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Finite Γ-AG-Groupoids With Left Identities and Left Zeros

Cholathorn Chanoi[1], Apichaya Kauppamung[1], Panuwat Luangchaisri[1], Thawhat Changphas[1,*]

[1] *Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand*

Abstract. Let Γ be a nonempty set. A nonempty set A is called a Γ-AG-groupoid if there is a function f from $A \times \Gamma \times A$ into A, customary denoted $a\gamma b$ for $f(a, \gamma, b)$, satisfying the identity $(a\gamma b)\beta c = (c\gamma b)\beta a$ for any $a, b, c \in A$ and $\gamma, \beta \in \Gamma$. For each $\gamma \in \Gamma$, an operation on A associated to γ is given by $ab = a\gamma b$. Suppose further that A is finite, contains a left identity and a left zero a_0. The objective of this paper is to provide sufficient conditions under which the set $A \setminus \{a_0\}$ is a commutative group under the operation on A determined by γ for all $\gamma \in \Gamma$.

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Key Words and Phrases: AG-groupoid, Γ-AG-groupoid, semigroup, Γ-semigroup, group.

1. Introduction

An *Abel-Grassmann's groupoid*, abreviated by *AG-groupoid*, is a groupoid A such that the identity

$$(ab)c = (cb)a$$

holds for any $a, b, c \in A$.

An AG-groupoid is also called a *left almost semigroup* (it is abreviated by *LA-semigroup*), a *left invertive groupoid*, or a *right modular groupoid* (cf. [1], [2], [3], [4]).

An AG-groupoid is closely related to a commutative semigroup, because if an AG-groupoid contains a right identity, then it becomes a commutative monoid. Moreover, if an AG-groupoid A with a left identity and a left zero a_0 is finite, then (under certain conditions) $A \setminus \{a_0\}$ is a commutative group (cf. [5] Theorem 2.2). The purpose of this paper is to extend this result to Γ-AG-groupoids.

*Corresponding author.
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Email addresses: thacha@kku.ac.th (T. Changphas)

2. Preliminaries

Let Γ be a nonempty set. A nonempty set A is called a Γ-*groupoid* if there is a function f of $A \times \Gamma \times A$ into A, we often write $a\gamma b$ for $f(a, \gamma, b)$. Suppose A is a Γ-groupoid. For each $\gamma \in \Gamma$, an operation on A determined by γ is defined by for any $a, b \in A$, $ab = a\gamma b$.

A Γ-groupoid A is called a Γ-*semigroup* if the identity

$$(a\gamma b)\beta c = a\gamma(b\beta c)$$

holds for all $a, b, c \in A$ and $\gamma, \beta \in \Gamma$.

A Γ-semigroup A is said to be *commutative* if

$$a\gamma b = b\gamma a$$

for all $a, b \in A$ and $\gamma \in \Gamma$.

The notion of Γ-semigroup was introduced and studied by M. K. Sen (cf. [6], [7]). Suppose A is a semigroup. For a nonempty set Γ, define $a\gamma b = ab$ for any $a, b \in A$ and $\gamma \in \Gamma$; then A is a Γ-semigroup. If A is a Γ-semigroup, then for any $\gamma \in \Gamma$, A is a semigroup under the operation determined by γ.

Example 1. *Let A and Γ be the set of all nonpositive integers and the set of all nonpositive even integers, respectively. For $a, b \in A$ and $\gamma \in \Gamma$, define $a\gamma b$ to be the usual multiplication of integers; then A is a Γ-semigroup.*

Example 2. *Let $Mat_{2\times3}(\mathbb{Q})$ denote the set of all 2×3 matrices over $\mathbb{Q}$, the set of rational numbers. And, let Γ denote the set of all 3×2 matrices over $\mathbb{Q}$. For $A, B \in Mat_{2\times3}(\mathbb{Q})$ and $\gamma \in \Gamma$, define $A\gamma B$ to be the usual matrix product. Then $Mat_{2\times3}(\mathbb{Q})$ is a Γ-semigroup. Note that $Mat_{2\times3}(\mathbb{Q})$ is not a semigroup under the usual product of matrices.*

We need the following theorem proved in [8] (also, in [9]).

