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Pythagorean Neutrosophic IUP-Algebras: Theoretical Foundations and Extensions

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Abstract. This paper introduces the concepts of Pythagorean neutrosophic IUP-subalgebras, Pythagorean neutrosophic IUP-ideals, Pythagorean neutrosophic IUP-filters, and Pythagorean neutrosophic strong IUP-ideals within the framework of IUP-algebras. We establish the fundamental properties of these structures and provide necessary and sufficient conditions under which a Pythagorean neutrosophic set qualifies as one of these algebraic subsets. Additionally, we explore the relationships between these subsets and their corresponding level subsets, offering a deeper understanding of their interconnections. By extending the theoretical foundation of IUP-algebras through the integration of Pythagorean neutrosophic sets, this study contributes to the broader field of algebraic structures and uncertainty modeling. Beyond its theoretical significance, this work promotes inclusive and equitable education by making abstract mathematical concepts more accessible to students, educators, and researchers at both the school and university levels. By fostering mathematical literacy and critical thinking, this research equips learners with essential problem-solving skills applicable to various fields, including mathematics, computer science, and artificial intelligence. To enhance accessibility, key concepts are presented in a structured and comprehensible manner, allowing students from diverse backgrounds to engage with complex algebraic theories more effectively. This study fosters lifelong learning and innovation by strengthening the integration of mathematical education across different learning environments.

2020 Mathematics Subject Classifications: 03G25, 03E72, 08A72
Key Words and Phrases: IUP-algebra, Pythagorean neutrosophic IUP-subalgebra, Pythagorean neutrosophic IUP-ideal, Pythagorean neutrosophic IUP-filter, and Pythagorean neutrosophic strong IUP-ideal

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1. Introduction

In 1965, Zadeh [1] defined the concept of fuzzy sets (FSs), an important idea that has been extensively built upon by many. Later, in 1986, Atanasov [2] defined the concept of intuitionistic fuzzy sets (IFSs), which is a generalization of fuzzy sets (FSs). Subsequently, in 2004, Smarandache defined the concept of neutrosophic sets, a generalization of intuitionistic fuzzy sets (IFSs). Subsequently, in 2004, Smarandache [3] introduced the concept of neutrosophic sets (NSs), which is a generalization of intuitionistic fuzzy sets (IFSs). Afterwards, in 2013, Yager [4] defined the concept of Pythagorean fuzzy subsets (PFSs), an extension of fuzzy sets. Later, in 2019, Jansi et al. [5] defined a highly significant and interesting concept known as the Pythagorean neutrosophic sets (PNSs). Since the idea of Pythagorean neutrosophic sets was defined, many researchers have continuously studied and expanded upon this idea. In 2019, Jansi et al. [5] introduced the notion of exploring the new concept of Pythagorean neutrosophic sets (PNSs) with T and F as dependent neutrosophic components. They introduced PNSs as a generalization of neutrosophic sets and Pythagorean fuzzy sets. They investigated the basic operations of PNSs and proposed a correlation measure for PNSs, proving some of their fundamental properties. They extended the concept of correlation measures from Pythagorean fuzzy sets and neutrosophic sets. Finally, they applied the correlation measure of PNSs to medical diagnosis. In 2021, Satirad et al. [6] extended the framework of UP-algebras by incorporating Pythagorean fuzzy sets to address uncertainty in algebraic reasoning. They define lower and upper approximations of Pythagorean fuzzy sets within UP-algebras and explore their structural properties. The study enriches fuzzy algebraic theory by offering a more flexible approximation model for managing vague or partial information. In 2023, Ismail et al. [7] put forth a set of algebraic operations that could be applied to PNSs. These operations included addition, multiplication, scalar multiplication, and power. These operations facilitated the efficient manipulation and combination of PNSs, thereby enhancing decision-making in scenarios characterized by uncertainty and vagueness. To demonstrate the efficacy of these operations, they presented several illustrative examples accompanied by corroborating proofs. The introduction of algebraic operations enhanced the capabilities of PNSs, thereby creating opportunities for their practical application. In 2024, Razak et al. [8] propose the Interval Valued Pythagorean Neutrosophic Set (IVPNS) to better capture uncertainty and imprecision in real-world data. They define key algebraic operations and validate their properties, offering a comparative analysis with related set models. This framework enhances the robustness of decision-making and modeling under complex uncertainty.

In 2022, Iampan et al. [9] defined a new concept called IUP-algebras, which is an algebraic structure with four special subsets: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. They also discovered additional properties, which have inspired many researchers to study and expand this knowledge further. This concept has sparked inspiration among mathematicians to explore this algebraic structure more deeply, such as in 2023, when Chanmanee et al. [10] introduced the concept of the direct product of infinite families of IUP-algebras. Their research unveiled the notion of weak direct products and presented pivotal findings regarding (anti-)IUP-homomorphisms within this context. These contributions significantly deepened the structural comprehension of IUP-algebras and established essential tools for further exploration of their properties. In 2024, Kuntama et al. [11] made substantial advancements by integrating FS theory into IUP-algebras. They introduced fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals, meticulously examining the properties and interactions of these subsets. Suayngam et al. [12] applied Fermatean fuzzy sets to IUP-algebras, focusing on Fermatean fuzzy IUP-subalgebras, IUP-ideals, IUP-filters, and strong IUP-ideals. They examined their properties, including characteristic Fermatean fuzzy sets and upper and lower t-(strong) level subsets, offering deeper insights into their structural relationships. This research expanded the applicability of IUP-algebras, offering fresh perspectives and mathematical tools that bridge algebraic structures with fuzzy logic. Further enriching this theoretical framework, Suayngam et al. [13] introduced the concept of intuitionistic fuzzy IUP-algebras in 2024. Their work amalgamated IFS theory with IUP-algebras, leading to the development of intuitionistic fuzzy IUP-subalgebras, ideals, filters, and strong ideals. In 2025, Suayngam et al. [14] introduced the notions of neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals of IUP-algebras and investigated their basic properties. They provided conditions for neutrosophic sets to be neutrosophic IUP-subalgebras, neutrosophic IUP-ideals, neutrosophic IUP-filters, and neutrosophic strong IUP-ideals of IUP-algebras. They considered relations between neutrosophic IUP-subalgebras (resp., neutrosophic IUP-ideals, neutrosophic IUP-filters, neutrosophic strong IUP-ideals) and their level subsets. Suayngam et al. [15] applied the concept of Pythagorean fuzzy sets to IUP-algebras and introduced the notions of Pythagorean fuzzy IUP-subalgebras, Pythagorean fuzzy IUP-ideals, Pythagorean fuzzy IUP-filters, and Pythagorean fuzzy strong IUP-ideals. They investigated their properties, including the characteristic Pythagorean fuzzy sets, the upper t-(strong) level subsets, and the lower t-(strong) level subsets of the Pythagorean fuzzy set. Suayngam et al. [16] advanced the study of IUP-algebras by incorporating intuitionistic neutrosophic sets to formalize algebraic reasoning under uncertainty. They introduced and characterized key substructures—such as IUP-subalgebras, ideals, filters, and strong ideals—establishing necessary and sufficient conditions within the INS framework. This work laid important theoretical foundations for extending IUP-algebras to contexts involving indeterminate or imprecise data. This innovative approach enhanced the study of IUP-algebras, presenting new hybrid structures with the potential to inspire a wide array of applications and future research.

From the literature review, it has been found that since the concept of Pythagorean neutrosophic sets (PNSs) was defined, many researchers have begun and continued to study this concept. Many researchers have studied IUP-algebras, which are highly interesting algebraic structures. Our researchers are therefore interested in applying the concept of PNSs to IUP-algebras. Thus, we will study the application of PNSs to the subsets of IUP-algebras, including IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals, and investigate their properties and relationships. We will examine the relationships between PNSs and these subsets.

The content of this paper is divided into four sections. The first section explains the related research and the inspiration for this paper. The second section introduces the definitions of PNSs, providing examples and key properties. Additionally, we will review the definitions of IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals, showing their relationships. The third section reviews the definitions of PNSs, including the concepts of Pythagorean neutrosophic IUP-subalgebras, Pythagorean neutrosophic IUP-filters, Pythagorean neutrosophic IUP-ideals, and Pythagorean neutrosophic strong IUP-ideals, with examples and explanations. The fourth section examines the characteristic functions for these concepts, finds general conclusions, and shows the relationships between characteristic functions, level subsets, and their PNSs. The final section summarizes the research findings and suggests further studies and expansions of this research.

2. Preliminaries

The study of algebraic structures has evolved significantly to accommodate the complexities of uncertainty and imprecision in mathematical modeling. Among these structures, IUP-algebras have been recognized for their unique properties, making them a valuable tool in various mathematical and computational applications. Simultaneously, the development of neutrosophic sets and their extensions, such as Pythagorean neutrosophic sets (PNSs), has provided a more refined approach to handling degrees of truth, indeterminacy, and falsity. The combination of these two frameworks—IUP-algebras and Pythagorean neutrosophic sets—offers a novel perspective on algebraic systems under uncertainty.

Before delving into the main results of this study, it is essential to establish a strong foundation by reviewing the fundamental concepts that underpin our research. This section introduces key definitions and properties of IUP-algebras and Pythagorean neutrosophic sets, ensuring clarity in their mathematical formulation. By revisiting these preliminary concepts, we aim to provide a comprehensive background that facilitates a deeper understanding of how Pythagorean neutrosophic structures can be integrated into IUP-algebras. These insights will serve as the basis for the subsequent theoretical developments presented in this study.

Definition 1. *[9] An algebra $X = (X, \star, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $\star$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 \star x = x) \tag{IUP-1}$$

$$(\forall x \in X)(x \star x = 0) \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x \star y) \star (x \star z) = y \star z) \tag{IUP-3}$$

For simplicity, we will refer to X as the IUP-algebra $X = (X, \star, 0)$ unless stated otherwise.

Example 1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\star$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	3	0	5	1	2	4
2	5	2	0	4	1	3
3	1	3	4	0	5	2
4	4	5	3	2	0	1
5	2	4	1	5	3	0

then $X = (X, \star, 0)$ is an IUP-algebra.

Example 2. *[9] Let $(G, \bullet, e)$ be a group with the identity element e in which each element is its own inverse. Under this condition, $(G, \bullet, e)$ inherently satisfies the axioms of an IUP-algebra.*

Example 3. *[9] Let X be a set, and let $\mathcal{P}(X)$ denote its power set. As shown in Example 2, $(\mathcal{P}(X), \triangle, \emptyset)$ forms an IUP-algebra, where $\triangle$ represents the symmetric difference between sets.*

Example 4. *[9] Let $(G, \cdot, e)$ be a group with the identity element e. Define a binary operation $\bullet$ on G by:*

$$(\forall x, y \in G)(x \bullet y = y \cdot x^{-1}) \tag{2.1}$$

Then $(G, \bullet, e)$ is an IUP-algebra.

