showcasing their relations with various Genocchi-type polynomials. Additionally, connections of these degenerate Apostol-Frobenius-type poly-Genocchi polynomials with degenerate Stirling numbers of the first and second kind, higher-order degenerate Bernoulli polynomials, and higher-order degenerate Frobenius-Euler polynomials are discussed.

Furthermore, this paper establishes significant links between the new polynomials and several well-studied degenerate sequences, including the degenerate Stirling numbers of the first and second kinds, higher-order degenerate Bernoulli polynomials [35, 36, 38, 41], and higher-order degenerate Frobenius-Euler polynomials [37]. These interrelations highlight the rich algebraic structure of the new family and position them within the expanding domain of probabilistic and degenerate special functions [39, 40, 42].

Finally, notable applications of Genocchi and poly-Genocchi polynomials include:

(i) Estimating the number of finite languages accepted by finite automata [50];

(ii) Deriving wavelet-based numerical solutions to fractional Rosenau-Hyman equations [51];

(iii) Solving fractional differential equations through operational matrix methods using poly-Genocchi polynomials [52].

2. Definition and Some Explicit Formulas

Similar to the definition of degenerate poly-Euler polynomials outlined in (26), we can establish the desired variation of Apostol-type poly-Genocchi polynomials by introducing the parameter u to encompass the concept of Frobenius polynomials, along with the parameters a and b as well as the degenerate exponential polynomials $e_\rho^y(t^2)$. The following presents the formal definition of these desired polynomials.

Definition 2.1. The type 2 degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b, denoted by $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$, are defined as coefficients of the following generating function:

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^x(t)e_\rho^y(t^2), \tag{27}$$

where $|t|<\frac{\sqrt{(\ln\left(\frac{\lambda}{u}\right))^2+4\pi^2}}{|\ln a+\ln b|}$. When $\alpha=1$, (27) yields

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k)}(x,y;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}e_\rho^x(t)e_\rho^y(t^2). \tag{28}$$

where $\widehat{\mathcal{G}}_n^{(k)}(x,y;\lambda,\rho,u,a,b)=\widehat{\mathcal{G}}_n^{(k,1)}(x,y;\lambda,\rho,u,a,b)$ denotes the degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b.

Now, if $x = (1-u)t\ln ab$, then (25) yields

$$\begin{aligned}
&\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab)) \\
&\quad = t\sum_{m=0}^{\infty}((1-u)\ln ab)^{m+1}\sum_{m_1+m_2+\ldots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}} \\
&\qquad \times \frac{b_{m_1,\rho}(\rho-1)}{m_1+1}\frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1}\frac{t^m}{m!}.
\end{aligned}$$

Also,

$$\begin{aligned}
e_\rho^x(t) &= \sum_{n=0}^{\infty}(x)_{n,\rho}\frac{t^n}{n!} \\
e_\rho^y(t^2) &= \sum_{n=0}^{\infty}(y)_{n,\rho}\frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty}(y)_{n,\rho}\frac{t^{2n}}{n!} = \sum_{m=0}^{\infty}(y)_{\frac{m}{2},\rho}\frac{t^m}{(\frac{m}{2})!}
\end{aligned}$$

$$\begin{aligned}
(y)_{\frac{m}{2},\rho} &= (y)(y-\rho)\cdots\left(y-\left(\frac{m}{2}-1\right)\rho\right) \\
&= \rho^{\frac{m}{2}}\left(\frac{y}{\rho}\right)\left(\frac{y}{\rho}-1\right)\cdots\left(\frac{y}{\rho}-\left(\frac{m}{2}-1\right)\right) = \rho^{\frac{m}{2}}\left(\frac{y}{\rho}\right)_{\frac{m}{2}} \\
\frac{(y)_{\frac{m}{2},\rho}}{(\frac{m}{2})!} &= \rho^{\frac{m}{2}}\binom{\frac{y}{\rho}}{\frac{m}{2}}
\end{aligned}$$

$$e_\rho^y(t^2) = \sum_{m=0}^{\infty}\rho^{\frac{m}{2}}m!\binom{\frac{y}{\rho}}{\frac{m}{2}}\frac{t^m}{m!} \quad , \binom{\frac{y}{\rho}}{i} = 0 \ , i \notin \mathbb{N}\cup\{0\}.$$

