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Inertial Iterative Method for Generalized Mixed Equilibrium Problem and Fixed Point Problem

Vahid Darvish[1], Grace Nnennaya Ogwo[2,*], Olawale Kazeem Oyewole[3,4], Hammed Anuoluwapo Abass[5,6], Amirbek Aminovich Ikramov[6]

[1] *Department of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing China*
[2] *School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China*
[3] *Department of Mathematics and Statistics, Tshwane University of Technology, PMB 007, Arcadia, Pretoria, South Africa*
[4] *Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602 105, Tamil Nadu, India*
[5] *Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University, P.O. Box 94, Pretoria 0204, South Africa*
[6] *Center of Research and Innovation, Asia International University, Yangiobod MFY, G'ijduvon Street, House 74, Bukhara, Uzbekistan*

Abstract. In this paper, we study the generalized mixed equilibrium problem and the fixed point problem. We propose an inertial iterative method for approximating the common solution of a generalized mixed equilibrium problem of a monotone mapping and a fixed point problem for a Bregman strongly nonexpansive mapping in the framework of real reflexive Banach spaces. Under certain mild conditions, we obtain a strong convergence result of the proposed method. Finally, we present numerical examples to illustrate the applicability of our method.

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Key Words and Phrases: Generalized mixed equilibrium problem, fixed point problem, inertial technique, Bregman strongly nonexpansive mapping

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1. Introduction

Let E be a real reflexive Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, $\Theta : C \times C \to \mathbb{R}$ a bifunction, $\Psi : C \to E^*$ a nonlinear

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Email addresses: vahid.darvish@mail.com, vdarvish@nuist.edu.cn (V. Darvish), graceogwo@zjnu.edu.cn (G. N. Ogwo), oyewoleolawalekazeem@gmail.com (O. K. Oyewole), hammedabass548@gmail.com,hammed.abass@smu.ac.za (H. A. Abass), amirbekikramov@oxu.uz (A. A. Ikramov)

mapping and $\varphi : C \to \mathbb{R}$ a real valued function. The *generalized mixed equilibrium problem* (GMEP) is defined as follows: Find $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \text{for all } \ y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GMEP(Θ, Ψ, φ), that is

$$\text{GMEP}(\Theta, \Psi, \varphi) = \{x \in C : \ \Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \text{for all } \ y \in C\}.$$

In particular, if $\Psi \equiv 0$, the problem (1.1) is reduced to the *mixed equilibrium problem* (MEP) [1] defined as follows: Find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) \geq \varphi(x), \quad \text{for all } \ y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $MEP(\Theta, \varphi)$.
If $\varphi \equiv 0$, the problem (1.1) is reduced to the *generalized equilibrium problem* (GEP) [2] defined as follows: Find $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x \rangle \geq 0, \quad \text{for all } \ y \in C. \tag{1.3}$$

The set of solution (1.3) is denoted by $GEP(\Theta, \Psi)$.
If $\Theta \equiv 0$, the problem (1.1) is reduced to the *mixed variational inequality of Browder type* (MVI) [3] defined as follows: Find $x \in C$ such that

$$\langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad \text{for all } \ y \in C. \tag{1.4}$$

The set of solution of (1.4) is denoted by $MVI(\varphi, \Psi)$.
If $\Psi \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced to the *equilibrium problem* (EP) [4] for finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \text{for all } \ y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $EP(\Theta)$.
We observe that (1.1) generalizes (1.2)-(1.5).
The equilibrium problem was introduced by Blum and Oettli [4] and Noor and Oettli [5] in 1994, and has had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provides a novel and unified treatment of a wide class of problems that arise in economics, finance, image reconstruction, ecology, transportation, networks, elasticity, and optimization. The EP was shown in [4] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibrium problems in noncooperative games. Some methods have been proposed to solve the equilibrium problem. These methods include the penalty and gap functions, regularization, extragradient methods and splitting methods (see [6–10] and other references therein).
For solving the generalized mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(Q_1) $\Theta(x,x) = 0$ for all $x \in C$;
(Q_2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leq 0$ for any $x, y \in C$;
(Q_3) for each $y \in C, x \mapsto \Theta(x,y)$ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \searrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x,y);$$

(Q_4) for each $x \in C, y \mapsto \Theta(x,y)$ is convex and lower semicontinuous (see [11]).
Several authors have studied and proposed various iterative methods for studying (1.1). Tuyen [12] introduced a hybrid projection method for solving systems of GMEP in a reflexive Banach space and defined it as follows:

$$\begin{cases} y_n^i = Res^f_{\Theta_i, \Psi_i, \varphi_i} x_n, \ i = 1, 2, \cdots, N \\ i_n := \arg\max_{i=1,2\cdots,N}\{D_f(y_n^i, x_n)\}, \ \bar{y}_n = y_n^{i_n} \\ C_n := \{z \in E : D_f(z, \bar{y}_n \leq D_f(z, x_n))\} \\ Q_n := \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\} \\ x_{n+1} = \text{Proj}_{C_n \cap Q_n}(x_0), \quad n \geq 0. \end{cases}$$

The author obtained a strong convergence result of the proposed method. The limitation OF this method is the fact that it requires the computation of subsets of C_n and Q_n which can be computationally expensive.
On the other hand, another problem of interest is the fixed point problem (FPP). Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is a fixed point of T if $Tx = x$. Let $F(T)$ denote the set of fixed points, that is

$$F(T) = \{x \in C : Tx = x\}.$$

Many problems in sciences and engineering can be transformed into a problem of finding the solution of a fixed point problem (FPP) of a nonlinear mapping. For more information on fixed point see [13–17]. Moudafi [18] introduced the viscosity approximation method for a nonexpansive mapping T and defined it as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and f is a contraction mapping.
Recently, several authors have studied iterative algorithms for finding a common solution of the FFP and GMEP. In particular, several authors have considered the following problem (see [19–21] and other references therein):

Find $x \in C$ such that

$$x \in F(T) \cap \text{GMEP}(\Theta, \Psi, \varphi).$$

The motivation for studying a common solution problem lies in its application to problems whose constraints can be reformulated as FPPs and GMEPs. For instance, in signal processing, network resource allocation, among others.

Recently, Takahashi and Takahashi [22] introduced the following iterative scheme for solving GEP and FPP of a nonexpansive mapping T in a Hilbert space. They defined the proposed method as follows: Find $x_1, z \in C$ and

$$\begin{cases} z_n \in C \quad \text{such that} \\ \Theta(z_n, y) + \langle Bx_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ y \in C \\ x_{n+1} = \beta_n x_n + (1-\beta_n)T\Big[\alpha_n z + (1-\alpha_n)z_n\Big], \ n \geq 1 \end{cases}$$

$\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{r_n\} \subset (0,+\infty)$ and B is an α-inverse strongly monotone mapping. The authors obtained a strong convergent result under certain conditions. Eskandari and Raeisi [23] introduced an iterative method for approximating a common solution of FPP of Bregman quasi-nonexpansive mappings and zeros of maximal monotone operators. They defined the algorithm as follows:

$$\begin{cases} x_1 \in E, \\ z_n = Res^f_{\lambda_n^N B_n} \circ \ldots \circ Res^f_{\lambda_n^1 B_1} x_n, \\ y_n = \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(T_n(z_n)), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(q_n) + (1-\alpha_n)y_n), \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$. Under certain standard conditions, the authors obtained a strong convergence result.

The ultimate aim of every researcher is to construct effective iterative methods with high convergence rates to the solutions of the optimization problem under consideration. To achieve this high rate of convergence, authors employ the inertial technique. Polyak [24] introduced the inertial extrapolation as an acceleration process to solve smooth convex minimization problems. It has been shown by several authors that the inertial term improves the performance of iterative algorithms numerically in terms of the number of iterations and CPU time. Several authors have studied and proposed iterative algorithms with the inertial technique for solving optimization problems (see to [25–27] and other references therein).

Motivated by the above mentioned methods in the literature and the ongoing research in this area, we introduce a new inertial iterative method for approximating the solutions of a generalized mixed equilibrium problem with a maximal monotone mapping and fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach space. Our method does not require us to compute subsets of C_n and Q_n. Under mild condition, we establish a strong convergence result for the proposed method. Finally, we present numerical examples to illustrate the applicability of our proposed method.

The rest of the paper is organized as follows: In Section 2, we present some basic definitions, concepts, lemmas, and results which will be required to obtain the convergence analysis of the proposed method. In Section 3, we present some required assumptions and introduce our proposed method. In Section 4, we present our convergence analysis. In Section 5, we present numerical experiments in comparisons with other related methods to illustrate the effectiveness of our proposed method. In Section 6, we present a brief summary of our result.

2. Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. We denote by dom$f$, the domain of f, that is the set $\{x \in E : f(x) < +\infty\}$. For a sequence $\{x_n\}$ in E, we denote the strong and weak convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.
Let $x \in \text{int}(\text{dom}f)$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \text{ for all } \ y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E, x^* \in E^*\}.$$

For any $x \in \text{int}(\text{dom}f)$, the right-hand derivative of f at x in the derivation $y \in E$ is defined by

$$f^{'}(x, y) := \lim_{t\searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t\searrow 0} \frac{f(x+ty)-f(x)}{t}$ exists for all $y \in E$. In this case, $f^{'}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient (∇f) of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom}f)$ and f is called Fréchet differentiable at x if this limit is attain uniformly for all y which satisfies $\|y\| = 1$. The function f is uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for any $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on int(dom$f$), then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on int(dom$f$) (see [28]).
Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$ defined as follows:

$$D_f(x, y) := f(x) - f(y) - \langle\nabla f(y), x - y\rangle, \quad \text{for all} \quad x \in \text{dom}(f),\ y \in \text{int}(\text{dom}(f)) \tag{2.1}$$

is called the Bregman distance with respect to f, [29].