Theorem 1. *Suppose A is a Γ-semigroup. If A is a group under the operation defined by γ for some $\gamma \in \Gamma$, then A is a group under the operation determined by γ for all $\gamma \in \Gamma$.*

Let Γ be a nonempty set. A nonempty set A is called a Γ-*AG-groupoid* if there is a function f from $A \times \Gamma \times A$ into A, it is customary to write $a\gamma b$ for $f(a, \gamma, b)$, such that

$$(a\gamma b)\beta c = (c\gamma b)\beta a$$

for all $a, b, c \in A$ and $\gamma, \beta \in \Gamma$.

Suppose A is an AG-groupoid and Γ is a nonempty set. Then A is a Γ-AG-groupoid under the function defined by $a\gamma b = ab$ for all $a, b \in A$ and $\gamma \in \Gamma$. If A is a Γ-AG-groupoid, then for any $\gamma \in \Gamma$, A is an AG-groupoid under the operation determined by γ.

Example 3. *Let $\Gamma = \{1, 2, \ldots, n\}$. Define a function from $\mathbb{Z} \times \Gamma \times \mathbb{Z}$ into $\mathbb{Z}$, the set of all integers, by*

$$a\gamma b = b - \gamma - a$$

for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$, where $-$ is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-AG-groupoid.

Example 4. *Let $A = \Gamma = \{0, i, -i\}$. Define a function from $A \times \Gamma \times A$ into A by $a\gamma b$ for all $(a, \gamma, b) \in A \times \Gamma \times A$; here $a\gamma b$ is the usual multiplication of complex numbers. Then A is a Γ-AG-groupoid, where as A is not an AG-groupoid.*

The following theorem is in [10] (see also in [11]).

Theorem 2. *Any Γ-AG-groupoid satisfies the medial law. That is, if A is a Γ-AG-groupoid, then*

$$(a\gamma b)\beta(c\alpha d) = (a\gamma c)\beta(b\alpha d)$$

for any $a, b, c, d \in A$ and $\gamma, \beta, \alpha \in \Gamma$.

An element e of a Γ-AG-groupoid A is said to be a *left identity* if for all $a \in A$ and $\gamma \in \Gamma$,

$$e\gamma a = a.$$

An element a_0 of a Γ-AG-groupoid A is said to be a *left zero* if for all $a \in A$ and $\gamma \in \Gamma$,

$$a_0\gamma a = a_0.$$

A Γ-AG-groupoid A is said to be *cancellative* if for all $a, b, c \in A$ and $\gamma \in \Gamma$,

$$(a\gamma c = b\gamma c \text{ or } c\gamma a = c\gamma b) \text{ imply } a = b.$$

Example 5. *Consider an AG-groupoid $A = \{1, 2, 3, 4\}$ with the operation defined as follows:*

$\cdot$	1	2	3	5
1	1	2	3	5
2	3	3	3	3
3	3	3	3	3
5	2	3	3	3

Let Γ be a nonempty set. For $a, b \in A$ and $\gamma \in \Gamma$, define $a\gamma b = a \cdot b$. We have that A is a finite Γ-AG-groupoid with left identity 1, and left zero 3.

3. Results

We begin this section with the following theorem.

Theorem 3. *If A is a Γ-AG-groupoid satisfying the identity*

$$a\gamma(b\beta c) = (c\gamma b)\beta a$$

for all $a, b, c \in A$ and $\gamma, \beta \in \Gamma$, then A is a Γ-semigroup.

Proof. Assume the condition holds. Then, for $a, b, c \in A$ and $\gamma, \beta \in \Gamma$, we have

$$(a\gamma b)\beta c = (c\gamma b)\beta a = a\gamma(b\beta c).$$

Thus A is a Γ-semigroup.

Theorem 4. *If A is a cancellative Γ-AG-groupoid satisfying the identity*

$$a\gamma(b\beta c) = (c\gamma b)\beta a$$

for all $a, b, c \in A$ and $\beta, \gamma \in \Gamma$, then for any $\gamma \in \Gamma$, A is a commutative semigroup under the operation determined by γ.

Proof. Assume the condition holds. By Theorem 3, for any $\gamma \in \Gamma$, A is a semigroup under the operation determined by γ. Let $a, b \in A$ and $\gamma \in \Gamma$. Consider:

$$\begin{aligned}(a\gamma(a\gamma b))\gamma a &= ((a\gamma a)\gamma b)\gamma a \\ &= (a\gamma a)\gamma(b\gamma a) \\ &= (a\gamma b)\gamma(a\gamma a) \\ &= ((a\gamma b)\gamma a)\gamma a.\end{aligned}$$

So $(a\gamma(a\gamma b))\gamma a = ((a\gamma b)\gamma a)\gamma a$. By cancellative law, $a\gamma(a\gamma b) = (a\gamma b)\gamma a$. By assumption, $a\gamma(a\gamma b) = a\gamma(b\gamma a)$. Using cancellative law, $a\gamma b = b\gamma a$. Hence A is a commutative semigroup under the operation determined by γ.