Proposition 1. *[9] In an IUP-algebra $X = (X, \star, 0)$, the following assertions are valid:*

$$(\forall x, y \in X)((x \star 0) \star (x \star y) = y) \tag{2.2}$$
$$(\forall x \in X)((x \star 0) \star (x \star 0) = 0) \tag{2.3}$$
$$(\forall x, y \in X)((x \star y) \star 0 = y \star x) \tag{2.4}$$
$$(\forall x \in X)((x \star 0) \star 0 = x) \tag{2.5}$$
$$(\forall x, y \in X)(x \star ((x \star 0) \star y) = y) \tag{2.6}$$
$$(\forall x, y \in X)(((x \star 0) \star y) \star x = y \star 0) \tag{2.7}$$
$$(\forall x, y, z \in X)(x \star y = x \star z \Leftrightarrow y = z) \tag{2.8}$$
$$(\forall x, y \in X)(x \star y = 0 \Leftrightarrow x = y) \tag{2.9}$$
$$(\forall x \in X)(x \star 0 = 0 \Leftrightarrow x = 0) \tag{2.10}$$
$$(\forall x, y, z \in X)(y \star x = z \star x \Leftrightarrow y = z) \tag{2.11}$$
$$(\forall x, y \in X)(x \star y = y \Rightarrow x = 0) \tag{2.12}$$
$$(\forall x, y, z \in X)((x \star y) \star 0 = (z \star y) \star (z \star x)) \tag{2.13}$$
$$(\forall x, y, z \in X)(x \star y = 0 \Leftrightarrow (z \star x) \star (z \star y) = 0) \tag{2.14}$$
$$(\forall x, y, z \in X)(x \star y = 0 \Leftrightarrow (x \star z) \star (y \star z) = 0) \tag{2.15}$$

the right and the left cancellation laws hold (2.16)

Within IUP-algebras, four fundamental subsets stand out: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets form a critical framework that deepens our understanding and facilitates the application of IUP-algebras across different mathematical contexts.

Definition 2. *[9] A nonempty subset S of X is called*

(i) *an IUP-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x \star y \in S) \tag{2.17}$$

(ii) *an IUP-filter of X if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S \tag{2.18}$$

$$(\forall x, y \in X)(x \star y \in S, x \in S \Rightarrow y \in S) \tag{2.19}$$

(iii) *an IUP-ideal of X if it satisfies the condition* (2.18) *and the following condition:*

$$(\forall x, y, z \in X)(x \star (y \star z) \in S, y \in S \Rightarrow x \star z \in S) \tag{2.20}$$

(iv) *a strong IUP-ideal of X if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \star y \in S) \tag{2.21}$$

According to [9], IUP-filters represent a unifying concept encompassing both IUP-ideals and IUP-subalgebras. These two subsets, IUP-ideals and IUP-subalgebras, are generalizations of strong IUP-ideals. Particularly, in an IUP-algebra X, strong IUP-ideals are equivalent to the entire algebra X itself. This hierarchical relationship among these subsets is visually represented in Figure 1, illustrating the structure of special subsets within IUP-algebras.

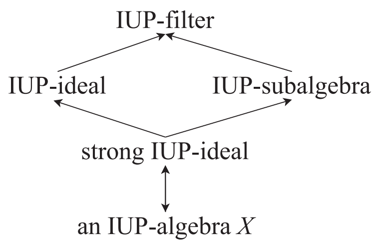

Figure 1: Special subsets of IUP-algebras

3. Main results

The study of algebraic structures has continually evolved to incorporate concepts that better represent uncertainty and imprecision in mathematical modeling. Among these structures, IUP-algebras have gained attention due to their unique operational properties and applicability in logic and computational mathematics. At the same time, Pythagorean neutrosophic sets (PNSs) have emerged as a powerful extension of fuzzy and neutrosophic sets, offering enhanced flexibility in handling degrees of truth, indeterminacy, and falsity. Given their potential, integrating Pythagorean neutrosophic sets into IUP-algebras presents an opportunity to develop a more robust algebraic framework capable of capturing a wider range of uncertain scenarios.

This section formalizes the key notions of Pythagorean neutrosophic IUP-subalgebras, Pythagorean neutrosophic IUP-ideals, Pythagorean neutrosophic IUP-filters, and Pythagorean neutrosophic strong IUP-ideals and explores their fundamental properties. By establishing essential definitions, theorems, and proofs, we aim to provide a structured foundation for understanding how Pythagorean neutrosophic sets interact with IUP-algebraic operations. Furthermore, we analyze the interrelations between these newly defined subsets and their respective level subsets, revealing deeper insights into their structural coherence. The results obtained not only extend the theoretical framework of IUP-algebras but also open new avenues for their application in areas such as fuzzy logic systems, artificial intelligence, and decision analysis under uncertainty.

Definition 3. *[5] Let X be a nonempty set (universe). A Pythagorean neutrosophic set (PNS) with $\mathcal{P}_T$ and $\mathcal{P}_F$ are dependent neutrosophic components on X is an object of the form*

$$\mathcal{P} = \{(x, \mathcal{P}_T(x), \mathcal{P}_I(x), \mathcal{P}_F(x)) \mid x \in X\}, \tag{3.1}$$

where $\mathcal{P}_T(x), \mathcal{P}_I(x), \mathcal{P}_F(x) \in [0,1]$, and $0 \leq (\mathcal{P}_T(x))^2 + (\mathcal{P}_I(x))^2 + (\mathcal{P}_F(x))^2 \leq 2$, for all x in X. $\mathcal{P}_T(x)$ is the degree of membership, $\mathcal{P}_I(x)$ is the degree of inderminancy, and $\mathcal{P}_F(x)$ is the degree of non-membership of the element x in the set $\mathcal{P}$. Here $\mathcal{P}_T(x)$ and $\mathcal{P}_F(x)$ are dependent components and $\mathcal{P}_I(x)$ is an independent component.

To streamline notation, we represent a PNS as $\mathcal{P} = (X, \mathcal{P}_T, \mathcal{P}_I, \mathcal{P}_F)$, where $\mathcal{P}$ is defined as $\{(x, \mathcal{P}_T(x), \mathcal{P}_I(x), \mathcal{P}_F(x)) \mid x \in X\}$.

Definition 4. *Let f be an FS in X and let n be a positive integer. The FS f_n defined by $f_n(x) = \frac{f(x)}{n}$ for all $x \in X$ is called the n-division of f in X.*

Definition 5. *Let $\mathcal{P}$ be a PNS in a nonempty set X and let n be a positive integer. The PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is called the n-division of $\mathcal{P}$ in X.*

Definition 6. *A PNS $\mathcal{P}$ in X is called a Pythagorean neutrosophic IUP-subalgebra of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\mathcal{P}_T(x \star y) \geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\}) \tag{3.2}$$

$$(\forall x, y \in X)(\mathcal{P}_I(x \star y) \leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\}) \tag{3.3}$$

$$(\forall x, y \in X)(\mathcal{P}_F(x \star y) \geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}) \tag{3.4}$$

Definition 7. *A PNS $\mathcal{P}$ in X is called a Pythagorean neutrosophic IUP-ideal of X if it satisfies the following properties:*

$$(\forall x \in X)(\mathcal{P}_T(0) \geq \mathcal{P}_T(x)) \tag{3.5}$$

$$(\forall x \in X)(\mathcal{P}_I(0) \leq \mathcal{P}_I(x)) \tag{3.6}$$

$$(\forall x \in X)(\mathcal{P}_F(0) \geq \mathcal{P}_F(x)) \tag{3.7}$$

$$(\forall x, y, z \in X)(\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}) \tag{3.8}$$

$$(\forall x, y, z \in X)(\mathcal{P}_I(x \star z) \leq \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}) \tag{3.9}$$

$$(\forall x, y, z \in X)(\mathcal{P}_F(x \star z) \geq \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}) \tag{3.10}$$

Definition 8. *A PNS $\mathcal{P}$ in X is called a Pythagorean neutrosophic IUP-filter of X if it satisfies* (3.5)*,* (3.6)*,* (3.7)*, and the following properties:*

$$(\forall x, y \in X)(\mathcal{P}_T(y) \geq \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}) \tag{3.11}$$

$$(\forall x, y \in X)(\mathcal{P}_I(y) \leq \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}) \tag{3.12}$$

$$(\forall x, y \in X)(\mathcal{P}_F(y) \geq \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}) \tag{3.13}$$

Definition 9. *A PNS $\mathcal{P}$ in X is called a Pythagorean neutrosophic strong IUP-ideal of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\mathcal{P}_T(x \star y) \geq \mathcal{P}_T(y)) \tag{3.14}$$

$$(\forall x, y \in X)(\mathcal{P}_I(x \star y) \leq \mathcal{P}_I(y)) \tag{3.15}$$

$$(\forall x, y \in X)(\mathcal{P}_F(x \star y) \geq \mathcal{P}_F(y)) \tag{3.16}$$

Lemma 1. *Every Pythagorean neutrosophic IUP-subalgebra of X satisfies* (3.5)*,* (3.6)*, and* (3.7)*.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned}
\mathcal{P}_T(0) &= \mathcal{P}_T(x \star x) && \text{(by (IUP-2))}\\
&\geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(x)\} && \text{(by (3.2))}\\
&= \mathcal{P}_T(x),\\
\mathcal{P}_I(0) &= \mathcal{P}_I(x \star x) && \text{(by (IUP-2))}\\
&\leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(x)\} && \text{(by (3.3))}\\
&= \mathcal{P}_I(x),\\
\mathcal{P}_F(0) &= \mathcal{P}_F(x \star x) && \text{(by (IUP-2))}\\
&\geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(x)\} && \text{(by (3.2))}\\
&= \mathcal{P}_F(x).
\end{aligned}$$

Hence, it satisfies (3.5), (3.6), and (3.7).

Theorem 1. *Every Pythagorean neutrosophic strong IUP-ideal of X satisfies* (3.5), (3.6), *and* (3.7).

Proof. Assume that Pythagorean neutrosophic strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned}
\mathcal{P}_T(0) &= \mathcal{P}_T(x \star x) && \text{(by (IUP-2))}\\
&\geq \mathcal{P}_T(x), && \text{(by (3.14))}\\
\mathcal{P}_I(0) &= \mathcal{P}_I(x \star x) && \text{(by (IUP-2))}\\
&\leq \mathcal{P}_I(x), && \text{(by (3.15))}\\
\mathcal{P}_F(0) &= \mathcal{P}_F(x \star x) && \text{(by (IUP-2))}\\
&\geq \mathcal{P}_F(x). && \text{(by (3.16))}
\end{aligned}$$

Hence, It satisfies (3.5), (3.6), and (3.7).

Theorem 2. *A Pythagorean neutrosophic strong IUP-ideal and constant PNS coincide.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned}
\mathcal{P}_T(x) &= \mathcal{P}_T((x \star 0) \star 0) && \text{(by (2.5))}\\
&\geq \mathcal{P}_T(0), && \text{(by (3.14))}\\
\mathcal{P}_I(x) &= \mathcal{P}_I((x \star 0) \star 0) && \text{(by (2.5))}\\
&\leq \mathcal{P}_I(0), && \text{(by (3.15))}\\
\mathcal{P}_F(x) &= \mathcal{P}_F((x \star 0) \star 0) && \text{(by (2.5))}\\
&\geq \mathcal{P}_F(0). && \text{(by (3.16))}
\end{aligned}$$

Hence, $\mathcal{P}$ is a constant of X.

Conversely, it is obvious that every constant PNF is a Pythagorean neutrosophic strong IUP-ideal.

Theorem 3. *Every Pythagorean neutrosophic strong IUP-ideal of X is a Pythagorean neutrosophic IUP-subalgebra of X.*

Proof. It is straightforward by Theorem 2.