Remark 1. *The Bregman distance has the following properties:*

(i) the three-point identity, for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle\nabla f(z) - \nabla f(y), x - y\rangle; \tag{2.2}$$

[30]

(ii) the four-point identity, for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle\nabla f(z) - \nabla f(x), y - w\rangle. \tag{2.3}$$

Definition 1. *A Gâteaux differentiable function f is said to be γ-strongly convex if there exists a constant $\gamma > 0$ such that*

$$f(x) \geq f(y) + \langle x - y, \nabla f(y)\rangle + \frac{\gamma}{2}\|x-y\|^2, \quad \text{for all} \quad x \in \mathrm{dom}(f), y \in \mathrm{int}(\mathrm{dom}(f)).$$

Lemma 1. *[31] Let f be a strongly convex function with constant $\gamma > 0$. Then for all $y \in \mathrm{dom}(f)$ and $x \in \mathrm{int}(\mathrm{dom}(f))$, we have:*

$$D_f(x,y) \geq \frac{\gamma}{2}\|x-y\|^2, \tag{2.4}$$

where $D_f(x,y)$ is the Bregman distance with respect to f.

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [32]. It is well known that in reflexive spaces, f is Legendre function if and only if it satisfies the following conditions:
(L_1) The interior of the domain of f, $\mathrm{int}(\mathrm{dom}f)$, is nonempty, f is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}f)$ and $\mathrm{dom}f = \mathrm{int}(\mathrm{dom}f)$;
(L_2) The interior of the domain of f^*, $\mathrm{int}(\mathrm{dom}f^*)$, is nonempty, f^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}f^*)$ and $\mathrm{dom}f^* = \mathrm{int}(\mathrm{dom}f^*)$.
Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see [28]). This , with (L_1) and (L_2), imply the following equalities:

$$\nabla f = (\nabla f^*)^{-1}, \quad \mathrm{ran}\nabla f = \mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom}f^*)$$

and

$$\mathrm{ran}\nabla f^* = \mathrm{dom}(\nabla f) = \mathrm{int}(\mathrm{dom}f),$$

where $\mathrm{ran}\nabla f$ denotes the range of ∇f.
When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$, [33]. By Bauschke et al. [32] the conditions (L_1) and (L_2) also yields that the function f and f^* are strictly convex on the interior of their respective domains.
If E is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := \frac{1}{p}\|x\|^p (1 < p < +\infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, i.e., $\nabla f = J_p (1 < p < +\infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. From now on we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre. In connection with Legendre functions, see also the recent paper [34].

Definition 2. *Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection of $x \in$ int(domf) onto the nonempty, closed and convex subset $C \subset$ domf is the necessary unique vector $proj_C^f(x) \in C$ satisfying*

$$D_f(proj_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

Remark 2. *If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2Jx$ for all $x \in E$, where J is the normalized duality* *mapping from E in to 2^{E^*}, and hence $D_f(x,y)$ reduced to $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$, which is the Lyapunov function introduced by Alber [35] and Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by*

$$\phi(\Pi_C(x), x) = \min_{y\in C} \phi(y, x).$$

If $E = H$, a Hilbert space, J is the identity mapping and hence Bregman projection $P_C^f(x)$ reduced to the metric projection of H onto C, $P_C(x)$.

Definition 3. *[36, 37] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:*

(i) totally convex at $x \in int(domf)$ if its modulus of total convexity at x, that is, the function $\nu_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in domf, \|y - x\| = t\},$$

is positive whenever $t > 0$;

(ii) totally convex if it is totally convex at every point $x \in int(domf)$;

(iii) totally convex on bounded sets if $\nu_f(B,t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : int(domf) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap domf\}.$$

The set $lev_{\leq}^f(r) = \{x \in E : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a sublevel of f.

Definition 4. *[37, 38] The function $f : E \to (-\infty, +\infty]$ is called;*

(i) cofinite if $domf^* = E^*$;

(ii) coercive [39] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty;$$

(iii) strongly coercive if $\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty$;

(iv) sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 2. *[40] The function f is totally convex on bounded subsets if and only if it is sequentially consistent.*

Lemma 3. *[38, Proposition 2.3] If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 4. *[40] Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points.Then the following statements hold:*

(i) f is sequentially consistent if and only if it is totally convex on bounded sets;

(ii) If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets;

(iii) If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and Fréchet derivative ∇f is uniformly continuous on bounded sets.

Lemma 5. *[41, Proposition 2.1] Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 6. *[38, Lemma 3.1] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$. A point $p \in C$ is called an asymptotic fixed point of T (see [42]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T.
A mapping $T : C \to \text{int}(\text{dom}f)$ with $F(T) \neq \emptyset$ is called:

(i) quasi-Bregman nonexpansive [38] with respect to f if

$$D_f(p, Tx) \leq D_f(p, x), \text{ for all } x \in C, p \in F(T).$$

(ii) Bregman relatively nonexpansive [38, 43] with respect to f if,

$$D_f(p, Tx) \leq D_f(p, x), \quad \text{for all } x \in C, p \in F(T), \quad \text{and } \widehat{F}(T) = F(T).$$

(iii) Bregman strongly nonexpansive (see [38, 44]) with respect to f and $\widehat{F}(T)$ if,

$$D_f(p, Tx) \leq D_f(p, x), \quad \text{for all } x \in C, p \in \widehat{F}(T)$$

and, if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{z\to+\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n\to+\infty} D_f(x_n, Tx_n) = 0.$$

(iv) Bregman firmly nonexpansive (for short BFNE [45]) with respect to f if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle$$

equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \quad (2.5)$$

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [42]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex. It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Lemma 7. *[40] Let C be a nonempty, closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$, then*
1) $z = proj_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \qquad \text{for all } \ y \in C.$$

2) $D_f(y, proj_C^f(x)) + D_f(proj_C^f(x), x) \leq D_f(y, x), \qquad$ for all $\ x \in E, y \in C$.

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [35] and [29], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x \rangle + f^*(x^*), \qquad \text{for all } \ x \in E, x^* \in E^*. \quad (2.6)$$

Then V_f is nonexpansive and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*) \quad (2.7)$$

for all $x \in E$ and $x^*, y^* \in E^*$ [46]. In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function (see [47]). Hence, V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \quad (2.8)$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Definition 5. *Let $B_r = \{x \in E : \|x\| \leq r\}$ for all $r > 0$. Then a function $g : E \longrightarrow \mathbb{R}$ is said to be uniformly convex on bounded sets of E [48] if $\rho_r(t) > 0$ for all $r, t > 0$ where $\rho_r : [0, +\infty) \longrightarrow [0, +\infty]$ is defined by*

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g[\alpha x + (1-\alpha)y]}{\alpha(1-\alpha)}$$

for all $t > 0$. The function ρ_r is called the gauge of uniform convexity of g. The function g is said to be uniformly convex if the function $\delta_g : [0, +\infty) \longrightarrow [0, +\infty]$, defined by

$$\delta_g(t) = \sup_{\|x-y\|=t} \left\{ \frac{1}{2}g(x) + \frac{1}{2}g(y) - g\left(\frac{x+y}{2}\right) \right\}$$

satisfies that $\lim_{t \downarrow 0} \dfrac{\delta_g(t)}{t} = 0$.

Lemma 8. *[49] Let C be a nonempty, closed and convex subset of int(dom$f$) and $T : C \to C$ be a quasi-Bregman nonexpansive mappings with respect to f. Then $F(T)$ is closed and convex.*

Definition 6. *Let C be a nonempty, closed and convex subsets of a real reflexive Banach space and let φ be a lower semicontinuous and convex functional from C to $\mathbb{R}$ and $\Psi : C \to E^*$ be a continuous monotone mapping. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunctional satisfying (A_1)-(A_4). The mixed resolvent of Θ is the operator $Res^f_{\Theta,\varphi,\Psi} : E \to 2^C$*

$$\begin{aligned} Res^f_{\Theta,\varphi,\Psi}(x) \;&=\; \{z \in C : \Theta(z,y) + \varphi(y) + \langle \Psi z, y - z\rangle + \langle \nabla f(z) - \nabla f(x), y - z\rangle \\ &\geq \varphi(z), \quad \text{for all} \;\; y \in C\}. \end{aligned} \tag{2.9}$$

Lemma 9. *[50] Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of E. Assume that $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional, $\Psi : C \to E^*$ be a continuous monotone mapping and the bifunctional $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then dom($Res^f_{\Theta,\varphi,\Psi}$) $= E$.*

Lemma 10. *[50] Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a closed and convex subset of E. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4), then*

(i) $Res^f_{\Theta,\varphi,\Psi}$ *is single-valued;*

(ii) $Res^f_{\Theta,\varphi,\Psi}$ *is a BFNE operator;*

(iii) $F\left(Res^f_{\Theta,\varphi,\Psi}\right) = GMEP(\Theta, \varphi, \Psi)$;

(iv) $GMEP(\Theta, \varphi, \Psi)$ *is closed and convex;*

(v) $D_f\left(p, Res^f_{\Theta,\varphi,\Psi}(x)\right) + D_f\left(Res^f_{\Theta,\varphi,\Psi}(x), x\right) \leq D_f(p, x)$, *for all* $p \in F\left(Res^f_{\Theta,\varphi,\Psi}\right), x \in E$.