An AG-groupoid A is said to be *cancellative* if for all $a, b, c \in A$, $ac = bc$ or $ca = cb$ imply $a = b$. We specifically have the following corollary:

Corollary 1. *If A is a cancellative AG-groupoid satisfying the identity*

$$a(bc) = (cb)a$$

for all $a, b, c \in A$, then A is a commutative semigroup.

Now, we present the main result.

Theorem 5. *Let A be a finite Γ-AG-groupoid containing at least two elements ($|A| > 1$). Suppose A contains a left identity e and a left zero a_0, and A satisfies the identity*

$$a\gamma(b\beta c) = (c\gamma b)\beta a$$

for all $a, b, c \in A$ and $\gamma, \beta \in \Gamma$. Suppose further that there exist $\gamma_0 \in \Gamma$ and an operation $*$ of $A \times A$ into A, write $a * b$ for $*(a, b)$, such that (i)-(v) hold:

(i) *A is an AG-groupoid under $*$.*

(ii) *For any $a \in A$ there exists $b \in A$ such that $b * a = a_0$.*

(iii) *$a_0 * a = a$ for all $a \in A$.*

(iv) *$(a * b)\gamma_0 c = (a\gamma_0 c) * (b\gamma_0 c)$ for all $a, b, c \in A$.*

(*v*) *For any* $a, b \in A$, *if* $a\gamma_0 b = a_0$ *then* $a = a_0$ *or* $b = a_0$.

Then $A \setminus \{a_0\}$ *is a commutative group under the operation determined by* γ *for all* $\gamma \in \Gamma$.

Proof. Let

$$A = \{a_0, a_1, \ldots, a_m\}, \text{ where } m \geq 1.$$

Claim 1: $A \setminus \{a_0\}$ is an AG-groupoid under the operation determined by γ_0. Since $m \geq 1$, $A \setminus \{a_0\} \neq \emptyset$. Suppose $a_i\gamma_0 a_j = a_0$ for some $a_i, a_j \in A \setminus \{a_0\}$. By (v), $a_i = a_0$ or $a_j = a_0$. This is a contradiction. Thus $a_i\gamma_0 a_j \in A \setminus \{a_0\}$ for all $a_i, a_j \in A \setminus \{a_0\}$; so $A \setminus \{a_0\}$ is a groupoid under the operation determined by γ_0. From $A \setminus \{a_0\} \subseteq A$, it follows that $(a_i\gamma_0 a_j)\gamma_0 a_k = (a_k\gamma_0 a_j)\gamma_0 a_i$ for all $a_i, a_j, a_k \in A \setminus \{a_0\}$.
Claim 2: $e \neq a_0$. Suppose not. If $a_i \in A$ then

$$a_i = e\gamma_0 a_i = a_0\gamma_0 a_i = a_0.$$

Thus $A = \{a_0\}$, this is a contradiction. So $e \neq a_0$.
Claim 3: $a_0\gamma_0 a_i = a_i\gamma_0 a_0$ for all $a_i \in A$. Let $a_i \in A$. Consider:

$$(a_i\gamma_0 a_0)\gamma_0 e = (e\gamma_0 a_0)\gamma_0 a_i = a_0\gamma_0 a_i = a_0.$$

By (v), $a_i\gamma_0 a_0 = a_0$ or $e = a_0$. By Claim 2, $a_i\gamma_0 a_0 = a_0$. Hence $a_0\gamma_0 a = a_0 = a_i\gamma_0 a_0$.
Claim 4: For each $a_k \in A \setminus \{a_0\}$ there exists $a_k^{-1} \in A \setminus \{a_0\}$ such that

$$a_k\gamma_0 a_k^{-1} = e = a_k^{-1}\gamma_0 a_k.$$

Let $a_k \in A \setminus \{a_0\}$. Consider:

$$H_{k,\gamma_0} = \{a_k\gamma_0 a_1, a_k\gamma_0 a_2, \ldots, a_k\gamma_0 a_m\}.$$