Example 5. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	4	2	0	1	5	3
3	3	4	5	0	1	2
4	2	3	1	5	0	4
5	1	5	3	4	2	0

Then X is an IUP-algebra. We define $\mathcal{P}$ on X as follows:

$$\mathcal{P}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0.8 & 0.4 & 0.4 & 0.4 & 0.8 \end{pmatrix}$$

$$\mathcal{P}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 \end{pmatrix}$$

$$\mathcal{P}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0.6 & 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix}$$

Then $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. Since $\mathcal{P}_T(3 \star 1) = \mathcal{P}_T(4) = 0.4 \ngeq 0.8 = \mathcal{P}_T(1)$, $\mathcal{P}_I(3 \star 0) = \mathcal{P}_I(3) = 0.3 \nleq 0 = \mathcal{P}_I(0)$, and $\mathcal{P}_F(4 \star 5) = \mathcal{P}_F(4) = 0.2 \ngeq 0.6 = \mathcal{P}_F(5)$. Hence, $\mathcal{P}$ is not a Pythagorean neutrosophic strong IUP-ideal of X.

Theorem 4. *Every Pythagorean neutrosophic strong IUP-ideal of X is a Pythagorean neutrosophic IUP-ideal of X.*

Proof. It is straightforward by Theorem 2.

Example 6. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	1	3	2
2	3	4	0	2	5	1
3	2	5	3	0	1	4
4	4	2	1	5	0	3
5	1	3	5	4	2	0

Then X is an IUP-algebra. We define $\mathcal{P}$ on X as follows:

$$\mathcal{P}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.1 & 0.1 & 0.6 & 0.6 & 0.1 \end{pmatrix}$$

$$\mathcal{P}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 1 & 0.9 & 0.9 & 1 & 1 \end{pmatrix}$$

$$\mathcal{P}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.2 & 0.4 & 0.4 & 0.2 & 0.2 \end{pmatrix}$$

Then $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X. Since $\mathcal{P}_T(5 \star 0) = \mathcal{P}_T(1) = 0.1 \ngeq 0.6 = \mathcal{P}_T(0)$, $\mathcal{P}_I(2 \star 0) = \mathcal{P}_I(3) = 0.9 \nleq 0.7 = \mathcal{P}_I(0)$, and $\mathcal{P}_F(4 \star 3) = \mathcal{P}_F(5) = 0.2 \ngeq 0.4 = \mathcal{P}_F(3)$. Hence, $\mathcal{P}$ is not a Pythagorean neutrosophic strong IUP-ideal of X.

Theorem 5. *Every Pythagorean neutrosophic IUP-ideal of X is a Pythagorean neutrosophic IUP-filter of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X. By the assumption, it satisfies (3.5), (3.6), and (3.7). Let $x, y \in X$. Then

$$\begin{aligned}
\mathcal{P}_T(y) &= \mathcal{P}_T(0 \star y) && \text{(by (IUP-1))} \\
&\geq \min\{\mathcal{P}_T(0 \star (x \star y)), \mathcal{P}_T(x)\} && \text{(by (3.8))} \\
&= \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}, && \text{(by (IUP-1))} \\
\mathcal{P}_I(y) &= \mathcal{P}_I(0 \star y) && \text{(by (IUP-1))} \\
&\leq \max\{\mathcal{P}_I(0 \star (x \star y)), \mathcal{P}_I(x)\} && \text{(by (3.9))} \\
&= \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}, && \text{(by (IUP-1))} \\
\mathcal{P}_F(y) &= \mathcal{P}_F(0 \star y) && \text{(by (IUP-1))} \\
&\geq \min\{\mathcal{P}_F(0 \star (x \star y)), \mathcal{P}_F(x)\} && \text{(by (3.10))} \\
&= \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}. && \text{(by (IUP-1))}
\end{aligned}$$

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X.

Example 7. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\star$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	3	4	2	1
2	2	3	0	1	5	4
3	3	4	5	0	1	2
4	4	2	1	5	0	3
5	1	5	4	2	3	0

Then X is an IUP-algebra. We define $\mathcal{P}$ on X as follows:

$$\mathcal{P}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.1 & 0.1 & 0.3 & 0.1 & 0.1 \end{pmatrix}$$

$$\mathcal{P}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.2 & 0.8 & 0.8 & 0.4 & 0.8 & 0.8 \end{pmatrix}$$

$$\mathcal{P}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0.3 & 0.3 & 0.7 & 0.3 & 0.3 \end{pmatrix}$$

Then $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X. Since $\mathcal{P}_T(5 \star 1) = \mathcal{P}_T(5) = 0.1 \ngeq 0.3 = \min\{0.3, 0.3\} = \min\{\mathcal{P}_T(3), \mathcal{P}_T(3)\} = \min\{\mathcal{P}_T(5 \star 4), \mathcal{P}_T(3)\} = \min\{\mathcal{P}_T(5 \star (3 \star 1)), \mathcal{P}_T(3)\}$, $\mathcal{P}_I(4 \star 2) = \mathcal{P}_I(1) = 0.8 \nleq 0.4 = \max\{0.4, 0.4\} = \max\{\mathcal{P}_I(3), \mathcal{P}_I(3)\} = \max\{\mathcal{P}_I(4 \star 5), \mathcal{P}_I(3)\} = \max\{\mathcal{P}_I(4 \star (3 \star 2)), \mathcal{P}_I(3)\}$, and $\mathcal{P}_F(1 \star 4) = \mathcal{P}_F(2) = 0.3 \ngeq 0.7 = \min\{1, 0.7\} = \min\{\mathcal{P}_F(0), \mathcal{P}_F(3)\} = \min\{\mathcal{P}_F(1 \star 1), \mathcal{P}_F(3)\} = \min\{\mathcal{P}_F(1 \star (3 \star 4)), \mathcal{P}_F(3)\}$. Hence, $\mathcal{P}$ is not a Pythagorean neutrosophic IUP-ideal of X.

Theorem 6. *Every Pythagorean neutrosophic IUP-subalgebra of X is a Pythagorean neutrosophic IUP-filter of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. By Lemma 1, it satisfies (3.5), (3.6), and (3.7). Let $x, y \in X$. Then

$$\begin{aligned}
\mathcal{P}_T(y) &= \mathcal{P}_T(0 \star y) && \text{(by (IUP-1))} \\
&= \mathcal{P}_T((x \star 0) \star (x \star y)) && \text{(by (IUP-3))} \\
&\geq \min\{\mathcal{P}_T(x \star 0), \mathcal{P}_T(x \star y)\} && \text{(by (3.2))} \\
&\geq \min\{\min\{\mathcal{P}_T(x), \mathcal{P}_T(0)\}, \mathcal{P}_T(x \star y)\} && \text{(by (3.2))} \\
&= \min\{\mathcal{P}_T(x), \mathcal{P}_T(x \star y)\}, && \text{(by (3.5))} \\
\mathcal{P}_I(y) &= \mathcal{P}_I(0 \star y) && \text{(by (IUP-1))} \\
&= \mathcal{P}_I((x \star 0) \star (x \star y)) && \text{(by (IUP-3))} \\
&\leq \max\{\mathcal{P}_I(x \star 0), \mathcal{P}_I(x \star y)\} && \text{(by (3.3))} \\
&\leq \max\{\max\{\mathcal{P}_I(x), \mathcal{P}_I(0)\}, \mathcal{P}_I(x \star y)\} && \text{(by (3.3))} \\
&= \max\{\mathcal{P}_I(x), \mathcal{P}_I(x \star y)\}, && \text{(by (3.6))} \\
\mathcal{P}_F(y) &= \mathcal{P}_F(0 \star y) && \text{(by (IUP-1))} \\
&= \mathcal{P}_F((x \star 0) \star (x \star y)) && \text{(by (IUP-3))} \\
&\geq \min\{\mathcal{P}_F(x \star 0), \mathcal{P}_F(x \star y)\} && \text{(by (3.4))} \\
&\geq \min\{\min\{\mathcal{P}_F(x), \mathcal{P}_F(0)\}, \mathcal{P}_F(x \star y)\} && \text{(by (3.4))} \\
&= \min\{\mathcal{P}_F(x), \mathcal{P}_F(x \star y)\}. && \text{(by (3.7))}
\end{aligned}$$

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X.

Example 8. *[9] Let $\mathbb{R}^*$ be the set of all nonzero real numbers. Define a binary operation $\star$ on $\mathbb{R}^*$ by:*

$$(\forall x, y \in \mathbb{R}^*)(x \star y = \frac{y}{x}).$$

Thus, $(\mathbb{R}^*, \star, 1)$ is an IUP-algebra.

Example 9. *From Example 8, let $G = \{x \in \mathbb{R}^* \mid x \geq 1\}$. Then $1 \in G$. Next, let $x, y, z \in \mathbb{R}^*$ be such that $x \star (y \star z) \geq 1$ and $y \geq 1$. Then $\frac{z}{yx} \geq 1$. Thus, $x \star z = \frac{z}{x} = (\frac{z}{yx})y \geq 1$, that is, $x \star z \in G$. Hence, G is an IUP-ideal of $\mathbb{R}^*$. Then G is an IUP-filter of $\mathbb{R}^*$. From Theorems 10 and 11, $\mathcal{P}^G\left[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}\right]$ is a Pythagorean neutrosophic IUP-ideal and a Pythagorean neutrosophic IUP-filter of $\mathbb{R}^*$. Thus, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal and a Pythagorean neutrosophic IUP-filter of $\mathbb{R}^*$. Since $1, 3 \in G$ but $3 \star 1 = \frac{1}{3} \in G$, we have G is not an IUP-subalgebra of $\mathbb{R}^*$. From Theorem 9, $\mathcal{P}^G\left[^{\alpha^+, \beta^-, \gamma^+}_{\alpha^-, \beta^+, \gamma^-}\right]$ is not a Pythagorean neutrosophic IUP-subalgebra of $\mathbb{R}^*$. Hence, $\mathcal{P}$ is not a Pythagorean neutrosophic IUP-subalgebra of $\mathbb{R}^*$.*

Example 10. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	4	2	3	1
2	2	4	0	5	1	3
3	3	2	1	0	5	4
4	4	3	5	1	0	2
5	1	5	3	4	2	0

Then X is an IUP-algebra. We define $\mathcal{P}$ on X as follows:

$$\mathcal{P}_T = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.2 & 0.2 & 0.2 & 0.6 & 0.2 \end{pmatrix}$$

$$\mathcal{P}_I = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0.3 & 0.3 & 0.3 & 0.1 & 0.3 \end{pmatrix}$$

$$\mathcal{P}_F = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.6 & 0.3 & 0.3 & 0.3 & 0.5 & 0.3 \end{pmatrix}$$

Then $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. Since $\mathcal{P}_T(5 \star 2) = \mathcal{P}_T(3) = 0.2 \ngeq 0.6 = \min\{0.8, 0.6\} = \min\{\mathcal{P}_T(0), \mathcal{P}_T(4)\} = \min\{\mathcal{P}_T(5 \star (4 \star 2)), \mathcal{P}_T(4)\}$, $\mathcal{P}_I(3 \star 1) = \mathcal{P}_I(2) = 0.3 \nleq 0 = \max\{\mathcal{P}_I(0), \mathcal{P}_I(4)\} = \max\{\mathcal{P}_I(3 \star (4 \star 1)), \mathcal{P}_I(4)\}$, and $\mathcal{P}_F(5 \star 1) = \mathcal{P}_F(5) = 0.3 \ngeq 0.5 = \min\{0.5, 0.5\} = \min\{\mathcal{P}_F(4), \mathcal{P}_F(4)\} = \min\{\mathcal{P}_F(5 \star (4 \star 1)), \mathcal{P}_F(4)\}$. Hence, $\mathcal{P}$ is not a Pythagorean neutrosophic IUP-ideal of X.