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction and define the mapping $B_\Theta : E \to 2^{E^*}$ in the following way:

$$B_\Theta(x) := \begin{cases} \{x^* \in E^* : \Theta(x,y) + \varphi(y) + \langle \Psi x, y-x\rangle \geq \langle x^*, y-x\rangle + \varphi(x) \quad \text{for all} \quad y \in C\}, \\ x \in C, \\ \emptyset \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \notin C. \end{cases} \tag{2.10}$$

Lemma 11. *[51] Let E be a Banach space and $f : E\mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequenecs in E. Then, $\lim_{n\to+\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to+\infty} \|x_n - y_n\| = 0$.*

Lemma 12. *[52] Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{+\infty} \alpha_n = +\infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad \textit{for all} \quad n \geq 1.$$

If $\limsup_{k\to+\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to+\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0$, then $\lim_{n\to+\infty} a_n = 0$.

3. Main result

In this section, we present the assumptions under which our convergence analysis will be obtained. Furthermore, we present our proposed algorithm.

Assumption 3.1. Assumption A:
(A1) E is a reflexive Banach space with dual E^* and C is a nonempty, closed and convex subset of $\text{int}(\text{dom}(f))$.
(A2) $T : C \to C$ is a Bregman strongly nonexpansive mapping such that $F(T) = \hat{F}(T)$ and T is uniformly continuous.
(A3) $f : E \to \mathbb{R}$ is a super coercive Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E.
(A4) $B_{\theta_j} : E \to 2^{E^*}, j = 1, 2, \ldots N$ is a maximal monotone mapping with $\text{dom}(B_{\theta_j}) \subset C$.
(A5) The solution set $\Omega = F(T) \cap \left(\bigcap_{j=1}^N B_{\theta_j}^{-1}(0^*)\right) \neq \emptyset$.

Assumption B:
(B1) Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying the following $\lim_{n\to+\infty} \beta_n = 0$ and $\sum_{n=1}^{+\infty} \beta_n = +\infty$. $0 < \liminf_{n\to+\infty} \alpha_n \leq \limsup_{n\to+\infty} \alpha_n < 1$.
(B2) Let $\theta > 0$ and $\{\xi_n\}$ be a positive sequence such that $\lim_{n\to+\infty} \frac{\xi_n}{\beta_n} = 0$.
(B3) Let $\{q_n\} \subset E$ such that $\lim_{n\to+\infty} q_n = q \in E$.

Algorithm 3.2. *Common solution of generalized mixed equilibrium problem and fixed point problem*

Step 0 *: Let $x_0, x_1 \in E$ be arbitrary initial points and set $n = 1$.*
Step 1*: Given the $(n-1)$th and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by*

$$\hat{\theta}_n = \begin{cases} \min\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\}, & \textit{If} \quad x_n \neq x_{n-1}, \\ \theta, & \textit{Otherwise.} \end{cases} \tag{3.1}$$

Step 2*: Compute*

$$\begin{cases} w_n = \nabla f^* \left(\nabla f(x_n) + \theta_n \left(\nabla f(x_{n-1}) - \nabla f(x_n)\right)\right) \\ z_n = Res^f_{B_{\theta_N}} \circ \cdots \circ \mathrm{Res}^f_{B_{\theta_1}} (w_n) \\ y_n = \nabla f^* \left(\beta_n \nabla f(q_n) + (1 - \beta_n) \nabla f(T(z_n))\right) \\ x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(w_n) + (1 - \alpha_n) \nabla f(T(y_n))\right). \end{cases}$$

Set $n := n + 1$ and return to **Step 1.**

Remark 3. *The inertial technique used in step 1 of Algorithm 1 can be easily implemented since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. We also note that the restrictive summability condition $\sum_{n=1}^{+\infty} \|x_n - x_{n-1}\| < +\infty$ often used by several authors when constructing initial algorithms is dispensed with in our proposed algorithm.*

4. Convergence Analysis

First, we present some lemmas which will be needed in obtaining our convergence result. In the following lemma, we obtain some results for the maximal operator B_Θ from the bifunction Θ. The main idea of the following lemma is from [53].

Lemma 13. *Let $f\colon E \to (-\infty, +\infty]$ be a supercoercive, Legendre, Fréchet differentiable and totally convex function. Let C be a closed and convex subset of E and assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A_1)-(A_4) and Ψ is monotone. Then*
(1) $\mathrm{GMEP}(\Theta, \varphi, \Psi) = B_\Theta^{-1}(0^*)$*;*
(2) B_Θ is a maximal monotone mapping;
(3) $Res^f_{\Theta,\varphi,\Psi} = Res^f_{B_\Theta}$.

Proof. (1) If $x \in C$ then from the definition of the mapping B_Θ (2.10) we have

$$x \in B_\Theta^{-1}(0^*) \Leftrightarrow \Theta(x, y) + \varphi(y) + \langle \Psi x, y - x\rangle \geq \varphi(x) \ \text{ for all } \ y \in C \Leftrightarrow x \in \mathrm{GMEP}(\Theta, \varphi, \Psi).$$

(2) We show that B_Θ is monotone mapping. Let (x_1, x_1^*) and (x_2, x_2^*) belong to the graph of B_Θ. By the definition of the mapping B_Θ, we have

$$\Theta(x_1, z) + \varphi(z) + \langle \Psi x_1, z - x_1\rangle \geq \langle x_1^*, z - x_1\rangle + \varphi(x_1)$$

and

$$\Theta(x_2, z) + \varphi(z) + \langle \Psi x_2, z - x_2\rangle \geq \langle x_2^*, z - x_2\rangle + \varphi(x_2)$$

for any $z \in C$. In particular we have that

$$\Theta(x_1, x_2) + \varphi(x_2) + \langle \Psi x_1, x_2 - x_1\rangle \geq \langle x_1^*, x_2 - x_1\rangle + \varphi(x_1) \tag{4.1}$$

and

$$\Theta(x_2, x_1) + \varphi(x_1) + \langle \Psi x_2, x_1 - x_2\rangle \geq \langle x_2^*, x_1 - x_2\rangle + \varphi(x_2). \tag{4.2}$$

Adding equations (4.1) and (4.2) together, we obtain

$$\begin{aligned}&\Theta(x_1, x_2) + \Theta(x_2, x_1) + \varphi(x_2) + \varphi(x_1) + \langle \Psi x_1, x_2 - x_1\rangle + \langle \Psi x_2, x_1 - x_2\rangle\\ &\geq \langle x_1^*, x_2 - x_1\rangle + \langle x_2^*, x_1 - x_2\rangle + \varphi(x_1) + \varphi(x_2).\end{aligned}$$

By (A_2), it is equivalent to write

$$\begin{aligned}0 &\geq \Theta(x_1, x_2) + \Theta(x_2, x_1) + \langle \Psi x_1 - \Psi x_2, x_2 - x_1\rangle\\ &\geq \langle x_1^* - x_2^*, x_2 - x_1\rangle.\end{aligned}$$

It means that $\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0$ which follows that B_Θ is a monotone mapping.

To show that B_Θ is maximal monotone mapping, it is enough to show that $ran(B_\Theta + \nabla f) = E^*$ ([54, Corollary 2.3]). Let $x^* \in E^*$ from [55, Proposition 2.3] and [48, Theorem 3.5.10], we have that f is cofinite and therefore $ran\nabla f = intdomf^* = E^*$ which follows that ∇f is surjective. So, there exists $x \in E$ such that $\nabla f(x) = x^*$. From Lemma 9 we know that $dom(Res^f_{\Theta,\varphi,\Psi}) = E$ and from the definition of $Res^f_{\Theta,\varphi,\Psi}$ we obtain

$$\begin{aligned}&\Theta\left(Res^f_{\Theta,\varphi,\Psi}(x_1), x_2\right) + \varphi(x_2) + \langle \Psi(Res^f_{\Theta,\varphi,\Psi}(x_1)), x_2 - Res^f_{\Theta,\varphi,\Psi}(x_1)\rangle\\ &+\langle \nabla f(Res^f_{\Theta,\varphi,\Psi}(x_1)) - \nabla f(x_1), x_2 - Res^f_{\Theta,\varphi,\Psi}(x_1)\rangle\\ &\geq \varphi(Res^f_{\Theta,\varphi,\Psi}(x_1))\end{aligned}$$

for any $x_2 \in C$. It follows that

$$\begin{aligned}&\Theta\left(Res^f_{\Theta,\varphi,\Psi}(x_1), x_2\right) + \varphi(x_2) + \langle \Psi(Res^f_{\Theta,\varphi,\Psi}(x_1)), x_2 - Res^f_{\Theta,\varphi,\Psi}(x_1)\rangle\\ &\geq \langle \nabla f(x_1) - \nabla f(Res^f_{\Theta,\varphi,\Psi}(x_1)), x_2 - Res^f_{\Theta,\varphi,\Psi}(x_1)\rangle\\ &+\varphi(Res^f_{\Theta,\varphi,\Psi}(x_1))\end{aligned}$$

for any $x_2 \in C$. This shows that $\nabla f(x_1) - \nabla f(Res^f_{\Theta,\varphi,\Psi}(x_1)) \in B_\Theta(Res^f_{\Theta,\varphi,\Psi}(x_1))$. Hence

$$x^* = \nabla f(x_1) \in (\nabla f + B_\Theta)\left(Res^f_{\Theta,\varphi,\Psi}(x_1)\right). \tag{4.3}$$

It follows that $x^* \in ran(B_\Theta + \nabla f)$.