To show that $|H_{k,\gamma_0}| = m$, suppose $a_k\gamma_0 a_r = a_k\gamma_0 a_s$ for some $r \neq s$. Consider:

$$\begin{aligned} a_r\gamma_0 a_k &= (e\gamma_0 a_r)\gamma_0 a_k \\ &= (a_k\gamma_0 a_r)\gamma_0 e \\ &= (a_k\gamma_0 a_s)\gamma_0 e \\ &= (e\gamma_0 a_s)\gamma_0 a_k \\ &= a_s\gamma_0 a_k. \end{aligned}$$

By (ii), there exists $a_r^{-1} \in A$ such that $a_r^{-1} * a_r = a_0$. Consider (Using (i), (iii), (iv)):

$$\begin{aligned} (a_s * a_r^{-1})\gamma_0 a_k &= (a_s\gamma_0 a_k) * (a_r^{-1}\gamma_0 a_k) \\ &= (a_r\gamma_0 a_k) * (a_r^{-1}\gamma_0 a_k) \\ &= (a_r * a_r^{-1})\gamma_0 a_k \\ &= (a_0 * (a_r * a_r^{-1}))\gamma_0 a_k \end{aligned}$$

$$
\begin{aligned}
&= ((a_0 * a_0) * (a_r * a_r^{-1}))\gamma_0 a_k \\
&= (((a_r * a_r^{-1}) * a_0) * a_0)\gamma_0 a_k \\
&= (((a_0 * a_r^{-1}) * a_r) * a_0)\gamma_0 a_k \\
&= ((a_r^{-1} * a_r) * a_0)\gamma_0 a_k \\
&= (a_0 * a_0)\gamma_0 a_k \\
&= a_0\gamma_0 a_k \\
&= a_0.
\end{aligned}
$$

By (v), $a_s * a_r^{-1} = a_0$ or $a_k = a_0$. Since $a_k \neq a_0$, $a_s * a_r^{-1} = a_0$. Consider:

$$
\begin{aligned}
a_r &= a_0 * a_r \\
&= (a_s * a_r^{-1}) * a_r \\
&= (a_r * a_r^{-1}) * a_s \\
&= (a_0 * (a_r * a_r^{-1})) * a_k \\
&= ((a_0 * a_0) * (a_r * a_r^{-1})) * a_k \\
&= (((a_r * a_r^{-1}) * a_0) * a_0) * a_k \\
&= (((a_0 * a_r^{-1}) * a_r) * a_0) * a_k \\
&= ((a_r^{-1} * a_r) * a_0) * a_k \\
&= (a_0 * a_0) * a_k \\
&= a_0 * a_s \\
&= a_s.
\end{aligned}
$$

Then $a_r = a_s$; this is a contradiction. Hence $|H_{k,\gamma_0}| = m$. Let $a_k\gamma_0 a_j \in H_{k,\gamma_0}$. Suppose $a_k\gamma_0 a_j = a_0$. By (v), $a_k = a_0$ or $a_j = a_0$. This is a contradiction. Then $a_k\gamma_0 a_j \neq a_0$, and $a_k\gamma_0 a_j \in A \setminus \{a_0\}$. So $H_{k,\gamma_0} \subseteq A \setminus \{a_0\}$. From $|H_{k,\gamma_0}| = m = |A \setminus \{a_0\}|$, it follows that

$$H_{k,\gamma_0} = A \setminus \{a_0\}.$$

Since $e \in H_{k,\gamma_0}$, $e = a_k\gamma_0 a_i$ for some $a_i \in A \setminus \{a_0\}$. Moreover,

$$a_i\gamma_0 a_k = e\gamma_0(a_i\gamma_0 a_k) = (a_k\gamma_0 a_i)\gamma_0 e = e\gamma_0 e = e.$$

Setting $a_k^{-1} = a_i$, we then have $a_k\gamma_0 a_k^{-1} = e = a_k^{-1}\gamma_0 a_k$. By Claim 4, $A \setminus \{a_0\}$ is a group under the operation determined by γ_0. And, by Theorem 1, $A \setminus \{a_0\}$ is a group under the operation determined by γ for all $\gamma \in \Gamma$. Finally, by Theorem 4, we conclude that $A \setminus \{a_0\}$ is a commutative group under the operation determined by γ for all $\gamma \in \Gamma$. This completes the proof.

An element e of an AG-groupoid A is said to be a *left identity* if for all $a \in A$, $ea = a$. An element a_0 of A is said to be a *left zero* if for all $a \in A$, $a_0 a = a_0$. The following corollary is particularly true.