The study identified a relationship among the four concepts: Pythagorean neutrosophic IUP-ideals and Pythagorean neutrosophic IUP-subalgebras are generalizations of Pythagorean neutrosophic strong IUP-ideals within IUP-algebras, where Pythagorean neutrosophic strong IUP-ideals can only be a constant PNS. Pythagorean Neutrosophic IUP-filters extend the generalization to include Pythagorean neutrosophic IUP-ideals and Pythagorean neutrosophic IUP-subalgebras. The relationships among these four concepts are summarized in Figure 2.

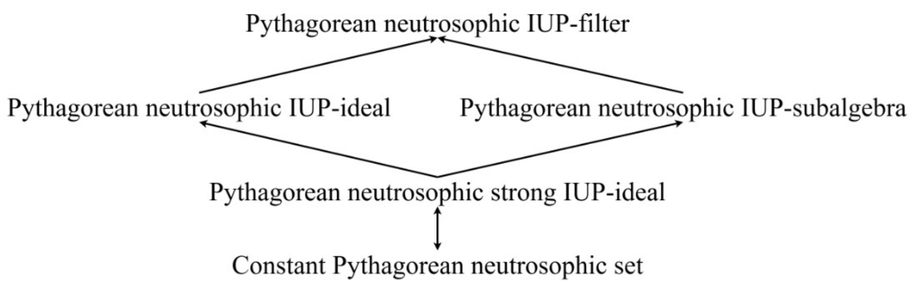

Figure 2: Pythagorean neutrosophic sets in IUP-algebras

Theorem 7. *If $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X satisfying the following condition:*

$$(\forall x, y \in X)\left(x \star y \neq 0 \Rightarrow \begin{cases} \mathcal{P}_T(x) \geq \mathcal{P}_T(y) \\ \mathcal{P}_I(x) \leq \mathcal{P}_I(y) \\ \mathcal{P}_F(x) \geq \mathcal{P}_F(y) \end{cases}\right) \tag{3.17}$$

then $\mathcal{P}$ is a Pythagorean neutrosophic strong IUP-ideal of X.

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X satisfying the condition (3.17). Let $x, y \in X$.

Case 1: Suppose $x \star y = 0$. Then

$$\begin{aligned} \mathcal{P}_T(x \star y) &= \mathcal{P}_T(0) \\ &\geq \mathcal{P}_T(y), && \text{(by (3.5))} \\ \mathcal{P}_I(x \star y) &= \mathcal{P}_I(0) \\ &\leq \mathcal{P}_I(y), && \text{(by (3.6))} \\ \mathcal{P}_F(x \star y) &= \mathcal{P}_F(0) \\ &\geq \mathcal{P}_F(y). && \text{(by (3.7))} \end{aligned}$$

Case 2: Suppose $x \star y \neq 0$. Then

$$\begin{aligned} \mathcal{P}_T(x \star y) &\geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\} && \text{(by (3.2))} \\ &= \mathcal{P}_T(y), \\ \mathcal{P}_I(x \star y) &\leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\} && \text{(by (3.3))} \\ &= \mathcal{P}_I(y), \\ \mathcal{P}_F(x \star y) &\geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\} && \text{(by (3.4))} \\ &= \mathcal{P}_F(y). \end{aligned}$$

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic strong IUP-ideal of X.

Theorem 8. *If $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X)\begin{pmatrix} \mathcal{P}_T(y \star (x \star z)) = \mathcal{P}_T(x \star (y \star z)) \\ \mathcal{P}_I(y \star (x \star z)) = \mathcal{P}_I(x \star (y \star z)) \\ \mathcal{P}_F(y \star (x \star z)) = \mathcal{P}_F(x \star (y \star z)) \end{pmatrix} \tag{3.18}$$

then $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X.

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X satisfying the condition (3.18). By the assumption, it satisfies (3.5), (3.6), and (3.7). Let $x, y, z \in X$. Then

$$\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(y \star (x \star z)), \mathcal{P}_T(y)\} \qquad \text{(by (3.11))}$$

$$
\begin{aligned}
&= \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}, \\
\mathcal{P}_I(x \star z) &\leq \max\{\mathcal{P}_I(y \star (x \star z)), \mathcal{P}_I(y)\} && \text{(by (3.12))} \\
&= \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}, \\
\mathcal{P}_F(x \star z) &\geq \min\{\mathcal{P}_F(y \star (x \star z)), \mathcal{P}_F(y)\} && \text{(by (3.13))} \\
&= \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}.
\end{aligned}
$$

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X.

For any fixed numbers $\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^- \in [0,1]$ such that $\alpha^+ > \alpha^-, \beta^+ > \beta^-, \gamma^+ > \gamma^-$ and a nonempty subset G of X, a PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}] = (X, \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}], \mathcal{P}^G_I[^{\beta^-}_{\beta^+}], \mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}])$ in X, where $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}], \mathcal{P}^G_I[^{\beta^-}_{\beta^+}]$, and $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}]$ are function on X which are given as follows:

$$
\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}] = \begin{cases} \alpha^+ & \text{if } x \in G \\ \alpha^- & \text{otherwise} \end{cases}
$$

$$
\mathcal{P}^G_I[^{\beta^-}_{\beta^+}] = \begin{cases} \beta^- & \text{if } x \in G \\ \beta^+ & \text{otherwise} \end{cases}
$$

$$
\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}] = \begin{cases} \gamma^+ & \text{if } x \in G \\ \gamma^- & \text{otherwise} \end{cases}
$$

Lemma 2. *Let G be a nonempty subset of X. Then the constant* 0 *of X is in G if and only if the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies* (3.5), (3.6), *and* (3.7).

Proof. Assume that the constant 0 of X is in G. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](0) = \alpha^+$, $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](0) = \beta^-$, and $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](0) = \gamma^+$. Thus, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](0) = \alpha^+ \geq \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)$, $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](0) = \beta^- \leq \mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x)$, and $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](0) = \gamma^+ \geq \mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x)$ for all $x \in X$, that is, $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies (3.5), (3.6), and (3.7).

Conversely, assume that $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ satisfies (3.5), (3.6), and (3.7). Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](0) \geq \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)$ for all $x \in X$. Since G is a nonempty subset of X, we let $a \in G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](0) \geq \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](a) = \alpha^+$, so $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](0) = \alpha^+$. Hence, the constant 0 of X is in G.

Theorem 9. *A nonempty subset G is an IUP-subalgebra of X if and only if the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-subalgebra of X.*

Proof. Assume that G is an IUP-subalgebra of X. Let $x, y \in X$. Then

Case 1 : Suppose $x, y \in G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$ and $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Since G is an IUP-subalgebra of X, we have $x \star y \in G$. Thus, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y) = \alpha^+ \geq \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 2 : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x) = \alpha^-$ or $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \alpha^-$. Thus, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y) \geq \alpha^- = \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 1' : Suppose $x, y \in G$. Then $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x) = \beta^-$ and $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^-$. Since G is an IUP-subalgebra of X, we have $x \star y \in G$. Thus, $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star y) = \beta^- \leq \beta^- = \max\{\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x), \mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 2' : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x) = \beta^+$ or $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^+$. Thus, $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star y) \leq \beta^+ = \max\{\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x), \mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 1" : Suppose $x, y \in G$. Then $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^+$ and $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^+$. Since G is an IUP-subalgebra of X, We have $x \star y \in G$. Thus, $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star y) = \gamma^+ \geq \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x), \mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Case 2" : Suppose $x \notin G$ or $y \notin G$. Then $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x) = \gamma^-$ or $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^-$. Thus, $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star y) \geq \gamma^- = \min\{\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x), \mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Hence, the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-subalgebra of X.

Conversely, assume that the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-subalgebra of X. Let $x, y \in G$. Then $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$ and $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. By (3.2), we have $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star y) \geq \min\{\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x), \mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](y)\} = \min\{\alpha^+, \alpha^+\} = \alpha^+$. Thus $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star y) = \alpha^+$, that is, $x \star y \in G$. Hence, G is an IUP-subalgebra of X.

Theorem 10. *A nonempty subset G is an IUP-ideal of X if and only if the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-ideal of X.*

Proof. Assume that G is an IUP-ideal of X. Since $0 \in G$, it follows from Lemma 2 that $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}]$, $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}]$, and $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}]$ satisfy (3.5), (3.6), and (3.7), respectively. Next, let $x, y, z \in X$.

Case 1 : Suppose $x \star (y \star z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \star z \in G$. Thus, $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star z) = \alpha^+ \geq \alpha^+ = \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star (y \star z)), \mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 2 : Suppose $x \star (y \star z) \notin G$ or $y \notin G$. Then $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star (y \star z)) = \alpha^-$ or $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](y) = \alpha^-$. Thus, $\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star z) \geq \alpha^- = \min\{\mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](x \star (y \star z)), \mathcal{P}_T^G[^{\alpha^+}_{\alpha^-}](y)\}$.

Case 1' : Suppose $x \star (y \star z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \star z \in G$. Thus, $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star z) = \beta^- \leq \beta^- = \max\{\beta^-, \beta^-\} = \max\{\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star (y \star z)), \mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 2' : Suppose $x \star (y \star z) \notin G$ or $y \notin G$. Then $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star (y \star z)) = \beta^+$ or $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y) = \beta^+$. Thus, $\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star z) \leq \beta^+ = \max\{\mathcal{P}_I^G[^{\beta^-}_{\beta^+}](x \star (y \star z)), \mathcal{P}_I^G[^{\beta^-}_{\beta^+}](y)\}$.

Case 1" : Suppose $x \star (y \star z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \star z \in G$. Thus, $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star z) = \gamma^+ \geq \gamma^+ = \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star (y \star z)), \mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Case 2" : Suppose $x \star (y \star z) \notin G$ or $y \notin G$. Then $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star (y \star z)) = \gamma^-$ or $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y) = \gamma^-$. Thus, $\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star z) \geq \gamma^- = \min\{\mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](x \star (y \star z)), \mathcal{P}_F^G[^{\gamma^+}_{\gamma^-}](y)\}$.

Hence, $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-ideal of X.

Conversely, assume that the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-ideal of X. Since $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}]$ satisfies (3.5), it follow from Lemma 2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \star (y \star z) \in G$ and $y \in G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star (y \star z)) = \alpha^+$ and $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Thus, $\min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star (y \star z)), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+$. By (3.8), we have $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star z) \geq \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star (y \star z)), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y)\} = \alpha^+$, that is, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star z) = \alpha^+$. Hence, $x \star z \in G$, so G is an IUP-ideal.

Theorem 11. *A nonempty subset G is an IUP-filter of X if and only if the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-filter of X.*

Proof. Assume that G is an IUP-filter of X. Since $0 \in G$, it follows from Lemma 2 that $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}]$, $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}]$, and $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}]$ satisfy (3.5), (3.6), and (3.7), respectively. Next, let $x, y \in X$.

Case 1 : Suppose $x \star y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \alpha^+ \geq \alpha^+ = \min\{\alpha^+, \alpha^+\} = \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)\}$.