$$\begin{aligned}
e_\rho^x(t)\ e_\rho^y(t^2) &= \left(\sum_{n=0}^{\infty}\rho^m_{n!}\binom{\frac{x}{\rho}}{n}\frac{t^n}{n!}\right)\ \left(\sum_{n=0}^{\infty}\rho^{\frac{n}{2}}_{n!}\binom{\frac{y}{\rho}}{\frac{n}{2}}\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\rho^{n-j}(n-j)!\binom{\frac{x}{\rho}}{n-j}\frac{t^{n-j}}{(n-j)!}\rho^{\frac{j}{2}}(j)!\binom{\frac{y}{\rho}}{\frac{j}{2}}\frac{t^j}{j!} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}n!\,\rho^{n-\frac{j}{2}}\binom{\frac{x}{\rho}}{n-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}\frac{t^n}{n!} \\
e_\rho^x(t)\ e_\rho^y(t^2) &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}n!\,\rho^{n-\frac{j}{2}}\binom{\frac{x}{\rho}}{n-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}\right)\frac{t^n}{n!}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\lambda b^t - ua^{-t}} &= \frac{-1}{ua^{-t}\left(1-\frac{\lambda}{u}(ab)^t\right)} = \frac{-a^t}{u}\,\frac{1}{1-\frac{\lambda}{u}(ab)^t}\\
&= \frac{-a^t}{u}\sum_{n=0}^{\infty}\left[\frac{\lambda}{u}(ab)^t\right]^n\\
&= \frac{-e^{t\,\log a}}{u}\;\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n e^{nt\,\log ab}\\
&= \frac{-1}{u}\left(\sum_{n=0}^{\infty}(\log a)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n\sum_{m=0}^{\infty}n^m(\log ab)^m\,\frac{t^m}{m!}\right)\\
&= \frac{-1}{u}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}(\log a)^j\frac{t^j}{j!}\;\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n\;n^{m-j}\,(\log ab)^{m-j}\,\frac{t^{(m-j)}}{(m-j)!}\right)\\
&= \frac{-1}{u}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\;\sum_{n=0}^{\infty}\binom{m}{j}\left(\frac{\lambda}{u}\right)^n\;(\log a)^j\,(n\log ab)^{m-j}\right)\frac{t^m}{m!}\\
&= \sum_{m=0}^{\infty}\left(-\frac{1}{u}\sum_{j=0}^{m}\;\sum_{n=0}^{\infty}\binom{m}{j}\left(\frac{\lambda}{u}\right)^n\;(\log a)^j\,(n\log ab)^{m-j}\right)\frac{t^m}{m!}
\end{aligned}$$

$$\begin{aligned}
\frac{e^x_\rho(t)\;e^y_\rho(t^2)}{\lambda b^t - ua^{-t}} &= \sum_{m=0}^{\infty}\sum_{i=0}^{m}\sum_{s}^{i} i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}\frac{t^i}{i!}\left(-\frac{1}{u}\right)\sum_{j=0}^{m-i}\sum_{n=0}^{\infty}\binom{m-i}{j}\\
&\qquad\left(\frac{\lambda}{u}\right)^n\;(\log a)^j\,(n\log ab)^{m-j}\,\frac{t^{m-j}}{(m-j)!}
\end{aligned}$$

$$\begin{aligned}
\frac{e^x_\rho(t)e^y_\rho(t^2)}{\lambda b^t - ua^{-t}} &= \sum_{m=0}^{\infty}\left\{\sum_{i=0}^{m}\binom{m}{i}\sum_{s=0}^{i}\sum_{j=0}^{m-i}\sum_{n=0}^{\infty}i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}\right.\\
&\qquad\left.\left(\frac{-1}{u}\right)\binom{m-i}{j}\times\left(\frac{\lambda}{u}\right)^n(\log a)^j\,(n\log ab)^{m-i-j}\right\}\frac{t^m}{m!}
\end{aligned}$$

$$\frac{e^x_\rho(t)}{\lambda b^t - ua^{-t}} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{\infty}\binom{m}{j}(x)_{j,\rho}\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-j}\frac{t^m}{m!}.$$