(3) It is easy to show that $Res^f_{B_\Theta}$ is single valued. From Lemma 9 we know that $Res^f_{\Theta,\varphi,\Psi}$ is single valued too. From (4.3) we have

$$Res^f_{B_\Theta} = (B_\Theta + \nabla f)^{-1} \circ \nabla f = Res^f_{\Theta,\varphi,\Psi}.$$

Lemma 14. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1 (A and B). Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$. Then, from Lemma 8 we have that $F(T)$ is closed and convex. From Lemma 10 and the definition of z_n we have

$$\begin{aligned} D_f(p, z_n) &= D_f\left(p, Res^f_{B_{\theta_N}} \circ \cdots \circ Res^f_{B_{\theta_2}} \circ Res^f_{B_{\theta_1}}(w_n)\right) \\ &\le D_f\left(p, Res^f_{B_{\theta_1}}(w_n)\right) \\ &\le D_f(p, w_n). \end{aligned} \tag{4.4}$$

Also, from the definition of w_n and (2.8), we obtain

$$\begin{aligned} D_f(p, w_n) &\le D_f(p, \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)))) \\ &= D_f(p, \nabla f^*((1-\theta_n)\nabla f(x_n) + \theta_n \nabla f(x_{n-1}))) \\ &\le (1-\theta_n) D_f(p, x_n) + \theta_n D_f(p, x_{n-1}). \end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(q_n) + (1-\beta_n)\nabla f(T(z_n)))) \\ &\le \beta_n D_f(p, q_n) + (1-\beta_n) D_f(p, T(z_n)) \\ &\le \beta_n D_f(p, q_n) + (1-\beta_n) D_f(p, z_n) \\ &\le \beta_n D_f(p, q_n) + (1-\beta_n) D_f(p, w_n) \end{aligned} \tag{4.6}$$

From the definition of x_{n+1} and (2.8), we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(w_n) + (1-\alpha_n)\nabla f(T(y_n)))) \\ &\le \alpha_n D_f(p, w_n) + (1-\alpha_n) D_f(p, T(y_n)) \\ &\le \alpha_n D_f(p, w_n) + (1-\alpha_n) D_f(p, y_n) \\ &\le \alpha_n D_f(p, w_n) + (1-\alpha_n)[\beta_n D_f(p, q_n) + (1-\beta_n) D_f(p, w_n)] \\ &= \beta_n(1-\alpha_n) D_f(p, q_n) + [\alpha_n + (1-\alpha_n)(1-\beta_n)] D_f(p, w_n) \\ &\le \beta_n(1-\alpha_n) D_f(p, q_n) + [1-\beta_n(1-\alpha_n)] \\ &[(1-\theta_n) D_f(p, x_n) + \theta_n D_f(p, x_{n-1})] \\ &\le \max\{D_f(p, q_n), D_f(p, x_n), D_f(p, x_{n-1})\}. \end{aligned} \tag{4.7}$$

Since $\{q_n\}$ is bounded and ∇f is bounded on bounded subset of E, there exists a real number $d > 0$ such that $D_f(p, q_n) \le d,$ for all $n \in \mathbb{N}$. Thus, by induction, we have

$$\begin{aligned} D_f(p, x_{n+1}) &\le \max\{d, D_f(p, x_n), D_f(p, x_{n-1})\} \\ &\vdots \\ &\le \max\left\{d, D_f(p, x_{N_0}), D_f\left(p, x_{N_{0-1}}\right)\right\}. \end{aligned}$$

This implies that $\{D_f(p, x_n)\}$ is bounded. Hence, from Lemma 6 we have that $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{z_n\}$ and $\{y_n\}$ are all bounded.

Lemma 15. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1 (A and B). Suppose that $p \in \Omega$. Then, the following holds:*

(i) $\lim_{n\to+\infty} \theta_n \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right) = 0.$

(ii) $\lim_{n\to+\infty} \frac{\theta_n}{\beta_n} \left[D_f(p, x_{n-1}) - D_f(p, x_n)\right] = 0.$

Proof.

(i) Let $p \in \Omega$. From (3.1), we have

$$\theta_n \|x_n - x_{n-1}\| \leq \xi_n, \text{ for each } n \geq 1. \tag{4.8}$$

From Assumption 3.1(B2), we have that $\lim_{n\to+\infty} \frac{\xi_n}{\beta_n} = 0$ and $\lim_{n\to+\infty} \beta_n = 0$. It follows that $\lim_{n\to+\infty} \xi_n = 0$. Hence, we have that

$$\lim_{n\to+\infty} \theta_n \|x_n - x_{n-1}\| \leq \lim_{n\to+\infty} \xi_n = 0. \tag{4.9}$$

Since ∇f is norm-to-norm continuous on subsets of E, we have that

$$\lim_{n\to+\infty} \theta_n \|\nabla f(x_n) - \nabla f(x_{n-1})\| = 0. \tag{4.10}$$

Using the three-point identity,

$$D_f(p, x_{n-1}) - D_f(p, x_n) = -D_f(x_{n-1}, x_n) + \langle \nabla f(x_n) - \nabla f(x_{n-1}), x_{n-1} - p\rangle. \tag{4.11}$$

Multiplying (4.11) by θ_n, we have

$$\theta_n[D_f(p, x_{n-1}) - D_f(p, x_n)] = -\theta_n D_f(x_{n-1}, x_n) + \theta_n\langle \nabla f(x_n) - \nabla f(x_{n-1}), x_{n-1} - p\rangle. \tag{4.12}$$

Since ∇f is bounded on bounded sets (Assumption A3), there exists $L > 0$ such that $D_f(x_{n-1}, x_n) \leq L\|x_{n-1} - x_n\|$. Thus,

$$\theta_n D_f(x_{n-1}, x_n) \leq L\theta_n\|x_{n-1} - x_n\| \to 0.$$

By Cauchy-Schwarz and boundedness of $\{x_n\}$, we have

$$|\theta_n\langle \nabla f(x_n) - \nabla f(x_{n-1}), x_{n-1} - p\rangle| \leq \theta_n\|\nabla f(x_n) - \nabla f(x_{n-1})\| \cdot \|x_{n-1} - p\| \to 0.$$

Hence, $\lim_{n\to\infty} \theta_n[D_f(p, x_{n-1}) - D_f(p, x_n)] = 0.$

(ii) Also, since $\lim_{n\to+\infty} \frac{\xi_n}{\beta_n} = 0$, we obtain from (4.8) that

$$\lim_{n\to+\infty} \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\| \le \lim_{n\to+\infty} \frac{\xi_n}{\beta_n} = 0. \tag{4.13}$$

Since ∇f is norm-to-norm continuous on subsets of E, we obtain

$$\lim_{n\to+\infty} \frac{\theta_n}{\beta_n} \|\nabla f(x_n) - \nabla f(x_{n-1})\| = 0. \tag{4.14}$$

Multiplying (4.11) by $\frac{\theta_n}{\beta_n}$, we have

$$\frac{\theta_n}{\beta_n}[D_f(p, x_{n-1}) - D_f(p, x_n)] = -\frac{\theta_n}{\beta_n} D_f(x_{n-1}, x_n) + \frac{\theta_n}{\beta_n}\langle \nabla f(x_n) - \nabla f(x_{n-1}), x_{n-1} - p\rangle.$$

From (4.11), (4.13), and (4.14), we have $\lim_{n\to\infty} \frac{\theta_n}{\beta_n}[D_f(p, x_{n-1}) - D_f(p, x_n)] = 0$, which completes the proof.

Lemma 16. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1(A and B). Suppose that $p \in \Omega$. Then, the following holds:*

$$D_f(p, x_{n+1}) \le [1 - \beta_n(1 - \alpha_n)]\, D_f(p, x_n) + \beta(1 - \alpha_n)b_n,$$

where $b_n = \frac{1-\beta_n(1-\alpha_n)}{1-\alpha_n} \cdot \frac{\theta_n}{\beta_n}\left[D_f(p, x_{n-1}) - D_f(p, x_n)\right] + \langle y_n - p, \nabla f(q_n) - \nabla f(p)\rangle$.

Proof. Let $p \in \Omega$. From (2.7), (4.4), (4.5), we have

$$\begin{aligned}
D_f\left(p, x_{n+1}\right) &= \nabla f\left(p, \nabla f^*\left(\alpha_n \nabla f\left(w_n\right)+\left(1-\alpha_n\right) \nabla f\left(T\left(y_n\right)\right)\right)\right) \\
&\leq \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right) D_f\left(p, T\left(y_n\right)\right) \\
&\leq \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[D_f\left(p, \nabla^* \beta_n \nabla f\left(q_n\right)+\left(1-\beta_n\right) \nabla f\left(T\left(z_n\right)\right)\right]\right. \\
&= \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[V_f\left(p, \beta_n \nabla f\left(q_n\right)+\left(1-\beta_n\right) \nabla f\left(T\left(z_n\right)\right)\right)\right] \\
&\leq \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[V_f\left(p, \beta_n \nabla f\left(q_n\right)+\left(1-\beta_n\right) \nabla f\left(T\left(z_n\right)\right)\right.\right. \\
&-\beta_n\left(\nabla f\left(q_n\right)-\nabla f(p)\right)-\left\langle y_n-p,-\beta_n\left(\nabla f\left(q_n\right)-\nabla f(p)\right\rangle\right] \\
&= \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[V_f\left(p, \beta_n \nabla f(p)+\left(1-\beta_n\right) \nabla f\left(T\left(z_n\right)\right)\right)\right. \\
&+\beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&\leq \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[\beta_n D_f(p, p)+\left(1-\beta_n\right) D_f\left(p, T\left(z_n\right)\right)\right. \\
&+\beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&= \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[\left(1-\beta_n\right) D_f\left(p, T\left(z_n\right)\right)\right. \\
&+\beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&\leq \alpha_n D_f\left(p, w_n\right)+\left(1-\alpha_n\right)\left[\left(1-\beta_n\right) D_f\left(p, z_n\right)\right. \\
&+\beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&\leq\left[\alpha_n+\left(1-\alpha_n\right)\left(1-\beta_n\right)\right] D_f\left(p, w_n\right) \\
&+\left(1-\alpha_n\right) \beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&=\left[1-\beta_n\left(1-\alpha_n\right)\right] D_f\left(p, w_n\right)+\left(1-\alpha_n\right) \beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&\leq\left[1-\beta_n\left(1-\alpha_n\right)\right]\left[\left(1-\theta_n\right) D_f\left(p, x_n\right)+\theta_n D_f\left(p, x_{n-1}\right)\right] \\
&+\left(1-\alpha_n\right) \beta_n\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&=\left[1-\beta_n\left(1-\alpha_n\right)\right] D_f\left(p, x_n\right) \\
&+\left[1-\beta_n\left(1-\alpha_n\right)\right] \theta_n\left[D_f\left(p, x_{n-1}\right)-D_f\left(p, x_n\right)\right] \\
&+\beta_n\left(1-\alpha_n\right)\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla f(p)\right\rangle \\
&=\left[1-\beta_n\left(1-\alpha_n\right)\right] D_f\left(p, x_n\right) \\
&+\beta_n\left(1-\alpha_n\right)\Big[\frac{1-\beta_n\left(1-\alpha_n\right)}{\left(1-\alpha_n\right)} \cdot \frac{\theta_n}{\beta_n}\left(D_f\left(p, x_{n-1}\right)-D_f\left(p, x_n\right)\right. \\
&+\left\langle y_n-p, \nabla f\left(q_n\right)-\nabla_f(p)\right\rangle\Big],
\end{aligned}$$

which completes the proof.