Case 2 : Suppose $x \star y \notin G$ or $x \notin G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y) = \alpha^-$ or $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x) = \alpha^-$. Thus, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) \geq \alpha^- = \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)\}$.

Case 1' : Suppose $x \star y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](y) = \beta^- \leq \beta^- = \max\{\beta^-, \beta^-\} = \max\{\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x \star y), \mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x)\}$.

Case 2' : Suppose $x \star y \notin G$ or $x \notin G$. Then $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x \star y) = \beta^+$ or $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x) = \beta^+$. Thus, $\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](y) \leq \beta^+ = \max\{\mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x \star y), \mathcal{P}^G_I[^{\beta^-}_{\beta^+}](x)\}$.

Case 1" : Suppose $x \star y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](y) = \gamma^+ \geq \gamma^+ = \min\{\gamma^+, \gamma^+\} = \min\{\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x \star y), \mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x)\}$.

Case 2" : Suppose $x \star y \notin G$ or $x \notin G$. Then $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x \star y) = \gamma^-$ or $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x) = \gamma^-$. Thus, $\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](y) \geq \gamma^- = \min\{\mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x \star y), \mathcal{P}^G_F[^{\gamma^+}_{\gamma^-}](x)\}$.

Hence, $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-filter of X.

Conversely, assume that the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic IUP-filter of X. Since $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}]$ satisfies (3.5), it follow from Lemma 2 that $0 \in G$. Next, let $x, y \in X$ be such that $x \star y \in G$ and $x \in G$. Then $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y) = \alpha^+$ and $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x) = \alpha^+$. Thus, $\min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^+$. By (3.11), we have $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \min\{\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x \star y), \mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](x)\} = \alpha^+$, that is, $\mathcal{P}^G_T[^{\alpha^+}_{\alpha^-}](y) = \alpha^+$. Hence, $y \in G$, so G is an IUP-filter of X.

Theorem 12. *A nonempty subset G is a strong IUP-ideal of X if and only if the characteristic PNS $\mathcal{P}^G[^{\alpha^+,\beta^-,\gamma^+}_{\alpha^-,\beta^+,\gamma^-}]$ is a Pythagorean neutrosophic strong IUP-ideal of X.*

Proof. It is straightforward by Theorem 2.

Lemma 3. *Let f be an FS in a nonempty set X and let n be a positive integer. Then the following statements hold:*

$$(\forall x, y \in X)(\frac{\min\{f(x), f(y)\}}{n} = \min\{\frac{f(x)}{n}, \frac{f(y)}{n}\}) \tag{3.19}$$

$$(\forall x, y \in X)(\frac{\max\{f(x), f(y)\}}{n} = \max\{\frac{f(x)}{n}, \frac{f(y)}{n}\}) \tag{3.20}$$

Lemma 4. *Let f be an FS in a nonempty set X and let n be a positive integer. Then the following statements hold:*

$$(\forall x, y, z \in X)(f(z) \geq \min\{f(x), f(y)\} \Leftrightarrow f_n(z) \geq \min\{f_n(x), f_n(y)\}) \tag{3.21}$$

$$(\forall x, y, z \in X)(f(z) \leq \max\{f(x), f(y)\} \Leftrightarrow f_n(z) \leq \max\{f_n(x), f_n(y)\}) \tag{3.22}$$

Proof. It is straightforward by Theorem 3.

Theorem 13. *A PNS $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X if and only if a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is Pythagorean neutrosophic IUP-subalgebra of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. Then

$$\mathcal{P}_T(x \star y) \geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\},$$
$$\mathcal{P}_I(x \star y) \leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\},$$
$$\mathcal{P}_F(x \star y) \geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}.$$

Thus,

$$\mathcal{P}_{Tn}(x \star y) \geq \min\{\mathcal{P}_{Tn}(x), \mathcal{P}_{Tn}(y)\}, \quad \text{(by (3.21))}$$
$$\mathcal{P}_{In}(x \star y) \leq \max\{\mathcal{P}_{In}(x), \mathcal{P}_{In}(y)\}, \quad \text{(by (3.22))}$$
$$\mathcal{P}_{Fn}(x \star y) \geq \min\{\mathcal{P}_{Fn}(x), \mathcal{P}_{Fn}(y)\}. \quad \text{(by (3.21))}$$

Hence, a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is a Pythagorean neutrosophic IUP-subalgebra of X.

Conversely, it is obvious to prove that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X.

Theorem 14. *A PNS $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X if and only if a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is Pythagorean neutrosophic IUP-ideal of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X. Then

$$\mathcal{P}_T(0) \geq \mathcal{P}_T(x),$$
$$\mathcal{P}_I(0) \leq \mathcal{P}_I(x),$$
$$\mathcal{P}_F(0) \geq \mathcal{P}_F(x),$$
$$\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\},$$

$$\mathcal{P}_I(x \star z) \leq \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\},$$
$$\mathcal{P}_F(x \star z) \geq \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}.$$

Thus,

$$\begin{aligned} \mathcal{P}_{Tn}(0) &\geq \mathcal{P}_{Tn}(x), \\ \mathcal{P}_{In}(0) &\leq \mathcal{P}_{In}(x), \\ \mathcal{P}_{Fn}(0) &\geq \mathcal{P}_{Fn}(x), \\ \mathcal{P}_{Tn}(x \star z) &\geq \min\{\mathcal{P}_{Tn}(x \star (y \star z)), \mathcal{P}_{Tn}(y)\}, && \text{(by (3.21))} \\ \mathcal{P}_{In}(x \star z) &\leq \max\{\mathcal{P}_{In}(x \star (y \star z)), \mathcal{P}_{In}(y)\}, && \text{(by (3.22))} \\ \mathcal{P}_{Fn}(x \star z) &\geq \min\{\mathcal{P}_{Fn}(x \star (y \star z)), \mathcal{P}_{Fn}(y)\}. && \text{(by (3.21))} \end{aligned}$$

Hence, a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is Pythagorean neutrosophic IUP-ideal of X.

Conversely, it is obvious to prove that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X.

Theorem 15. *A PNS $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X if and only if a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is Pythagorean neutrosophic IUP-filter of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X. Then

$$\begin{aligned} \mathcal{P}_T(0) &\geq \mathcal{P}_T(x), \\ \mathcal{P}_I(0) &\leq \mathcal{P}_I(x), \\ \mathcal{P}_F(0) &\geq \mathcal{P}_F(x), \\ \mathcal{P}_T(y) &\geq \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}, \\ \mathcal{P}_I(y) &\leq \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}, \\ \mathcal{P}_F(y) &\geq \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{P}_{Tn}(0) &\geq \mathcal{P}_{Tn}(x), \\ \mathcal{P}_{In}(0) &\leq \mathcal{P}_{In}(x), \\ \mathcal{P}_{Fn}(0) &\geq \mathcal{P}_{Fn}(x), \\ \mathcal{P}_{Tn}(y) &\geq \min\{\mathcal{P}_{Tn}(x \star y), \mathcal{P}_{Tn}(x)\}, && \text{(by (3.21))} \\ \mathcal{P}_{In}(y) &\leq \max\{\mathcal{P}_{In}(x \star y), \mathcal{P}_{In}(x)\}, && \text{(by (3.22))} \\ \mathcal{P}_{Fn}(y) &\geq \min\{\mathcal{P}_{Fn}(x \star y), \mathcal{P}_{Fn}(x)\}. && \text{(by (3.21))} \end{aligned}$$

Hence, a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is a Pythagorean neutrosophic IUP-filter of X.

Conversely, it is obvious to prove that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X.

Theorem 16. *A PNS $\mathcal{P}$ is a Pythagorean neutrosophic strong IUP-ideal of X if and only if a PNS $\mathcal{P}_n = (X, \mathcal{P}_{Tn}, \mathcal{P}_{In}, \mathcal{P}_{Fn})$ is Pythagorean neutrosophic strong IUP-ideal of X.*

Proof. It is straightforward by Theorem 2.

Definition 10. *[17] Let f be an FS in a nonempty set X. For any $t \in [0,1]$, the sets*

$$U(f;t) = \{x \in X \mid f(x) \geq t\}, \tag{3.23}$$

$$L(f;t) = \{x \in X \mid f(x) \leq t\}, \tag{3.24}$$

$$E(f;t) = \{x \in X \mid f(x) = t\} \tag{3.25}$$

are called an upper t-level subset and a lower t-level subset of f, respectively. The sets

$$U^{+}(f;t) = \{x \in X \mid f(x) > t\}, \tag{3.26}$$

$$L^{-}(f;t) = \{x \in X \mid f(x) < t\} \tag{3.27}$$

are called an upper t-strong level subset and a lower t-strong level subset of f, respectively.

Theorem 17. *A PNS $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{P}_T;\alpha)$, $L(\mathcal{P}_I;\beta)$, and $U(\mathcal{P}_F;\gamma)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X. Let $\alpha \in [0,1]$ be such that $U(\mathcal{P}_T;\alpha) \neq \emptyset$. Let $x, y \in U(\mathcal{P}_T;\alpha)$. Then $\mathcal{P}_T(x) \geq \alpha$ and $\mathcal{P}_T(y) \geq \alpha$. Thus, $\min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\} \geq \alpha$. By (3.2), we have $\mathcal{P}_T(x \star y) \geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\} \geq \alpha$, that is, $\mathcal{P}_T(x \star y) \geq \alpha$. Thus, $x \star y \in U(\mathcal{P}_T;\alpha)$. Hence, $U(\mathcal{P}_T;\alpha)$ is an IUP-subalgebra of X.

Let $\beta \in [0,1]$ be such that $L(\mathcal{P}_I;\beta) \neq \emptyset$. Let $x, y \in L(\mathcal{P}_I;\beta)$. Then $\mathcal{P}_I(x) \leq \beta$ and $\mathcal{P}_I(y) \leq \beta$. Thus, $\max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\} \leq \beta$. By (3.3), we have $\mathcal{P}_I(x \star y) \leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\} \leq \beta$, that is, $\mathcal{P}_I(x \star y) \leq \beta$. Thus, $x \star y \in L(\mathcal{P}_I;\beta)$. Hence, $L(\mathcal{P}_I;\beta)$ is an IUP-subalgebra of X.

Let $\gamma \in [0,1]$ be such that $U(\mathcal{P}_F;\gamma) \neq \emptyset$. Let $x, y \in U(\mathcal{P}_F;\gamma)$. Then $\mathcal{P}_F(x) \geq \gamma$ and $\mathcal{P}_F(y) \geq \gamma$. Thus, $\min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\} \geq \gamma$. By (3.4), we have $\mathcal{P}_F(x \star y) \geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\} \geq \gamma$, that is, $\mathcal{P}_F(x \star y) \geq \gamma$. Thus, $x \star y \in U(\mathcal{P}_F;\gamma)$. Hence, $U(\mathcal{P}_F;\gamma)$ is an IUP-subalgebra of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{P}_T;\alpha)$, $L(\mathcal{P}_I;\beta)$, and $U(\mathcal{P}_F;\gamma)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Let $\alpha = \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\}$. Then $\mathcal{P}_T(x) \geq \alpha$ and $\mathcal{P}_T(y) \geq \alpha$. Thus, $x, y \in U(\mathcal{P}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_T;\alpha)$ is an IUP-subalgebra of X. By (2.17), we have $x \star y \in U(\mathcal{P}_T;\alpha)$. Thus, $\mathcal{P}_T(x \star y) \geq \alpha = \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\}$.