With

$$\begin{aligned}
&\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(m,\rho-1)\\
&\quad= ((1-u)\ln ab)^{m+1}\sum_{m_1+m_2+\ldots+m_{k-1}=m}\binom{m}{m_1,\ldots,m_{k-1}}\\
&\qquad\times\frac{b_{m_1,\rho}(\rho-1)}{m_1+1}\frac{b_{m_2,\rho}(\rho-1)}{m_1+m_2+1}\cdots\frac{b_{m_{k-1},\rho}(\rho-1)}{m_1+\ldots+m_{k-1}+1},
\end{aligned}\tag{29}$$

we have

$$\sum_{m=0}^{\infty}\widehat{\mathcal{G}}_m^{(k)}(x,y;\lambda,\rho,u,a,b)\frac{t^m}{m!}=\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}e_\rho^x(t)e_\rho^y(t^2)$$

$$=t\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)\Big\{\sum_{i=0}^{m-l}\binom{m-l}{i}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\sum_{n=0}^{\infty}i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}$$
$$\left(\frac{-1}{u}\right)\binom{m-i}{j}\times\left(\frac{\lambda}{u}\right)^n(\log a)^j\,(n\log ab)^{m-i-j}\Big\}\frac{t^m}{m!}$$
$$=t\sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\sum_{n=0}^{\infty}\binom{m}{l}\binom{m-l}{i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}$$
$$\left(\frac{-1}{u}\right)\binom{m-i}{j}\left(\frac{\lambda}{u}\right)^n(\log a)^j\,(n\log ab)^{m-i-j}\,\frac{t^m}{m!}$$

Note that the right-hand side of the preceding equation has no constant term. Hence, when $m=0$, $\widehat{\mathcal{G}}_0^{(k)}(x,y;\lambda,\rho,u,a,b)=0$. Moreover,

$$\sum_{m=0}^{\infty}\frac{1}{m}\widehat{\mathcal{G}}_m^{(k)}(x,y;\lambda,\rho,u,a,b)\frac{t^{m-1}}{(m-1)!}$$
$$=\sum_{m=0}^{\infty}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\sum_{n=0}^{\infty}\binom{m}{l}\binom{m-l}{i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}$$
$$\left(\frac{-1}{u}\right)\binom{m-i}{j}\left(\frac{\lambda}{u}\right)^n(\log a)^j\,(n\log ab)^{m-i-j}\,\frac{t^m}{m!}.$$

By comparing the coefficients of $\frac{t^m}{m!}$, we obtain the following explicit formula:

$$\frac{1}{m+1}\widehat{\mathcal{G}}_{m+1}^{(k)}(x,y;\lambda,\rho,u,a,b)$$
$$=\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\sum_{n=0}^{\infty}\binom{m}{l}\binom{m-l}{i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}$$
$$\left(\frac{-1}{u}\right)\binom{m-i}{j}\left(\frac{\lambda}{u}\right)^n(\log a)^j\,(n\log ab)^{m-i-j}\,.$$

Furthermore, using the arithmetic-geometric series formula in [53, p.245], we can write

$$\sum_{n=0}^{\infty}\left(\frac{u}{\lambda}\right)^n n^{m-i-j}=\frac{A_{m-i-j}\left(\frac{u}{\lambda}\right)}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}},$$

where $A_n(u)$ is the Eulerian polynomial

$$A_n(u)=\sum_{k=0}^{n}A(n,k)u^k \tag{30}$$

with $A(n,k)$, the Eulerian number, satisfying

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k).$$

Thus,

$$\begin{aligned}&\frac{1}{m+1}\widehat{\mathcal{G}}^{(k)}_{m+1}(x,y;\lambda,\rho,u,a,b)\\&=\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\binom{m}{l}\binom{m-l}{i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)A_{m-i-j}\left(\frac{\lambda}{u}\right)\\&\qquad i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}\left(\frac{-1}{u}\right)\binom{m-i}{j}\frac{(\log a)^j\,(\log ab)^{m-i-j}}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}}.\end{aligned}$$

To state formally this result, we have the following theorem.