Theorem 1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1 (A and B). Then, $\{x_n\}$ converges strongly to p^* in Ω.*

Proof. Let $p^* \in \Omega$. Then, from Lemma 16, we have

$$\begin{aligned}
D_f(p, x_{n+1}) \quad \leq \quad & \left[1-\beta_n\left(1-\alpha_n\right)\right] D_f\left(p, x_n\right) \\
& +\beta_n\left(1-\alpha_n\right)\Big[\frac{1-\beta_n(1-\alpha_n)}{\left(1-\alpha_n\right)} \cdot \frac{\theta_n}{\beta_n}\left[D_f\left(p, x_{n-1}\right)-D_f\left(p, x_n\right)\right] \qquad (4.15)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \langle y_n - p, \nabla f(q_n) - \nabla_f(p)\rangle\Big] \\
&= \quad [1-\beta_n(1-\alpha_n)]\, D_f(p,x_n) + \beta_n(1-\alpha_n)\, b_n, \qquad (4.16)
\end{aligned}
$$

where $b_n = \frac{1-\beta_n(1-\alpha_n)}{1-\alpha_n} \cdot \frac{\theta_n}{\beta_n}\left[D_f(p,x_{n-1}) - D_f(p,x_n)\right] + \langle y_n - p, \nabla f(q_n) - \nabla f(p)\rangle$. Next, we show that $\{D_f(p^*,x_n)\}$ converges to zero. To show this, by Lemma 12, we need to show that $\limsup_{k\to+\infty} b_{n_k} \leq 0$ for every subsequence $\{D_f(p^*,x_{n_k})\}$ of $\{D_f(p^*,x_n)\}$ satisfying

$$
\liminf_{k\to+\infty}\Big(D_f(p^*,x_{n_{k+1}}) - D_f(p^*,x_{n_k})\Big) \geq 0. \qquad (4.17)
$$

Suppose $\{D_f(p^*,x_{n_k})\}$ is a subsequence of $\{D_f(p^*,x_n)\}$ such that (4.17) holds.
From Lemma 7, Lemma 15 and the definition of z_{n_k}, we have

$$
\begin{aligned}
\lim_{k\to+\infty} D_f(x_{n_k}, z_{n_k}) &= \lim_{k\to+\infty} D_f\Big(x_{n_k}, Res^f_{B_{\theta_N}} \circ \ldots \circ Res^f_{B_{\theta_1}}(w_{n_k})\Big) \\
&\leq \lim_{k\to+\infty} D_f\Big(x_{n_k}, Res^f_{B_{\theta_{N-1}}} \circ \cdots \circ Res^f_{B_{\theta_1}}(w_{n_k})\Big) \\
&\leq \lim_{k\to+\infty} D_f\Big(x_{n_k}, Res^f_{B_{\theta_1}}(w_{n_k})\Big) \qquad (4.18) \\
&\leq \lim_{k\to+\infty} \Big[D_f\Big(p^*, Res^f_{B_{\theta_1}}(w_{n_k})\Big) - D_f(p^*, x_{n_k})\Big] \\
&\leq \lim_{k\to+\infty} \left[D_f(p^*, w_{n_k}) - D_f(p^*, x_{n_k})\right] \qquad (4.19) \\
&\leq \lim_{k\to+\infty} \left[(1-\theta_{n_k}) D_f(p^*, x_{n_k}) + \theta_{n_k} D_f\left(p^*, x_{n_{k-1}}\right) - D_f(p^*, x_{n_k})\right] \\
&= \lim_{k\to+\infty} \left[\theta_{n_k} D_f\left(p^*, x_{n_{k-1}}\right) - \theta_{n_k} D_f(p^*, x_{n_k})\right] \\
&= \lim_{k\to+\infty} \theta_{n_k}\left[D_f\left(p^*, x_{n_{k-1}}\right) - D_f(p^*, x_{n_k})\right] \\
&= 0. \qquad (4.20)
\end{aligned}
$$

From Lemma 1, we obtain

$$
\lim_{k\to+\infty} \|x_{n_k} - z_{n_k}\| = 0. \qquad (4.21)
$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 5, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. Hence,

$$
\lim_{k\to+\infty} \|\nabla f(x_{n_k}) - \nabla f(z_{n_k})\|_\star = 0. \qquad (4.22)
$$

Also, since f is uniformly Fréchet differentiable, it is also uniformly continuous, hence we obtain that

$$
\lim_{k\to+\infty} \|f(x_{n_k}) - f(z_{n_k})\| = 0. \qquad (4.23)
$$

Using the Bregman distance, we obtain

$$
D_f(p^*, x_{n_k}) - D_f(p^*, z_{n_k})
$$

$$\begin{aligned}
&= f(p^*) - f(x_{n_k}) - \langle \nabla f(x_{n_k}), p^* - x_{n_k} \rangle - f(p^*) + f(z_{n_k}) + \langle \nabla f(z_{n_k}), p^* - z_{n_k} \rangle \\
&= f(z_{n_k}) - f(x_{n_k}) + \langle \nabla f(z_{n_k}), p^* - z_{n_k} \rangle - \langle \nabla f(x_{n_k}), p^* - x_{n_k} \rangle \\
&= f(z_{n_k}) - f(x_{n_k}) + \langle \nabla f(z_{n_k}), x_{n_k} - z_{n_k} \rangle - \langle \nabla f(z_{n_k}) - \nabla f(x_{n_k}), p^* - x_{n_k} \rangle,
\end{aligned}$$

for $p^* \in F(T)$.
From (4.21) and (4.23), we obtain

$$\lim_{k\to+\infty} [D_f(p, x_{n_k}) - D_f(p, z_{n_k})] = 0. \tag{4.24}$$

Also, since $\beta_{n_k} \to 0$ as $k \to +\infty$, we obtain

$$\begin{aligned}
D_f(z_{n_k}, y_{n_k}) &= D_f(p^*, y_{n_k}) - D_f(p^*, z_{n_k}) \\
&= D_f(p^*, \nabla f^*(\beta_{n_k} \nabla f(q_{n_k}) + (1-\beta_{n_k}) \nabla f(T(z_{n_k})))) - D_f(p^*, z_{n_k}) \\
&\leq \beta_{n_k} D_f(p^*, q_{n_k}) + (1-\beta_{n_k}) D_f(p^*, T(z_{n_k})) - D_f(p^*, z_{n_k}) \\
&\leq \beta_{n_k} D_f(p^*, q_{n_k}) + (1-\beta n_k) D_f(p^*, z_{n_k}) - D_f(p^*, z_{n_k}) \\
&= \beta_{n_k} [D_f(p^*, q_{n_k}) - D_f(p^*, z_{n_k})] \\
&\to 0, \quad \text{as} \quad k \to +\infty.
\end{aligned} \tag{4.25}$$

Hence, $\lim_{k\to+\infty} D_f(z_{n_k}, y_{n_k}) = 0$. From 2.4, we obtain

$$\lim_{k\to+\infty} \|z_{n_k} - y_{n_k}\| = 0. \tag{4.26}$$

Consequently, we have

$$\lim_{k\to+\infty} \|\nabla f(z_{n_k}) - \nabla f(y_{n_k})\| = \lim_{k\to+\infty} \|f(z_{n_k}) - f(y_{n_k})\| = 0.$$

From (4.21) and (4.26), we obtain

$$\begin{aligned}
\|x_{n_k} - y_{n_k}\| &= \|x_{n_k} - z_{n_k} + z_{n_k} - y_{n_k}\| \\
&\leq \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - y_{n_k}\| \\
&= 0,
\end{aligned}$$

as $k \to +\infty$.
Therefore,

$$\lim_{k\to+\infty} \|x_{n_k} - y_{n_k}\| = 0. \tag{4.27}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, by Lemma 5, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. Hence

$$\lim_{k\to+\infty} \|\nabla f(x_{n_k}) - \nabla f(y_{n_k})\|_* = 0. \tag{4.28}$$