Let $x, y \in X$. Let $\beta = \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\}$. Then $\mathcal{P}_I(x) \leq \beta$ and $\mathcal{P}_I(y) \leq \beta$. Thus, $x, y \in L(\mathcal{P}_I;\beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{P}_I;\beta)$ is an IUP-subalgebra of X. By (2.17), we have $x \star y \in L(\mathcal{P}_I;\beta)$. Thus, $\mathcal{P}_I(x \star y) \leq \beta = \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\}$.

Let $x, y \in X$. Let $\gamma = \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}$. Then $\mathcal{P}_F(x) \geq \gamma$ and $\mathcal{P}_F(y) \geq \gamma$. Thus, $x, y \in U(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_F; \gamma)$ is an IUP-subalgebra of X. By (2.17), we have $x \star y \in U(\mathcal{P}_F; \gamma)$. Thus, $\mathcal{P}_F(x \star y) \geq \gamma = \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}$.

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X.

Theorem 18. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0, 1]$, the sets $U(\mathcal{P}_T; \alpha)$, $L(\mathcal{P}_I; \beta)$, and $U(\mathcal{P}_F; \gamma)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-ideal of X. Let $\alpha \in [0, 1]$ be such that $U(\mathcal{P}_T; \alpha) \neq \emptyset$. Let $a \in U(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(a) \geq \alpha$. By (3.5), we have $\mathcal{P}_T(0) \geq \mathcal{P}_T(a) \geq \alpha$. Thus, $0 \in U(\mathcal{P}_T; \alpha)$. Let $x, y, z \in X$ be such that $x \star (y \star z) \in U(\mathcal{P}_T; \alpha)$ and $y \in U(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(x \star (y \star z)) \geq \alpha$ and $\mathcal{P}_T(y) \geq \alpha$. Thus, $\min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\} \geq \alpha$. By (3.8), we have $\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\} \geq \alpha$. Thus, $x \star z \in U(\mathcal{P}_T; \alpha)$. Hence, $U(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X.

Let $\beta \in [0, 1]$ be such that $L(\mathcal{P}_I; \beta) \neq \emptyset$. Let $b \in L(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(b) \leq \beta$. By (3.6), we have $\mathcal{P}_I(0) \leq \mathcal{P}_I(b) \leq \beta$. Thus, $0 \in L(\mathcal{P}_I; \beta)$. Let $x, y, z \in X$ be such that $x \star (y \star z) \in L(\mathcal{P}_I; \beta)$ and $y \in L(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(x \star (y \star z)) \leq \beta$ and $\mathcal{P}_I(y) \leq \beta$. Thus, $\max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\} \leq \beta$. By (3.9), we have $\mathcal{P}_I(x \star z) \leq \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\} \leq \beta$. Thus, $x \star z \in L(\mathcal{P}_I; \beta)$. Hence, $L(\mathcal{P}_I; \beta)$ is an IUP-ideal of X.

Let $\gamma \in [0, 1]$ be such that $U(\mathcal{P}_F; \gamma) \neq \emptyset$. Let $c \in U(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(c) \geq \gamma$. By (3.7), we have $\mathcal{P}_F(0) \geq \mathcal{P}_F(c) \geq \gamma$. Thus, $0 \in U(\mathcal{P}_F; \gamma)$. Let $x, y, z \in X$ be such that $x \star (y \star z) \in U(\mathcal{P}_F; \gamma)$ and $y \in U(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(x \star (y \star z)) \geq \gamma$ and $\mathcal{P}_F(y) \geq \gamma$. Thus, $\min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\} \geq \gamma$. By (3.10), we have $\mathcal{P}_F(x \star z) \geq \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\} \geq \gamma$. Thus, $x \star z \in U(\mathcal{P}_F; \gamma)$. Hence, $U(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0, 1]$, the sets $U(\mathcal{P}_T; \alpha)$, $L(\mathcal{P}_I; \beta)$, and $U(\mathcal{P}_F; \gamma)$ are either empty or IUP-ideals of X. Let $x \in X$. Let $\alpha = \mathcal{P}_T(x)$. Then $\mathcal{P}_T(x) \geq \alpha$. Thus, $x \in U(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(0) \geq \alpha = \mathcal{P}_T(x)$. Let $x, y, z \in X$. Let $\alpha = \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}$. Then $\mathcal{P}_T(x \star (y \star z)) \geq \alpha$ and $\mathcal{P}_T(y) \geq \alpha$. Thus, $x \star (y \star z), y \in U(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in U(\mathcal{P}_T; \alpha)$. Thus, $\mathcal{P}_T(x \star z) \geq \alpha = \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}$.

Let $x \in X$. Let $\beta = \mathcal{P}_I(x)$. Then $\mathcal{P}_I(x) \leq \beta$. Thus, $x \in L(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{P}_I; \beta)$ is an IUP-ideal of X. By (2.18), we have $0 \in L(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(0) \leq \beta = \mathcal{P}_I(x)$. Let $x, y, z \in X$. Let $\beta = \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}$. Then $\mathcal{P}_I(x \star (y \star z)) \leq \beta$ and $\mathcal{P}_I(y) \leq \beta$. Thus, $x \star (y \star z), y \in L(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_I; \beta)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in L(\mathcal{P}_I; \beta)$. Thus, $\mathcal{P}_I(x \star z) \leq \beta = \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}$.

Let $x \in X$. Let $\gamma = \mathcal{P}_F(x)$. Then $\mathcal{P}_F(x) \geq \gamma$. Thus, $x \in U(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X. By (2.18), we have $0 \in U(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(0) \geq \gamma = \mathcal{P}_F(x)$. Let $x, y, z \in X$. Let $\gamma = \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}$. Then $\mathcal{P}_F(x \star (y \star z)) \geq \gamma$ and $\mathcal{P}_F(y) \geq \gamma$. Thus, $x \star (y \star z), y \in U(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in U(\mathcal{P}_F; \gamma)$. Thus,

$\mathcal{P}_F(x \star z) \geq \gamma = \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}$. Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-ideal of X.

Theorem 19. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{P}_T;\alpha)$, $L(\mathcal{P}_I;\beta)$, and $U(\mathcal{P}_F;\gamma)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-filter of X. Let $\alpha \in [0,1]$ be such that $U(\mathcal{P}_T;\alpha) \neq \emptyset$. Let $a \in U(\mathcal{P}_T;\alpha)$. Then $\mathcal{P}_T(a) \geq \alpha$. By (3.5), we have $\mathcal{P}_T(0) \geq \mathcal{P}_T(a) \geq \alpha$. Thus, $0 \in U(\mathcal{P}_T;\alpha)$. Let $x, y \in X$ be such that $x \star y \in U(\mathcal{P}_T;\alpha)$ and $x \in U(\mathcal{P}_T;\alpha)$. Then $\mathcal{P}_T(x \star y) \geq \alpha$ and $\mathcal{P}_T(x) \geq \alpha$. Thus, $\min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\} \geq \alpha$. By (3.11), we have $\mathcal{P}_T(y) \geq \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\} \geq \alpha$. Thus, $y \in U(\mathcal{P}_T;\alpha)$. Hence, $U(\mathcal{P}_T;\alpha)$ is an IUP-filter of X.

Let $\beta \in [0,1]$ be such that $L(\mathcal{P}_I;\beta) \neq \emptyset$. Let $b \in L(\mathcal{P}_I;\beta)$. Then $\mathcal{P}_I(b) \leq \beta$. By (3.6), we have $\mathcal{P}_I(0) \leq \mathcal{P}_I(b) \leq \beta$. Thus, $0 \in L(\mathcal{P}_I;\beta)$. Let $x, y \in X$ be such that $x \star y \in L(\mathcal{P}_I;\beta)$ and $x \in L(\mathcal{P}_I;\beta)$. Then $\mathcal{P}_I(x \star y) \leq \beta$ and $\mathcal{P}_I(x) \leq \beta$. Thus, $\max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\} \leq \beta$. By (3.12), we have $\mathcal{P}_I(y) \leq \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\} \leq \beta$. Thus, $y \in L(\mathcal{P}_I;\beta)$. Hence, $L(\mathcal{P}_I;\beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U(\mathcal{P}_F;\gamma) \neq \emptyset$. Let $c \in U(\mathcal{P}_F;\gamma)$. Then $\mathcal{P}_F(c) \geq \gamma$. By (3.7), we have $\mathcal{P}_F(0) \geq \mathcal{P}_F(c) \geq \gamma$. Thus, $0 \in U(\mathcal{P}_F;\gamma)$. Let $x, y \in X$ be such that $x \star y \in U(\mathcal{P}_F;\gamma)$ and $x \in U(\mathcal{P}_F;\gamma)$. Then $\mathcal{P}_F(x \star y) \geq \gamma$ and $\mathcal{P}_F(x) \geq \gamma$. Thus, $\min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\} \geq \gamma$. By (3.13), we have $\mathcal{P}_F(y) \geq \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\} \geq \gamma$. Thus, $y \in U(\mathcal{P}_F;\gamma)$. Hence, $U(\mathcal{P}_F;\gamma)$ is an IUP-filter of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{P}_T;\alpha)$, $L(\mathcal{P}_I;\beta)$, and $U(\mathcal{P}_F;\gamma)$ are either empty or IUP-filters of X. Let $x \in X$. Let $\alpha = \mathcal{P}_T(x)$. Then $\mathcal{P}_T(x) \geq \alpha$. Thus, $x \in U(\mathcal{P}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_T;\alpha)$ is an IUP-filter of X. By (2.18), we have $0 \in U(\mathcal{P}_T;\alpha)$. Then $\mathcal{P}_T(0) \geq \alpha = \mathcal{P}_T(x)$. Let $x, y \in X$. Let $\alpha = \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}$. Then $\mathcal{P}_T(x \star y) \geq \alpha$ and $\mathcal{P}_T(x) \geq \alpha$. Thus, $x \star y, x \in U(\mathcal{P}_T;\alpha) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_T;\alpha)$ is an IUP-filter of X. By (2.19), we have $y \in U(\mathcal{P}_T;\alpha)$. Thus, $\mathcal{P}_T(y) \geq \alpha = \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}$.

Let $x \in X$. Let $\beta = \mathcal{P}_I(x)$. Then $\mathcal{P}_I(x) \leq \beta$. Thus, $x \in L(\mathcal{P}_I;\beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{P}_I;\beta)$ is an IUP-filter of X. By (2.18), we have $0 \in L(\mathcal{P}_I;\beta)$. Then $\mathcal{P}_I(0) \leq \beta = \mathcal{P}_I(x)$. Let $x, y \in X$. Let $\beta = \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}$. Then $\mathcal{P}_I(x \star y) \leq \beta$ and $\mathcal{P}_I(x) \leq \beta$. Thus, $x \star y, x \in L(\mathcal{P}_I;\beta) \neq \emptyset$. By the assumption, we have $L(\mathcal{P}_I;\beta)$ is an IUP-filter of X. By (2.19), we have $y \in L(\mathcal{P}_I;\beta)$. Thus, $\mathcal{P}_I(y) \leq \beta = \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}$.