Theorem 2.2. *The type 2 degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b are equal to*

$$\begin{aligned}&\frac{1}{m+1}\widehat{\mathcal{G}}^{(k)}_{m+1}(x,y;\lambda,\rho,u,a,b)\\&=\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{s=0}^{i}\sum_{j=0}^{m-l-i}\binom{m}{l}\binom{m-l}{i}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(l,\rho-1)A_{m-i-j}\left(\frac{\lambda}{u}\right)\\&\qquad i!\rho^{i-\frac{s}{2}}\binom{\frac{x}{\rho}}{i-s}\binom{\frac{y}{\rho}}{\frac{s}{2}}\left(\frac{-1}{u}\right)\binom{m-i}{j}\frac{(\log a)^j\,(\log ab)^{m-i-j}}{\left(1-\frac{u}{\lambda}\right)^{m-i-j+1}},\end{aligned}$$

where $\widehat{\mathcal{G}}^{(k,\alpha)}_0(x,y;\lambda,\rho,u,a,b)=0$, $A_n(u)$ is the Eulerian polynomial and $\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)$ satisfies (29).

Now, let us extend the explicit formula to higher order degenerate Apostol-Frobenius-type poly-Genocchi polynomials with parameters a and b. First, we have to find the expansion of the following function:

$$\begin{aligned}&\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\\&=t\sum_{m=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{n=0}^{\infty}\frac{1}{m!}\binom{m}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^n(-n\log ab)^{m-j}\right\}t^m.\end{aligned}$$

Raising this to power α gives

$$\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^\alpha$$

$$= t^{\alpha}\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_\alpha=m}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\right.$$
$$\left.\left(\frac{u}{\lambda}\right)^{n}(-n\log ab)^{k_i-j}\right\}t^m.$$

Hence,

$$\sum_{m=0}^{\infty}\widehat{\mathcal{G}}_m^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\frac{t^m}{m!}$$
$$=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^x(t)e_\rho^y(t^2)$$
$$=t^{\alpha}\sum_{m=0}^{\infty}\sum_{q=0}^{m}\binom{m}{q}\sum_{j=0}^{m}q!\,\rho^{q-\frac{j}{2}}\binom{\frac{x}{\rho}}{q-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}(m-q)!$$
$$\times\sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\right.$$
$$\left.\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^{n}(-n\log ab)^{k_i-j}\right\}\frac{t^m}{m!}.$$

Note that, when $0\le m\le\alpha-1$, $\widehat{\mathcal{G}}_m^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)=0$. Now, we can further rewrite the preceding equation as follows:

$$\sum_{m=-\alpha}^{\infty}\widehat{\mathcal{G}}_{m+\alpha}^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\frac{t^m}{(m+\alpha)!}$$
$$=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^x(t)e_\rho^y(t^2)$$
$$=\sum_{m=0}^{\infty}\sum_{q=0}^{m}\sum_{j=0}^{m}\binom{m}{q}q!\,\rho^{q-\frac{j}{2}}\binom{\frac{x}{\rho}}{q-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}(m-q)!$$
$$\times\sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\sum_{n=0}^{\infty}\frac{1}{k_i!}\binom{k_i}{j}\right.$$
$$\left.\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)\left(\frac{u}{\lambda}\right)^{n}(-n\log ab)^{k_i-j}\right\}\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ and using the arithmetic-geometric formula yield the following explicit formula.

Theorem 2.3. *The type 2 degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b are equal to*

$$\frac{1}{(m+\alpha)_\alpha}\widehat{\mathcal{G}}_{m+\alpha}^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$$

$$= \sum_{q=0}^{m}\sum_{j=0}^{m}\binom{m}{q}q!\,\rho^{q-\frac{j}{2}}\binom{\frac{x}{\rho}}{q-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}(m-q)!$$
$$\times \sum_{k_1+k_2+\ldots+k_\alpha=m-q}\prod_{i=1}^{\alpha}\left\{\sum_{j=0}^{k_i}\frac{1}{k_i!}\binom{k_i}{j}\right.$$
$$\left.\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(j,\rho-1)A_{k_i-j}\left(\frac{u}{\lambda}\right)\frac{(-\log ab)^{k_i-j}}{(1-\frac{u}{\lambda})^{k_i-j+1}}\right\}.$$

where $\widehat{\mathcal{G}}_m^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)=0$ *for* $m=0,1,\ldots,\alpha-1$, $A_n(u)$ *is the Eulerian polynomial defined in* (30) *and* $\mathcal{B}_{m_1,m_2,\ldots,m_{k-1}}(i,\rho-1)$ *satisfies* (29).