On the other hand, since f is uniformly Fréchet differentiable, we have that f is also uniformly continuous. Hence,

$$\lim_{k\to+\infty} \|f(x_{n_k}) - f(y_{n_k})\| = 0. \tag{4.29}$$

Applying the Bregman distance, we obtain

$$\begin{aligned} & D_f\left(p^*, w_{n_k}\right)-D_f\left(p^*, y_{n_k}\right) \\ = & f(p^*)-f\left(w_{n_k}\right)-\left\langle\nabla f\left(w_{n_k}\right), p^*-w_{n_k}\right\rangle-f(p^*)+f\left(y_{n_k}\right)+\left\langle\nabla f\left(y_{n_k}\right), p^*-y_{n_k}\right\rangle \\ = & f\left(y_{n_k}\right)-f\left(w_{n_k}\right)+\left\langle\nabla f\left(y_{n_k}\right), p^*-y_{n_k}\right\rangle-\left\langle\nabla f\left(w_{n_k}\right), p^*-w_{n_k}\right\rangle \\ = & f\left(y_{n_k}\right)-f\left(w_{n_k}\right)+\left\langle\nabla f\left(y_{n_k}\right), w_{n_k}-y_{n_k}\right\rangle+\left\langle\nabla f\left(y_{n_k}\right)-\nabla f\left(w_{n_k}\right), p^*-w_{n_k}\right\rangle, \end{aligned} \tag{4.30}$$

for $p^* \in F(T)$. From the definition of w_{n_k} and (4.10), we have

$$\begin{aligned} \left\|\nabla f\left(w_{n_k}\right)-\nabla f\left(x_{n_k}\right)\right\| & =\left\|\nabla f\left(x_{n_k}\right)+\theta_{n_k}\left(\nabla f\left(x_{n_{k-1}}\right)-\nabla f\left(x_{n_k}\right)\right)-\nabla f\left(x_{n_k}\right)\right\| \\ & =\theta_{n_k}\left\|\nabla f\left(x_{n_{k-1}}\right)-\nabla f\left(x_{n_k}\right)\right\| \rightarrow 0 \text{ as } k \rightarrow+\infty. \end{aligned} \tag{4.31}$$

From (4.28) and (4.31), we can write

$$\lim_{k \rightarrow+\infty}\left\|\nabla f\left(w_{n_k}\right)-\nabla f\left(y_{n_k}\right)\right\| \leq \lim_{k \rightarrow+\infty}\left[\left\|\nabla f\left(w_{n_k}\right)-\nabla f\left(x_{n_k}\right)\right\|+\| \nabla f(x_{n_k})-\nabla f(y_{n_k})\right]=0 \tag{4.32}$$

Combining (4.30) and (4.32), we have

$$\lim_{k \rightarrow+\infty}\left(D_f\left(p^*, w_{n_k}\right)-D_f\left(p^*, y_{n_k}\right)\right)=0. \tag{4.33}$$

Also, from (4.27) and (4.32), we have that

$$\lim_{k \rightarrow+\infty}\left\|w_{n_k}-x_{n_k}\right\|=0. \tag{4.34}$$

From (4.33) and the definition of $x_{n_{k+1}}$, we have

$$\begin{aligned} D_f\left(y_{n_k}, x_{n_{k+1}}\right) & =D_f\left(p^*, x_{n_{k+1}}\right)-D_f\left(p^*, y_{n_k}\right) \\ & =D_f\left(p^*, \nabla f^*\left(\alpha_{n_k} \nabla f\left(w_{n_k}\right)+\left(1-\alpha_{n_k}\right) \nabla f\left(T\left(y_{n_k}\right)\right)\right)\right)-D_f\left(p^*, y_{n_k}\right) \\ & \leq D_f\left(p^*, \nabla f^*\left(\alpha_{n_k} \nabla f\left(w_{n_k}\right)+\left(1-\alpha_{n_k}\right) \nabla f\left(T\left(y_{n_k}\right)\right)-D_f\left(p, y_{n_k}\right)\right)\right) \\ & \leq \alpha_{n_k} D_f\left(p^*, w_{n_k}\right)+\left(1-\alpha_{n_k}\right) D_f\left(p^*, T\left(y_{n_k}\right)\right)-D_f\left(p^*, y_{n_k}\right) \\ & \leq \alpha_{n_k} D_f\left(p^*, w_{n_k}\right)+\left(1-\alpha_{n_k}\right) D_f\left(p^*, y_{n_k}\right)-D_f\left(p^*, y_{n_k}\right) \\ & =\alpha_{n_k}\left[D_f\left(p^*, w_{n_k}\right)-D_f\left(p^*, y_{n_k}\right)\right] \\ & \rightarrow 0 \text{ as } k \rightarrow+\infty. \end{aligned}$$

From Lemma 1, he have

$$\lim_{k \rightarrow+\infty}\left\|y_{n_k}-x_{n_{k+1}}\right\|=0. \tag{4.35}$$

From (4.25), and the fact that $\beta_{n_k} \rightarrow 0$ as $k \rightarrow+\infty$, we have

$$\begin{aligned} D_f\left(T y_{n_k}, y_{n_k}\right) & =D_f\left(p^*, y_{n_k}\right)-D_f\left(p^*, T y_{n_k}\right) \\ & \leq \beta_{n_k} D_f\left(p^*, q_{n_k}\right)+\left(1-\beta_{n_k}\right) D_f\left(p^*, T\left(z_{n_k}\right)\right)-D_f\left(p^*, T y_{n_k}\right) \\ & \leq \beta_{n_k} D_f\left(p^*, q_{n_k}\right)+\left(1-\beta_{n_k}\right) D_f\left(p^*, z_{n_k}\right)-D_f\left(p^*, y_{n_k}\right) \\ & =\beta_{n_k}\left[D_f\left(p^*, q_{n_k}\right)-D_f\left(p^*, z_{n_k}\right)\right]+\left[D_f\left(p^*, z_{n_k}\right)-D_f\left(p^*, y_{n_k}\right)\right] \\ & \rightarrow 0 \text{ as } k \rightarrow+\infty. \end{aligned}$$

Therefore, $\lim_{k\to+\infty} D_f\left(Ty_{n_k}, y_{n_k}\right) = 0.$
From Lemma 1, we obtain

$$\lim_{k\to+\infty} \|Ty_{n_k} - y_{n_k}\| = 0. \tag{4.36}$$

From (4.35) and (4.36), we have

$$\left\|x_{n_{k+1}} - Ty_{n_k}\right\| \le \left\|x_{n_{k+1}} - y_{n_k}\right\| + \|y_{n_k} - Ty_{n_k}\| \to 0, \text{ as } k \to +\infty.$$

In other words,

$$\lim_{k\to+\infty} \left\|x_{n_{k+1}} - Ty_{n_k}\right\| = 0. \tag{4.37}$$

From (4.27) and (4.36) we have

$$\|x_{n_k} - Tx_{n_k}\| \le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - Ty_{n_k}\| + \|Ty_{n_k} - Tx_{n_k}\| \to 0$$

as $k \to +\infty$. Hence,

$$\lim_{k\to+\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{4.38}$$

From (4.27)-(4.38) we have

$$\lim_{n\to+\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{4.39}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup p^*$. From (4.38), we have $\|x_{n_k} - T(x_{n_k})\| \to 0$ as $k \to +\infty$. Hence, $p^* \in F(T)$. For any $w \in \left(F(T) \cap \left(\bigcap_{j=1}^{N} B_{\theta_j}^{-1}(0^*)\right)\right)$, it follows from the there point identity that

$$\begin{aligned}
&|D_f(w, w_{n_k}) - D_f(w, y_{n_k})| \\
&= |D_f(w, y_{n_k}) + D_f(y_{n_k}, w_{n_k}) + \langle w - y_{n_k}, \nabla f(y_{n_k}) - \nabla f(w_{n_k})\rangle - D_f(w, y_{n_k})| \\
&= |D_f(y_{n_k}, w_{n_k}) + \langle w - y_{n_k}, \nabla f(y_{n_k}) - \nabla f(w_{n_k})\rangle| \\
&\le D_f(y_{n_k}, w_{n_k}) + \|w - y_{n_k}\| \|\nabla f(y_{n_k}) - \nabla f(w_{n_k})\| \\
&\le \|y_{n_k} - w_{n_k}\| + \|w - y_{n_k}\| \|\nabla f(y_{n_k}) - \nabla f(w_{n_k})\| \\
&\to 0 \quad \text{as} \quad k \to +\infty \qquad \text{by (4.32).}
\end{aligned}$$

Hence,

$$\lim_{k\to+\infty} |D_f(w, w_{n_k}) - D_f(w, y_{n_k})| = 0.$$

Since $Res_{B_\theta}^f$ is BQFNE, we have

$$\begin{aligned}
&D_f\left(Res_{B_{\theta_j}}^f \circ \cdots \circ Res_{B_{\theta_1}}^f(w_{n_k}), Res_{B_{\theta_{j-1}}}^f \circ \ldots \circ Res_{B_{\theta_1}}^f(w_{n_k})\right) \\
&= D_f\left(Res_{B_{\theta_j}}^f \circ Res_{B_{\theta_{j-1}}}^f \circ \ldots \circ Res_{B_{\theta_1}}^f(w_{n_k}), Res_{B\theta_{j-1}}^f \circ \ldots \circ Res_{B_{\theta_1}}^f(w_{n_k})\right) \\
&\le D_f(w, w_{n_k}) - D_f(w, w_{n_k}) \\
&\to 0
\end{aligned}$$

as $k \to +\infty$ for all $j \in \{1, 2, \ldots, N\}$. It then follows that

$$\lim_{k\to+\infty} D_f\left((Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k}), w_{n_k})\right) = 0$$

for all $j \in \{1, 2, \ldots, N\}$. So,

$$\lim_{k\to+\infty} \|Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k}) - w_{n_k}\| = 0 \tag{4.40}$$

for all $j \in \{1, 2, \ldots, N\}$.
From the definition of the f-resolvent, we have

$$\nabla f\left(Res^f_{B_{\Theta_{j-1}}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k})\right) \in (\nabla f + \lambda^j_{n_k} B_{\Theta_j})\left(Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k})\right).$$