Let $x \in X$. Let $\gamma = \mathcal{P}_F(x)$. Then $\mathcal{P}_F(x) \geq \gamma$. Thus, $x \in U(\mathcal{P}_F;\gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_F;\gamma)$ is an IUP-filter of X. By (2.18), we have $0 \in U(\mathcal{P}_F;\gamma)$. Then $\mathcal{P}_F(0) \geq \gamma = \mathcal{P}_F(x)$. Let $x, y \in X$. Let $\gamma = \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}$. Then $\mathcal{P}_F(x \star y) \geq \gamma$ and $\mathcal{P}_F(x) \geq \gamma$. Thus, $x \star y, x \in U(\mathcal{P}_F;\gamma) \neq \emptyset$. By the assumption, we have $U(\mathcal{P}_F;\gamma)$ is an IUP-filter of X. By (2.19), we have $y \in U(\mathcal{P}_F;\gamma)$. Thus, $\mathcal{P}_F(y) \geq \gamma = \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}$. Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X.

Theorem 20. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U(\mathcal{P}_T;\alpha)$, $L(\mathcal{P}_I;\beta)$, and $U(\mathcal{P}_F;\gamma)$ are either empty or strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 2.

Theorem 21. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic strong IUP-ideal of X if and only if the sets $E(\mathcal{P}_T;\mathcal{P}_T(0))$, $E(\mathcal{P}_I;\mathcal{P}_I(0))$, and $E(\mathcal{P}_F;\mathcal{P}_F(0))$ are strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 2.

Theorem 22. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T;\alpha)$, $L^-(\mathcal{P}_I;\beta)$, and $U^+(\mathcal{P}_F;\gamma)$ are either empty or IUP-subalgebras of X.*

Proof. Assume that $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-subalgebra of X. Let $\alpha \in [0,1]$ be such that $U^+(\mathcal{P}_T;\alpha) \neq \emptyset$. Let $x, y \in U^+(\mathcal{P}_T;\alpha)$. Then $\mathcal{P}_T(x) > \alpha$ and $\mathcal{P}_T(y) > \alpha$. Thus, $\min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\} > \alpha$. By (3.2), we have $\mathcal{P}_T(x \star y) \geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\} > \alpha$. Thus, $x \star y \in U^+(\mathcal{P}_T;\alpha)$. Hence, $U^+(\mathcal{P}_T;\alpha)$ is an IUP-subalgebra of X.

Let $\beta \in [0,1]$ be such that $L^-(\mathcal{P}_I;\beta) \neq \emptyset$. Let $x, y \in L^-(\mathcal{P}_I;\beta)$. Then $\mathcal{P}_I(x) < \beta$ and $\mathcal{P}_I(y) < \beta$. Thus, $\max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\} < \beta$. By (3.3), we have $\mathcal{P}_I(x \star y) \leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\} < \beta$. Thus, $x \star y \in L^-(\mathcal{P}_I;\beta)$. Hence, $L^-(\mathcal{P}_I;\beta)$ is an IUP-subalgebra of X.

Let $\gamma \in [0,1]$ be such that $U^+(\mathcal{P}_F;\gamma) \neq \emptyset$. Let $x, y \in U^+(\mathcal{P}_F;\gamma)$. Then $\mathcal{P}_F(x) > \gamma$ and $\mathcal{P}_F(y) > \gamma$. Thus, $\min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\} > \gamma$. By (3.2), we have $\mathcal{P}_F(x \star y) \geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\} > \gamma$. Thus, $x \star y \in U^+(\mathcal{P}_F;\gamma)$. Hence, $U^+(\mathcal{P}_F;\gamma)$ is an IUP-subalgebra of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T;\alpha)$, $L^-(\mathcal{P}_I;\beta)$, and $U^+(\mathcal{P}_F;\gamma)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Assume that $\mathcal{P}_T(x \star y) < \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\}$. Let $\alpha = \mathcal{P}_T(x \star y)$. Then $\mathcal{P}_T(x) > \alpha$ and $\mathcal{P}_T(y) > \alpha$. Thus, $x, y \in U^+(\mathcal{P}_T;\alpha)$. By the assumption, we have $U^+(\mathcal{P}_T;\alpha)$ is an IUP-subalgebra. By (2.17), we have $x \star y \in U^+(\mathcal{P}_T;\alpha)$. So $\mathcal{P}_T(x \star y) > \alpha = \mathcal{P}_T(x \star y)$, which is a contradiction. Thus, $\mathcal{P}_T(x \star y) \geq \min\{\mathcal{P}_T(x), \mathcal{P}_T(y)\}$.

Let $x, y \in X$. Assume that $\mathcal{P}_I(x \star y) > \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\}$. Let $\beta = \mathcal{P}_I(x \star y)$. Then $\mathcal{P}_I(x) < \beta$ and $\mathcal{P}_I(y) < \beta$. Thus, $x, y \in L^-(\mathcal{P}_I;\beta)$. By the assumption, we have $L^-(\mathcal{P}_I;\beta)$ is an IUP-subalgebra. By (2.17), we have $x \star y \in L^-(\mathcal{P}_I;\beta)$. So $\mathcal{P}_I(x \star y) < \beta = \mathcal{P}_I(x \star y)$, which is a contradiction. Thus, $\mathcal{P}_I(x \star y) \leq \max\{\mathcal{P}_I(x), \mathcal{P}_I(y)\}$.

Let $x, y \in X$. Assume that $\mathcal{P}_F(x \star y) < \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}$. Let $\gamma = \mathcal{P}_F(x \star y)$. Then $\mathcal{P}_F(x) > \gamma$ and $\mathcal{P}_F(y) > \gamma$. Thus, $x, y \in U^+(\mathcal{P}_F;\gamma)$. By the assumption, we have $U^+(\mathcal{P}_F;\gamma)$ is an IUP-subalgebra. By (2.17), we have $x \star y \in U^+(\mathcal{P}_F;\gamma)$. So $\mathcal{P}_F(x \star y) > \gamma = \mathcal{P}_F(x \star y)$, which is a contradiction. Thus, $\mathcal{P}_F(x \star y) \geq \min\{\mathcal{P}_F(x), \mathcal{P}_F(y)\}$.

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-subalgebra of X.

Theorem 23. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T; \alpha)$, $L^-(\mathcal{P}_I; \beta)$, and $U^+(\mathcal{P}_F; \gamma)$ are either empty or IUP-ideals of X.*

Proof. Assume that $\mathcal{P}$ in X is a Pythagorean grneutrosophic IUP-ideal of X. Let $\alpha \in [0,1]$ be such that $U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. Let $a \in U^+(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(a) > \alpha$. By (3.5), we have $\mathcal{P}_T(0) \geq \mathcal{P}_T(a) > \alpha$. Thus, $0 \in U^+(\mathcal{P}_T; \alpha)$. Let $x, y, z \in U^+(\mathcal{P}_T; \alpha)$ be such that $x \star (y \star z), y \in U^+(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(x \star (y \star z)) > \alpha$ and $\mathcal{P}_T(y) > \alpha$. Thus, $\min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\} > \alpha$. By (3.8), we have $\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\} > \alpha$. Thus, $x \star z \in U^+(\mathcal{P}_T; \alpha)$. Hence, $U^+(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X.

Let $\beta \in [0,1]$ be such that $L^-(\mathcal{P}_I; \beta) \neq \emptyset$. Let $b \in L^-(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(b) < \beta$. By (3.6), we have $\mathcal{P}_I(0) \leq \mathcal{P}_I(b) < \beta$. Thus, $0 \in L^-(\mathcal{P}_I; \beta)$. Let $x, y, z \in L^-(\mathcal{P}_I; \beta)$ be such that $x \star (y \star z), y \in L^-(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(x \star (y \star z)) < \beta$ and $\mathcal{P}_T(y) < \beta$. Thus, $\max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\} < \beta$. By (3.9), we have $\mathcal{P}_I(x \star z) \leq \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\} > \beta$. Thus, $x \star z \in L^-(\mathcal{P}_I; \beta)$. Hence, $L^-(\mathcal{P}_I; \beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. Let $c \in U^+(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(c) > \gamma$. By (3.7), we have $\mathcal{P}_F(0) \geq \mathcal{P}_F(c) > \gamma$. Thus, $0 \in U^+(\mathcal{P}_F; \gamma)$. Let $x, y, z \in U^+(\mathcal{P}_F; \gamma)$ be such that $x \star (y \star z), y \in U^+(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(x \star (y \star z)) > \gamma$ and $\mathcal{P}_F(y) > \gamma$. Thus, $\min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\} > \gamma$. By (3.10), we have $\mathcal{P}_F(x \star z) \geq \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\} > \gamma$. Thus, $x \star z \in U^+(\mathcal{P}_F; \gamma)$. Hence, $U^+(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T; \alpha)$, $L^-(\mathcal{P}_I; \beta)$, and $U^+(\mathcal{P}_F; \gamma)$ are either empty or IUP-ideals of X. Let $x \in X$. Assume that $\mathcal{P}_T(0) < \mathcal{P}_T(x)$. Let $\alpha = \mathcal{P}_T(0)$. Then $x \in U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{P}_T; \alpha)$. So $\mathcal{P}_T(0) > \alpha = \mathcal{P}_T(0)$, which is a contradiction. Thus, $\mathcal{P}_T(0) \geq \mathcal{P}_T(x)$. Let $x, y, z \in X$. Assume that $\mathcal{P}_T(x \star z) < \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}$. Let $\alpha = \mathcal{P}_T(x \star z)$. Then $x \star (y \star z), y \in U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in U^+(\mathcal{P}_T; \alpha)$. So $\mathcal{P}_T(x \star z) > \alpha = \mathcal{P}_T(x \star z)$, which is a contradiction. Thus, $\mathcal{P}_T(x \star z) \geq \min\{\mathcal{P}_T(x \star (y \star z)), \mathcal{P}_T(y)\}$.

Let $x \in X$. Assume that $\mathcal{P}_I(0) > \mathcal{P}_I(x)$. Let $\beta = \mathcal{P}_I(0)$. Then $x \in L^-(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{P}_I; \beta)$ is an IUP-ideal of X. By (2.18), we have $0 \in L^-(\mathcal{P}_I; \beta)$. So $\mathcal{P}_I(0) < \beta = \mathcal{P}_I(0)$, which is a contradiction. Thus, $\mathcal{P}_I(0) \leq \mathcal{P}_I(x)$. Let $x, y, z \in X$. Assume that $\mathcal{P}_I(x \star z) > \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}$. Let $\beta = \mathcal{P}_I(x \star z)$. Then $x \star (y \star z), y \in L^-(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{P}_I; \beta)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in L^-(\mathcal{P}_I; \beta)$. So $\mathcal{P}_I(x \star z) < \beta = \mathcal{P}_I(x \star z)$, which is a contradiction. Thus, $\mathcal{P}_I(x \star z) \leq \max\{\mathcal{P}_I(x \star (y \star z)), \mathcal{P}_I(y)\}$.

Let $x \in X$. Assume that $\mathcal{P}_F(0) < \mathcal{P}_F(x)$. Let $\gamma = \mathcal{P}_F(0)$. Then $x \in U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{P}_F; \gamma)$. So $\mathcal{P}_F(0) > \gamma = \mathcal{P}_F(0)$, which is a contradiction. Thus, $\mathcal{P}_F(0) \geq \mathcal{P}_F(x)$. Let $x, y, z \in X$. Assume that $\mathcal{P}_F(x \star z) < \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}$. Let $\gamma = \mathcal{P}_F(x \star z)$. Then $x \star (y \star z), y \in U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X. By (2.20), we have $x \star z \in U^+(\mathcal{P}_F; \gamma)$. So $\mathcal{P}_F(x \star z) > \gamma = \mathcal{P}_F(x \star z)$, which is a contradiction. Thus, $\mathcal{P}_F(x \star z) \geq \min\{\mathcal{P}_F(x \star (y \star z)), \mathcal{P}_F(y)\}$.