Remark 2.4. It can easily be seen that, when $\alpha=1$, the explicit formula in Theorem 2.3 reduces to that in Theorem 2.2.

3. Relation With Some Genocchi-type Polynomials

By giving special values to the parameters involved, $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$ reduces to some interesting Genocchi-type polynomials.

(i) Using (23), when $k=1$, (27) yields

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(\alpha)}(x,y;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\left(\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}\right)^{\alpha}e_\rho^x(t)e_\rho^y(t^2), \tag{31}$$

where the polynomials $\widehat{\mathcal{G}}_n^{(\alpha)}(x,y;\lambda,\rho,u,a,b)=\widehat{\mathcal{G}}_n^{(1,\alpha)}(x;\lambda,\rho,u,a,b)$ are called the degenerate Hermite-based Apostol-Frobenius-type Genocchi polynomials of higher order with parameters a, b and c. When $\alpha=1$, (31) yields

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(1)}(x,y;\lambda,\rho,u,a,b)\frac{t^n}{n!}=\frac{(1-u)t\ln ab}{\lambda b^t-ua^{-t}}e_\rho^x(t)e_\rho^y(t^2), \tag{32}$$

where the polynomials $\widehat{\mathcal{G}}_n^{(1)}(x,y;\lambda,\rho,u,a,b)$ are called the degenerate Hermite-based Apostol-Frobenius-type Genocchi polynomials with parameters a and b.

(ii) When $x=y=0$, equation (27) reduces to

$$\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^n}{n!}=\left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t-ua^{-t}}\right)^{\alpha}, \tag{33}$$

the type 2 degenerate Apostol-Frobenius-type poly-Genocchi numbers with parameters a and b.

(iii) When $a = 1, b = e$, (27) will reduce to

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x, y; \lambda, \rho, u)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1 + (1 - u)t))}{\lambda e^t - u}\right)^\alpha e_\rho^x(t)e_\rho^y(t^2), \tag{34}$$

and call $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x, y; \lambda, \rho, u)$, the type 2 degenerate Apostol-Frobenius-type poly-Genocchi polynomials of higher order. When $x = y = 0$, we get

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda, \rho, u)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1 + (1 - u)t)}{\lambda e^t - u}\right)^\alpha, \tag{35}$$

the type 2 degenerate Apostol-Frobenius-type Genocchi numbers of higher order.

(iv) When $\rho \to 0$, equation (27) reduces to

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x, y; \lambda, 0, u, a, b)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,0}(\log_0(1 + (1 - u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^\alpha e_0^x(t)e_0^y(t^2)$$

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x, y; \lambda, u, a, b)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k}(\log(1 + (1 - u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^\alpha e^{xt+yt^2}, \tag{36}$$

where the polynomials $\mathcal{G}_n^{(k,\alpha)}(x, y; \lambda, u, a, b)$ are the type 2 Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a, b and c with $c = e$ in [29].

(v) When $\lambda = 1$, (34) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(k,\alpha)}(x, y; u, 1, e)\frac{t^n}{n!} = \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1 + (1 - u)t))}{e^t - u}\right)^\alpha e^{xt+yt^2}. \tag{37}$$

which is the higher order version of equation (8) and are called the higher order type 2 Hermite-based poly-Genocchi polynomials. We may use $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x; u)$ to denote $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x; u, 1, e)$.

(vi) When $k = 1$, $a = 1$ and $b = e$, (36) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(1,\alpha)}(x, y; \lambda, u)\frac{t^n}{n!} = \left(\frac{(1 - u)t}{\lambda e^t - u}\right)^\alpha e^{xt+yt^2}, \tag{38}$$

and when $\lambda = 1$, (38) gives

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n^{(1,\alpha)}(x, y; 1, u)\frac{t^n}{n!} = \left(\frac{(1 - u)t}{e^t - u}\right)^\alpha e^{xt+yt^2},$$

where $\widehat{\mathcal{G}}_n^{(1,\alpha)}(x, y; \lambda, u) = \widehat{\mathcal{G}}_n^{(\alpha)}(x, y; \lambda, u)$ and $\widehat{\mathcal{G}}_n^{(1,\alpha)}(x; 1, u) = \widehat{\mathcal{G}}_n^{(\alpha)}(x; u)$ are called the degenerate Hermite-based Apostol-Frobenius-type Genocchi polynomials and