Hence

$$\zeta^j_{n_k} := \frac{1}{\lambda^j_{n_k}}\left(\nabla f(Res^f_{B_{\Theta_{j-1}}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k})) - \nabla f(Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k}))\right)$$

for all $j \in \{1, 2, \ldots, N\}$. It follows from the above equations that $\lim_{k\to+\infty} \|\zeta^j_{n_k}\| = 0$ for any $j \in \{1, 2, \ldots, N\}$. Since $x_{n_k} \rightharpoonup p^*$, we obtain from (4.34) that $w_{n_k} \rightharpoonup p^*$. From (4.40) and the fact that $w_{n_k} \rightharpoonup p^*$, we obtain

$$Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(w_{n_k}) \rightharpoonup p^*$$

for any $j \in \{1, 2, \ldots, N\}$.
Consequently, we have

$$Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(x_{n_k}) \rightharpoonup p^*$$

for any $j \in \{1, 2, \ldots, N\}$. From the monotonicity of B_Θ, we have

$$\langle \eta - \zeta^j_{n_k}, x - Res^f_{B_{\Theta_j}} \circ \ldots \circ Res^f_{B_{\Theta_1}}(x_{n_k})\rangle \geq 0.$$

for all $(x, \eta) \in graph(B_{\Theta_j})$. This implies that $\langle \eta, x - p^*\rangle \geq 0$ for all $(x, \eta) \in graph(B_{\Theta_j})$ and for all $j \in \{1, 2, \ldots, N\}$. So, by the maximal monotonicity of B_{Θ_j} we have $p^* \in B^{-1}_{\Theta_j}(0)$ for all $j \in \{1, 2, \ldots, N\}$. Therefore $p^* \in \cap^N_{j=1} B^{-1}_{\Theta_j}(0)$.
Hence, we have shown that $p^* \in \left(F(T) \cap \left(\cap^N_{j=1} B^{-1}_{\Theta_j}(0^*)\right)\right)$.
Since E is reflexive and $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\} \rightharpoonup u \in C$ and

$$\begin{aligned}
\lim_{j\to+\infty}\left\langle \nabla f(q_{n_k}) - \nabla f(\hat{p}), x_{n_{k_j}} - \hat{p}\right\rangle &= \limsup_{k\to+\infty}\langle \nabla f(q_{n_k}) - \nabla f(\hat{p}), x_{n_k} - \hat{p}\rangle \\
&= \limsup_{k\to+\infty}\langle \nabla f(q_{n_k}) - \nabla f(\hat{p}), y_{n_k} - \hat{p}\rangle.
\end{aligned}$$

It follows from the definition of the Bregman projection that

$$\limsup_{k\to+\infty}\langle\nabla f(q_{n_k})-\nabla f(\hat{p}),y_{n_k}-\hat{p}\rangle = \lim_{j\to+\infty}\left\langle\nabla f(q_{n_k})-\nabla f(\hat{p}),x_{n_{k_j}}-\hat{p}\right\rangle \quad (4.41)$$
$$= \langle\nabla f(q)-\nabla f(\hat{p}),u-\hat{p}\rangle\leq 0. \quad (4.42)$$

By Lemma 15, (4.41) and the condition on α, we can conclude that $\limsup_{k\to+\infty} b_{n_k}\leq 0$. From Lemma 4.17 and (4.15) we have $\lim_{n\to+\infty} D_f(\hat{p},x_n)=0$. Therefore, by Lemma 11, we have $\lim_{n\to+\infty} x_n=\hat{p}$. This completes the proof.

Using Theorem 1 and Lemma 13, we have the following result. Let $\varphi=\Psi=\theta_n=0$ and $T=I$ then we have the following corollary which was obtained in [56].

Corollary 1. *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of $int(domf)$. Let $f:E\to\mathbb{R}$ be a super coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $B_{\Theta_j}:E\to 2^{E^*}$, $j=1,2,\ldots,N$ be N maximal monotone mapping with $dom(B)\subset C$. Assume that $\left(\cap_{j=1}^N B_{\Theta_j}^{-1}(0^*)\right)\neq\emptyset$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:*

(i) $\lim_{n\to+\infty}\beta_n=0$;

(ii) $\sum_{n=1}^{+\infty}\beta_n=+\infty$;

(iii) $0<\liminf_{n\to+\infty}\alpha_n\leq\limsup_{n\to+\infty}\alpha_n<1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned}
&u\in E, x_1\in E \quad \text{chosen arbitrarily},\\
&z_n=Res^f_{B_{\Theta_N}}\circ\ldots\circ Res^f_{B_{\Theta_1}}(x_n),\\
&y_n=\nabla f^*\left(\beta_n\nabla f(q_n)+(1-\beta_n)\nabla f(z_n)\right)\\
&x_{n+1}=\nabla f^*\left(\alpha_n\nabla f(x_n)+(1-\alpha_n)\nabla f(y_n)\right),
\end{aligned} \quad (4.43)$$

where ∇f is the gradient of f. Then the sequence $\{x_n\}$ generated by (4.43) converges to $proj^f_{\cap_{j=1}^N B_{\Theta_j}^{-1}(0^*)}x$ as $n\to+\infty$.

5. Numerical Experiment

In this section, we present numerical experiments to illustrate the performance of our proposed method. In all our experiments, we use $\|x_{n+1}-x_n\|<10^{-4}$ as our stopping criterion. All the numerical computations were carried out using using Matlab version R2024(b).

Being a non-accelerated version of our method, we made a comparison with Algorithm (1.5) in [23] with a short name "EsRa".

Example 1. *Let $E = \mathbb{R}$ and $C = [-1, 1]$. Let $\Theta_i(x, y) = -9ix^2+xy+(9i-1)y^2$, $\Psi_i(x, y) = (9i-3)x$, $\varphi_i(x, y) = (9i-6)x$, $i = 1, 2, 3, \cdots, N$, we have $Res^f_{B\Theta_i}(x) = \frac{x}{5(9i-3)}$. Let $f = \|x\|^2$ and $T(x) = P_C(x)$ where*

$$P_C(x) = \begin{cases} -1, & x < -1 \\ x, & x \in [-1, 1] \\ 1, & x > 1. \end{cases}$$

Clearly, we observe that the bifunction Θ satisfies (A1)-(A4), Ψ is monotone and T is Bregman strongly nonexpansive mapping. In this example, we select $\alpha_n = \frac{1}{2n+3}$, $\beta_n = \frac{n}{n+1}$, $q_n = \frac{1}{n+1}$, $\epsilon = \frac{1}{n^{3.1}}$, $\theta = 0.1$ and set $N = 100$. As a stopping criterion, we use $\|x_{n+1} - x_n\| \leq \epsilon$ where $\epsilon = 10^{-4}$. The experiment was conducted for the following initial values of x_0 and x_1 given as Cases I-IV:

(Case I) : $x_0 = 0.96$ *and* $x_1 = 0.59$;

(Case II) : $x_0 = 1.1$ *and* $x_1 = 1.3$;

(Case III) : $x_0 = 2.1$ *and* $x_1 = 1.7$;

(Case IV) : $x_0 = 0.69$ *and* $x_1 = 0.09$.

The report appears in the form of Figures 1- 4 showing that our method converges faster in terms of number iteration than the non-accelerated version.

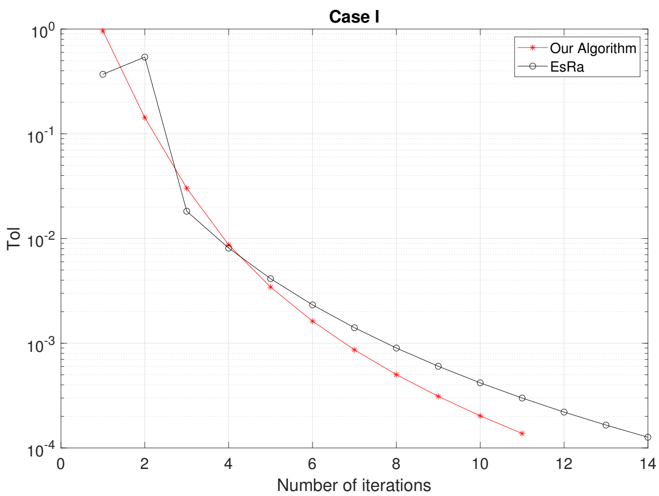

Figure 1: Example 1. Case I.

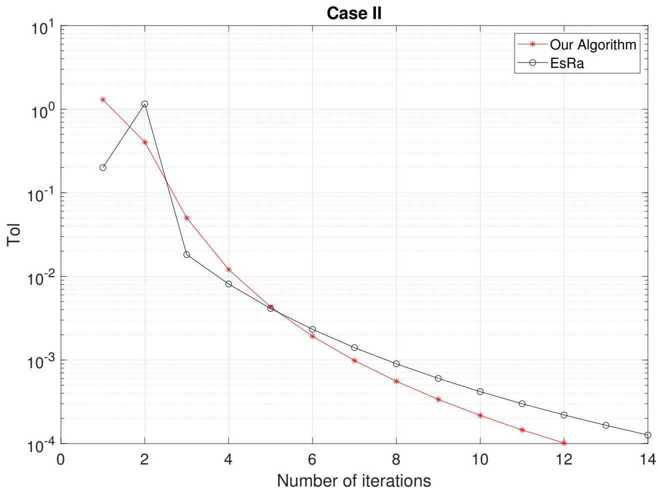

Figure 2: Example 1. Case II.

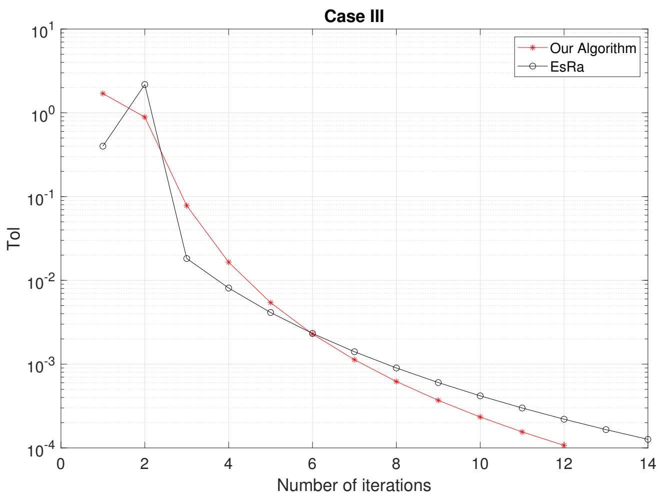

Figure 3: Example 1. Case III.