Hence, $\mathcal{P}$ is a neutrosophic IUP-ideal of X.

Theorem 24. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T; \alpha)$, $L^-(\mathcal{P}_I; \beta)$, and $U^+(\mathcal{P}_F; \gamma)$ are either empty or IUP-filters of X.*

Proof. Assume that $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-filter of X. Let $\alpha \in [0,1]$ be such that $U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. Let $a \in U^+(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(a) > \alpha$. By (3.5), we have $\mathcal{P}_T(0) \geq \mathcal{P}_T(a) > \alpha$. Thus, $0 \in U^+(\mathcal{P}_T; \alpha)$. Let $x, y \in U^+(\mathcal{P}_T; \alpha)$ be such that $x \star y, x \in U^+(\mathcal{P}_T; \alpha)$. Then $\mathcal{P}_T(x \star y) > \alpha$ and $\mathcal{P}_T(x) > \alpha$. Thus, $\min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\} > \alpha$. By (3.11), we have $\mathcal{P}_T(y) \geq \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\} > \alpha$. Thus, $y \in U^+(\mathcal{P}_T; \alpha)$. Hence, $U^+(\mathcal{P}_T; \alpha)$ is an IUP-filter of X.

Let $\beta \in [0,1]$ be such that $L^-(\mathcal{P}_I; \beta) \neq \emptyset$. Let $b \in L^-(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(b) < \beta$. By (3.6), we have $\mathcal{P}_I(0) \leq \mathcal{P}_I(b) < \beta$. Thus, $0 \in L^-(\mathcal{P}_I; \beta)$. Let $x, y \in L^-(\mathcal{P}_I; \beta)$ be such that $x \star y, x \in L^-(\mathcal{P}_I; \beta)$. Then $\mathcal{P}_I(x \star y) < \beta$ and $\mathcal{P}_T(x) < \beta$. Thus, $\max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\} < \beta$. By (3.12), we have $\mathcal{P}_I(y) \leq \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\} > \beta$. Thus, $y \in L^-(\mathcal{P}_I; \beta)$. Hence, $L^-(\mathcal{P}_I; \beta)$ is an IUP-ideal of X.

Let $\gamma \in [0,1]$ be such that $U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. Let $c \in U^+(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(c) > \gamma$. By (3.7), we have $\mathcal{P}_F(0) \geq \mathcal{P}_F(c) > \gamma$. Thus, $0 \in U^+(\mathcal{P}_F; \gamma)$. Let $x, y \in U^+(\mathcal{P}_F; \gamma)$ be such that $x \star y, x \in U^+(\mathcal{P}_F; \gamma)$. Then $\mathcal{P}_F(x \star y) > \gamma$ and $\mathcal{P}_F(x) > \gamma$. Thus, $\min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\} > \gamma$. By (3.13), we have $\mathcal{P}_F(y) \geq \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\} > \gamma$. Thus, $y \in U^+(\mathcal{P}_F; \gamma)$. Hence, $U^+(\mathcal{P}_F; \gamma)$ is an IUP-ideal of X.

Conversely, assume that for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^+(\mathcal{P}_T; \alpha)$, $L^-(\mathcal{P}_I; \beta)$, and $U^+(\mathcal{P}_F; \gamma)$ are either empty or IUP-filters of X. Let $x \in X$. Assume that $\mathcal{P}_T(0) < \mathcal{P}_T(x)$. Let $\alpha = \mathcal{P}_T(0)$. Then $x \in U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_T; \alpha)$ is an IUP-ideal of X. By (2.18), we have $0 \in U^+(\mathcal{P}_T; \alpha)$. So $\mathcal{P}_T(0) > \alpha = \mathcal{P}_T(0)$, which is a contradiction. Thus, $\mathcal{P}_T(0) \geq \mathcal{P}_T(x)$. Let $x, y \in X$. Assume that $\mathcal{P}_T(y) < \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}$. Let $\alpha = \mathcal{P}_T(y)$. Then $x \star y, x \in U^+(\mathcal{P}_T; \alpha) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_T; \alpha)$ is an IUP-filter of X. By (2.19), we have $y \in U^+(\mathcal{P}_T; \alpha)$. So $\mathcal{P}_T(y) > \alpha = \mathcal{P}_T(y)$, which is a contradiction. Thus, $\mathcal{P}_T(y) \geq \min\{\mathcal{P}_T(x \star y), \mathcal{P}_T(x)\}$.

Let $x \in X$. Assume that $\mathcal{P}_I(0) > \mathcal{P}_I(x)$. Let $\beta = \mathcal{P}_I(0)$. Then $x \in L^-(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{P}_I; \beta)$ is an IUP-filter of X. By (2.18), we have $0 \in L^-(\mathcal{P}_I; \beta)$. So $\mathcal{P}_I(0) < \beta = \mathcal{P}_I(0)$, which is a contradiction. Thus, $\mathcal{P}_I(0) \leq \mathcal{P}_I(x)$. Let $x, y \in X$. Assume that $\mathcal{P}_I(y) > \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}$. Let $\beta = \mathcal{P}_I(y)$. Then $x \star y, x \in L^-(\mathcal{P}_I; \beta) \neq \emptyset$. By the assumption, we have $L^-(\mathcal{P}_I; \beta)$ is an IUP-filter of X. By (2.19), we have $y \in L^-(\mathcal{P}_I; \beta)$. So $\mathcal{P}_I(y) < \beta = \mathcal{P}_I(y)$, which is a contradiction. Thus, $\mathcal{P}_I(y) \leq \max\{\mathcal{P}_I(x \star y), \mathcal{P}_I(x)\}$.

Let $x \in X$. Assume that $\mathcal{P}_F(0) < \mathcal{P}_F(x)$. Let $\gamma = \mathcal{P}_F(0)$. Then $x \in U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_F; \gamma)$ is an IUP-filter of X. By (2.18), we have $0 \in U^+(\mathcal{P}_F; \gamma)$. So $\mathcal{P}_F(0) > \gamma = \mathcal{P}_F(0)$, which is a contradiction. Thus, $\mathcal{P}_F(0) \geq \mathcal{P}_F(x)$. Let $x, y \in X$. Assume that $\mathcal{P}_F(y) < \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}$. Let $\gamma = \mathcal{P}_F(y)$. Then $x \star y, x \in U^+(\mathcal{P}_F; \gamma) \neq \emptyset$. By the assumption, we have $U^+(\mathcal{P}_F; \gamma)$ is an IUP-filter of X. By (2.19), we have $y \in U^+(\mathcal{P}_F; \gamma)$. So $\mathcal{P}_F(y) > \gamma = \mathcal{P}_F(y)$, which is a contradiction. Thus, $\mathcal{P}_F(y) \geq \min\{\mathcal{P}_F(x \star y), \mathcal{P}_F(x)\}$.

Hence, $\mathcal{P}$ is a Pythagorean neutrosophic IUP-filter of X.

Theorem 25. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the sets $U^{+}(\mathcal{P}_T;\alpha)$, $L^{-}(\mathcal{P}_I;\beta)$, and $U^{+}(\mathcal{P}_F;\gamma)$ are either empty or strong IUP-ideals of X.*

Proof. It is straightforward by Theorem 2.

Definition 11. *[18] Let $\mathcal{P}$ be a PNS in X. For any $\alpha, \beta, \gamma \in [0,1]$, the sets*

$$ULU_{\mathcal{P}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{P}_T(x) \geq \alpha, \mathcal{P}_I(x) \leq \beta, \mathcal{P}_F(x) \geq \gamma\}, \tag{3.28}$$

$$LUL_{\mathcal{P}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{P}_T(x) \leq \alpha, \mathcal{P}_I(x) \geq \beta, \mathcal{P}_F(x) \leq \gamma\}, \tag{3.29}$$

$$E_{\mathcal{P}}(\alpha,\beta,\gamma) = \{x \in X \mid \mathcal{P}_T(x) = \alpha, \mathcal{P}_I(x) = \beta, \mathcal{P}_F(x) = \gamma\} \tag{3.30}$$

are called a ULU-(α,β,γ)-level subset, an LUL-(α,β,γ)-level subset, and an E-(α,β,γ)-level subset of $\mathcal{P}$, respectively.

The following five corollaries are derived directly by applying Theorems 17, 18, 19, 20, and 21, respectively.

Corollary 1. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-subalgebra of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{P}}(\alpha,\beta,\gamma)$ is either empty or an IUP-subalgebra of X.*

Corollary 2. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{P}}(\alpha,\beta,\gamma)$ is either empty or an IUP-ideal of X.*

Corollary 3. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic IUP-filter of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{P}}(\alpha,\beta,\gamma)$ is either empty or an IUP-filter of X.*

Corollary 4. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic strong IUP-ideal of X if and only if for all $\alpha, \beta, \gamma \in [0,1]$, the set $ULU_{\mathcal{P}}(\alpha,\beta,\gamma)$ is either empty or an strong IUP-ideal of X.*

Corollary 5. *A PNS $\mathcal{P}$ in X is a Pythagorean neutrosophic strong IUP-ideal of X if and only if the set $E_{\mathcal{P}}(\mathcal{P}_T(0), \mathcal{P}_I(0), \mathcal{P}_F(0))$ is a strong IUP-ideal of X, that is, $E(\mathcal{P}_T, \mathcal{P}_T(0)) = X$, $E(\mathcal{P}_I, \mathcal{P}_I(0)) = X$ and $E(\mathcal{P}_F, \mathcal{P}_F(0)) = X$.*

4. Conclusion

In this study, we introduced and examined the concepts of Pythagorean neutrosophic IUP-subalgebras, Pythagorean neutrosophic IUP-ideals, Pythagorean neutrosophic IUP-filters, and Pythagorean neutrosophic strong IUP-ideals within the framework of IUP-algebras. We established their fundamental properties and provided necessary and sufficient conditions for Pythagorean neutrosophic sets to qualify as these algebraic subsets.

Additionally, we investigated the relationships between these subsets and their level subsets, revealing significant structural interdependencies.

By integrating Pythagorean neutrosophic sets into IUP-algebras, this research extends the theoretical foundation of algebraic structures that incorporate uncertainty and imprecision. These findings contribute to the broader fields of algebraic logic and fuzzy set theory, offering a more robust framework for mathematical modeling in decision-making, artificial intelligence, and computational uncertainty analysis.

Future research could further extend this study by incorporating soft set theory, which provides a flexible mathematical approach to dealing with parameterized uncertainties. The combination of Pythagorean neutrosophic IUP-algebras with soft sets could enhance decision-support systems in real-world applications where parameter dependency plays a crucial role. Additionally, exploring cubic set theory, which generalizes both fuzzy and neutrosophic sets by considering interval-valued membership and non-membership functions, could lead to deeper insights into algebraic structures that handle multi-dimensional uncertainty. Investigating these extensions could open new directions for theoretical advancements and practical implementations in fields such as machine learning, expert systems, and multi-criteria decision-making.

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