Hermite-based Frobenius-Genocchi polynomials of higher order in (5) and (3), respectively. Furthermore, when $\alpha = 1$, we have

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n(x;\lambda,u)\frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u} e^{xt+yt^2}, \tag{39}$$

and

$$\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_n(x,y;u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t - u} e^{xt+yt^2},$$

where $\widehat{\mathcal{G}}_n(x;\lambda,u)$ and $\widehat{\mathcal{G}}_n(x;u)$ are called the Hermite-based Apostol-Frobenius-type Genocchi polynomials and Hermite-based Frobenius-Genocchi polynomials

Now, let us consider some some relations of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$ with other Genocchi-type polynomials. First is to establish a kind of addition formula for $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$ expressing them as polynomials in x.

Theorem 3.1. *The type 2 degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b satisfy the relation*

$$\begin{aligned}&\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\\ &\quad= \sum_{m=0}^{n}\sum_{j=0}^{m}\binom{n}{m} m!\,\rho^{m-\frac{j}{2}}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\binom{\frac{x}{\rho}}{m-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}.\end{aligned} \tag{40}$$

Moreover, the expression of $\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)$ as polynomial in x and y is given by

$$\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b) = \sum_{i=0}^{n}\sum_{l=0}^{i}\widehat{\mathcal{G}}_{n,i,l,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)x^{i-l}y^l, \tag{41}$$

where

$$\begin{aligned}\widehat{\mathcal{G}}_{n,i,l,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b) &= \sum_{m=i}^{n}\sum_{\substack{j=0\\ j\ is\ even}}^{m}\binom{n}{m}\frac{m!}{(m-j)!(j/2)!}\\ &\quad\times \widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l).\end{aligned}$$

Proof. Using (33), we can write (27) as follows:

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(x;\lambda,\rho,u,a,b)\frac{t^n}{n!}\\ &\quad= \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha} e_\rho^x(t)e_\rho^y(t^2)\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty}\widehat{\mathcal{G}}_n^{(k,\alpha)}(\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n} n!\,\rho^{n-\frac{j}{2}}\binom{\frac{x}{\rho}}{n-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}\right\}\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\left\{\sum_{j=0}^{m} m!\,\rho^{m-\frac{j}{2}}\binom{\frac{x}{\rho}}{m-j}\binom{\frac{y}{\rho}}{\frac{j}{2}}\right\}\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (46). To prove (41), we first recall that the r-Whitney numbers of the first kind, denoted by $w_{m,r}(n,k)$ were defined by Mező [54] by means of the following horizontal generating function:

$$m^n(x)_n = \sum_{j=0}^{n} w_{m,r}(n,j)(mx+r)^j, \tag{42}$$

where $(x)_n = x(x-1)(x-2)\dots(x-n+1)$. Replacing x with x/m and letting $r=0$ yield

$$(x)_{n,m} = \sum_{j=0}^{n}\widetilde{w}_m(n,j)x^j,$$

where $\widetilde{w}_m(n,j) = w_{m,0}(n,j)$, a certain of generalization of Stirling numbers of the first kind, i.e. $S_1(n,j) = \widetilde{w}_0(n,j)$. Using (42), equation (46) can further be written as polynomial in x

$$\begin{aligned}&\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\\ &\quad= \sum_{m=0}^{n}\sum_{\substack{j=0\\ j\ is\ even}}^{m}\binom{n}{m}\frac{m!}{(m-j)!(j/2)!}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)(x)_{m-j,\rho}(y)_{j/2,\rho}\end{aligned} \tag{43}$$

$$(x)_{m-j,\rho}(y)_{j/2,\rho} = \left(\sum_{i=0}^{\infty}\widetilde{w}_\rho(m-j,i)x^i\right)\left(\sum_{i=0}^{\infty}\widetilde{w}_\rho(j/2,i)y^i\right) \tag{44}$$
$$= \sum_{i=0}^{\infty}\sum_{l=0}^{i}\widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l)x^{i-l}y^l. \tag{45}$$