Example 2. *Let* $E = (\ell_2(\mathbb{R}), \|\cdot\|)$, *where* $\ell_2(\mathbb{R}) := \left\{x : x = \{x_i\}_{i=1}^{+\infty}, \sum_{i=1}^{+\infty} |x_i|^2 < +\infty\right\}$, *with an inner product* $\langle\cdot,\cdot\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ *given by* $\langle x, y\rangle = \sum_{i=1}^{+\infty} x_i y_i$ *where* $x = \{x_i\}_{i=1}^{+\infty}$, $y = \{y_i\}_{i=1}^{+\infty}$ *and the norm* $\|\cdot\| : \ell_2 \to \mathbb{R}$ *is given by* $\|x\|_2 = \sqrt{\left(\sum_{i=1}^{+\infty} |x_i|^2\right)}$. *Let* $f(x) = \frac{x^2}{2}$, *then* f *satisfies Assumption 3.1 Let* $C := \{x \in \ell_2(\mathbb{R}) : \|x\|_2 \leq 1\}$ *and* $\Theta_i : E \times E \to \mathbb{R}$ *be defined by* $\Theta_i(x, y) = -3ix^2 + 2ixy + iy^2$ *for all* i *and* $x, y \in \ell_2$. *Let* $\Psi_i : \ell_2 \to \mathbb{R}$ *and* $\varphi_i : \ell_2 \to \mathbb{R}$ *for all* i *and* $x \in \ell_2$ *be given by* $\Psi_i = ix^2$ *and* $\varphi_i = ix$, *respectively. We have that* $Res^f_{B_{\Theta_i}}(x) = \frac{x}{1+7j}$. *Let* $T = \frac{x+2}{2}$. *Clearly, we observe that the bifunction* Θ *satisfies*

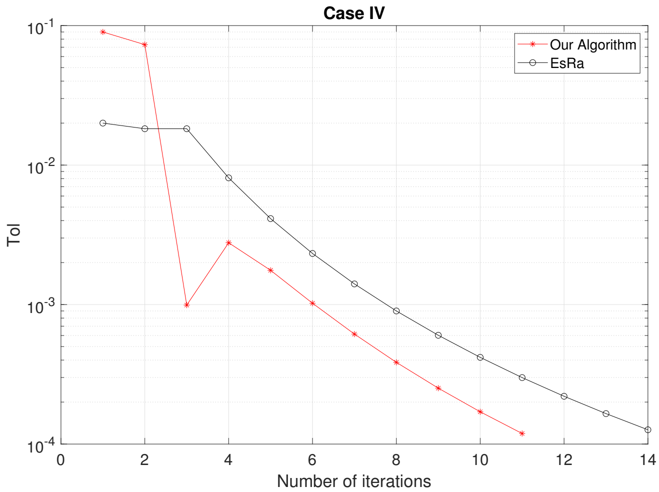

Figure 4: Example 1. Case IV.

(A1)-(A4), Ψ is monotone and T is Bregman strongly nonexpansive mapping. For example 2, we choose $\alpha_n = \frac{1}{2n+3}$, $\beta_n = \frac{1}{n+1}$, $q_n = \frac{1}{5}$, $\epsilon = \frac{1}{n^{3.1}}$, $\theta = 0.1$ and set $N = 100$. As a stopping criterion, we use $\|x_{n+1} - x_n\| \leq \epsilon$ where $\epsilon = 10^{-4}$. The experiment was conducted for the following initial values of x_0 and x_1 given as Cases A-D:

(Case A) : $x_0 = [0.25, 0.25, 0.33, \cdots, 0, 0, \cdots]$ *and* $x_1 = [0.2, 0.2, 0.3, \cdots, 0, 0, \cdots]$;

(Case B) : $x_0 = [1, 0.5, 0.25, \cdots, 0, 0, \cdots]$ *and* $x_1 = [0.2, 0.25, 0.125, \cdots, 0, 0, \cdots]$;

(Case C) : $x_0 = [0.91, 0.55, 0.53, \cdots, 0, 0, \cdots]$ *and* $x_1 = [0.85, 0.65, 0.65, \cdots, 0, 0, \cdots]$;

(Case D) : $x_0 = [1.2, 0, 0.38, \cdots, 0, 0, \cdots]$ *and* $x_1 = [0.8, 0, 1.2, \cdots, 0, 0, \cdots]$.

The report of this experiment is displayed Figures 5- 8 showing that our method converges faster in terms of number iteration than the non-accelerated version presented in [23].

6. Conclusion

In this paper, we studied the generalized mixed equilibrium problem and the fixed point problem in the framework of real reflexive Banach spaces. We introduce an inertial method for approximating the common solution of the above mentioned problems. Under mild conditions, we obtained the strong convergence of our proposed method. Finally, we present numerical experiments in comparison with an algorithm in the literature to illustrate the applicability of our proposed method.

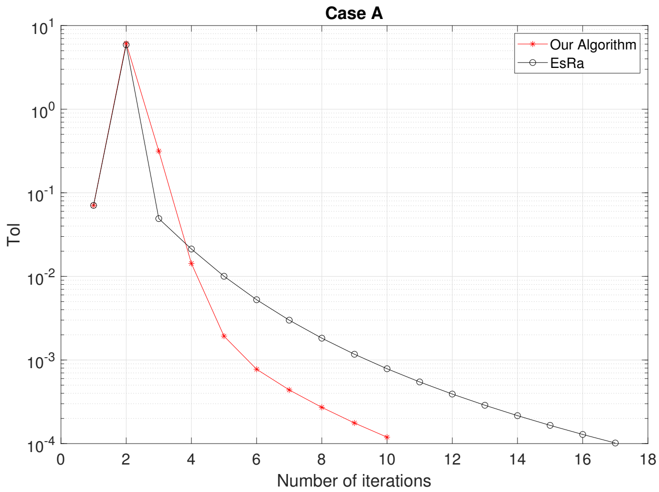

Figure 5: Example 2. Case A.

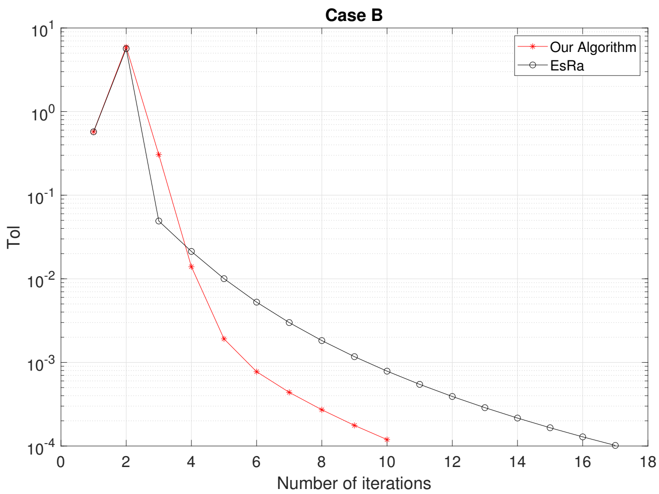

Figure 6: Example 2. Case B.

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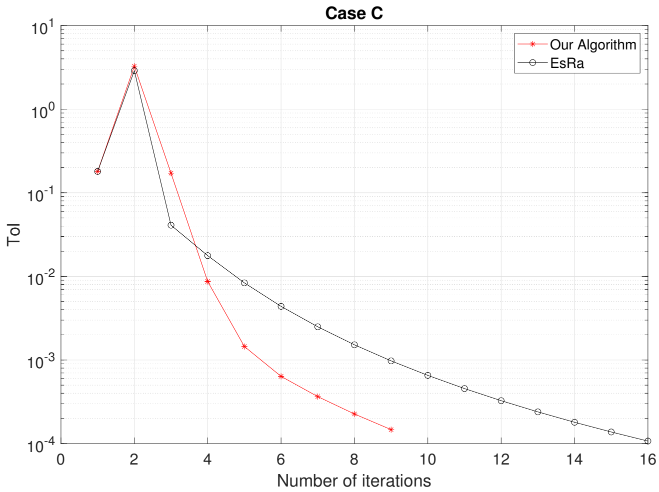

Figure 7: Example 2. Case C.

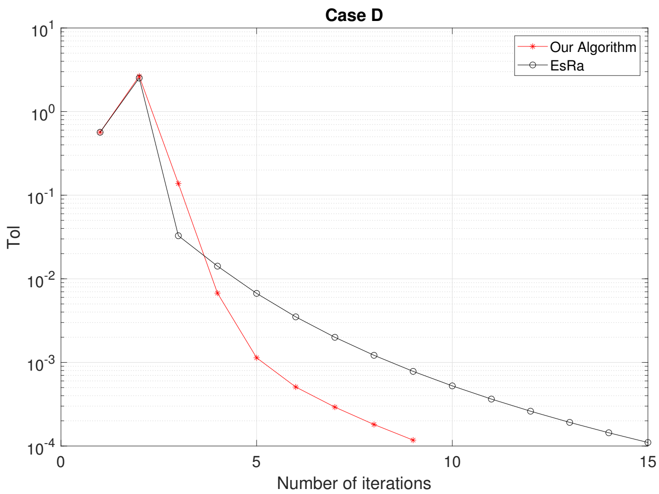

Figure 8: Example 2. Case D.

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