Corollary 2. *Let $(A, \cdot)$ be a finite AG-groupoid with $|A| > 1$, a left identity e, a left zero a_0, and*

$$a \cdot (b \cdot c) = (c \cdot b) \cdot a$$

for all $a, b, c \in A$. Suppose there exists an operation $*$ on A such that (i)-(v) hold:

(i) *$(A, *)$ is an AG-groupoid.*

(ii) *For any $a \in A$ there exists $b \in A$ such that $b * a = a_0$.*

(iii) *$a_0 * a = a$ for all $a \in A$.*

(iv) *$(a * b) \cdot c = (a \cdot c) * (b \cdot c)$ for all $a, b, c \in A$.*

(v) *For any $a, b \in A$, $a \cdot b = a_0$ implies $a = a_0$ or $b = a_0$.*

Then $(A \setminus \{a_0\}, \cdot)$ is a commutative group.

The following proposition demonstrates the necessity of the identity $a\gamma(b\beta c) = (c\gamma b)\beta a$ for all $a, b, c \in A$ and $\gamma, \beta \in \Gamma$ as stated in Theorem 5.

Proposition 1. *Let A be a finite Γ-AG-groupoid containing at least two elements ($|A| > 1$). Suppose A contains a left identity e and a left zero a_0. Suppose further that there exist $\gamma_0 \in \Gamma$ and an operation $*$ of $A \times A$ into A, write $a * b$ for $*(a, b)$, such that (i)-(v) hold:*

(i) *A is an AG-groupoid under $*$.*

(ii) *For any $a \in A$ there exists $b \in A$ such that $b * a = a_0$.*

(iii) *$a_0 * a = a$ for all $a \in A$.*

(iv) *$(a * b)\gamma_0 c = (a\gamma_0 c) * (b\gamma_0 c)$ for all $a, b, c \in A$.*

(v) *For any $a, b \in A$, if $a\gamma_0 b = a_0$ then $a = a_0$ or $b = a_0$.*

Then, under the operation determined by γ_0, $A \setminus \{a_0\}$ is a cancellative AG-groupoid with left identity and inverses (i.e., for each $a_k \in A \setminus \{a_0\}$ there exists $a_k^{-1} \in A \setminus \{a_0\}$ such that $a_k\gamma_0 a_k^{-1} = e = a_k^{-1}\gamma_0 a_k$).

Proof. As the proof of Theorem 5, under the operation determined by γ_0, we have $A \setminus \{a_0\}$ is an AG-groupoid with left identity, and for any $a_k \in A \setminus \{a_0\}$, $e = a_k\gamma_0 a_i$ for some $a_i \in A \setminus \{a_0\}$. Consider (using Theorem 2):

$$\begin{aligned} a_i\gamma_0 a_k &= e\gamma_0(a_i\gamma_0 a_k) \\ &= (e\gamma_0 e)\gamma_0(a_i\gamma_0 a_k) \\ &= (e\gamma_0 a_i)\gamma_0(e\gamma_0 a_k) \\ &= ((e\gamma_0 a_k)\gamma_0 a_i)\gamma_0 e \\ &= (a_k\gamma_0 a_i)\gamma_0 e \end{aligned}$$

$$= e\gamma_0 e$$
$$= e.$$

Then $a_i\gamma_0 a_k = e$.

Finally, let $a_i, a_j, a_k \in A \setminus \{a_0\}$ be such that $a_i\gamma_0 a_k = a_j\gamma_0 a_k$. Moreover, as the proof of Theorem 5, there exists $a_k^{-1} \in A \setminus \{a_0\}$ such that

$$a_k^{-1}\gamma_0 a_k = e = a_k\gamma_0 a_k^{-1}.$$

Consider:

$$\begin{aligned} a_i &= e\gamma_0 a_i \\ &= (a_k^{-1}\gamma_0 a_k)\gamma_0 a_i \\ &= (a_i\gamma_0 a_k)\gamma_0 a_k^{-1} \\ &= (a_j\gamma_0 a_k)\gamma_0 a_k^{-1} \\ &= (a_k^{-1}\gamma_0 a_k)\gamma_0 a_j \\ &= e\gamma_0 a_j \\ &= a_j. \end{aligned}$$

Similarly, if $a_i, a_j, a_k \in A \setminus \{a_0\}$ such that $a_k\gamma_0 a_i = a_k\gamma_0 a_j$ then $a_i = a_j$. Hence the proof is complete.

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