$$\begin{aligned}&\widehat{\mathcal{G}}_n^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b)\\ &\quad= \sum_{m=0}^{n}\sum_{\substack{j=0\\ j\ is\ even}}^{m}\binom{n}{m}\frac{m!}{(m-j)!(j/2)!}\widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\\ &\quad\times\sum_{i=0}^{m}\sum_{l=0}^{i}\widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l)x^{i-l}y^l\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \sum_{i=0}^{m} \sum_{l=0}^{i} \sum_{\substack{j=0 \\ j \text{ is even}}}^{m} \binom{n}{m} \frac{m!}{(m-j)!(j/2)!} \widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b) \\
&\quad \times \widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l)x^{i-l}y^l \\
&= \sum_{i=0}^{n} \sum_{l=0}^{i} \sum_{m=i}^{n} \sum_{\substack{j=0 \\ j \text{ is even}}}^{m} \binom{n}{m} \frac{m!}{(m-j)!(j/2)!} \widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b) \\
&\quad \times \widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l)x^{i-l}y^l
\end{aligned}
$$

$$
\widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x,y;\lambda,\rho,u,a,b) = \sum_{i=0}^{n} \sum_{l=0}^{i} \widehat{\mathcal{G}}_{n,i,l,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b)x^{i-l}y^l
$$

with coefficients

$$
\begin{aligned}
\widehat{\mathcal{G}}_{n,i,l,\widetilde{w}}^{(k,\alpha)}(\lambda,\rho,u,a,b) &= \sum_{m=i}^{n} \sum_{\substack{j=0 \\ j \text{ is even}}}^{m} \binom{n}{m} \frac{m!}{(m-j)!(j/2)!} \\
&\quad \times \widehat{\mathcal{G}}_{n-m}^{(k,\alpha)}(\lambda,\rho,u,a,b)\widetilde{w}_\rho(m-j,i-l)\widetilde{w}_\rho(j/2,l)
\end{aligned}
$$

the convolution of $\widehat{\mathcal{G}}_{n}^{(k,\alpha)}(\lambda,\rho,u,a,b)$ and $\widetilde{w}_\rho(n,k)$.

The next result is a kind of addition formula for $\widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x;\lambda,\rho,u,a,b)$.

Theorem 3.2. *The type 2 degenerate Hermite-based Apostol-Frobenius-type poly-Genocchi polynomials of higher order with parameters a and b satisfy the relation*

$$
\begin{aligned}
&\widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x_1+x_2,y_1+y_2;\lambda,\rho,u,a,b) \\
&\quad = \sum_{q=0}^{n} \sum_{\substack{j=0 \\ j \text{ is even}}}^{n-q} \frac{(n-q)!\widehat{\mathcal{G}}_{q}^{(k,\alpha)}(x_1,y_1;\lambda,\rho,u,a,b)}{(n-q-j)!(j/2)!}(x_2)_{n-q,\rho}(y_2)_{j/2,\rho}.
\end{aligned} \tag{46}
$$

Proof. We can write (27) as follows:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x_1+x_2,y_1+y_2;\lambda,\rho,u,a,b)\frac{t^n}{n!} \\
&= \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha} e_\rho^{x_1+x_2}(t)e_\rho^{y_1+y_2}(t^2) \\
&= \left(\frac{\mathrm{Ei}_{k,\rho}(\log_\rho(1+(1-u)t\ln ab))}{\lambda b^t - ua^{-t}}\right)^{\alpha} e_\rho^{x_1}(t)e_\rho^{y_1}(t^2)e_\rho^{x_2}(t)e_\rho^{y_2}(t^2) \\
&= \left(\sum_{n=0}^{\infty} \widehat{\mathcal{G}}_{n}^{(k,\alpha)}(x_1,y_1;\lambda,\rho,u,a,b)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} n!\,\rho^{n-\frac{j}{2}}\binom{\frac{x_2}{\rho}}{n-j}\binom{\frac{y_2}{\rho}}{\frac{j}{2}}\right)\frac{t^n}{n!}\right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}\sum_{j=0}^{n-q}(n-q)!\,\rho^{n-q-\frac{j}{2}}\binom{\frac{x_2}{\rho}}{n-q-j}\binom{\frac{y_2}{\rho}}{\frac{j}{2}}\widehat{\mathcal{G}}_q^{(k,\alpha)}(x_1,y_1;\lambda,\rho,u,a,b)\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem.

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