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Topological Homomorphisms in a Topological Dual B-Algebra

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Abstract. This paper introduced the topological dual B-homomorphism in a topological dual B-algebras and its example. Moreover, this paper also presented a Python program to check the homomorphism condition of the topological dual B-homomoprphism. Furthermore, the topology for the quotient dual B-algebra was presented using the natural dual B-homomorphism. This quotient dual B-topological space was proven to be a topological dual B-algebra. Consequently, properties of a topological homomorphism were determined. The topological dual B-isomorphism among topological dual B-algebras was also introduced, and some results were obtained.

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1. Introduction

Over the years, the studies of type (2,0) algebras have remained a rich subject of exploration (see [1], [2], [3], [4]). In particular, J. Neggers and H.S. Kim introduced the B-algebras and its characteristics [5]. Some subsequent studies on B-algebras have drawn parallel results with group theory (see [6], [7], [8], [9]). In 2022, the dual of the B-algebras was initiated by K. Belleza and J.R. Albaracin and some of its special subsets, namely, the dual B-subalgebra, dual B-filters, and normal subsets. Moreover, the researchers constructed a congruence relation on a dual B-algebra [10]. The normal dual B-subalgebra and congruence relation were used by J.E Bolima and K.B. Fuentes to form the quotient dual B-algebra and the homomorphism map from a dual B-algebra to a quotient dual B-algebra. The properties of this mapping were also obtained [11].

Several studies on establishing the homomorphism maps from a type (2,0) algebras to their respective quotient type (2,0) algebras using ideals have been noted (see [12], [13],

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[4], [14]). Remarkably, in [8] the canonical projection from a B-algebra to the quotient B-algebra was used to prove some isomorphism theorems. Moreover, this canonical projection map was also used to prove some fundamental properties of the topological B-algebra, which is a B-algebra equipped with a topology that makes the binary operation of the B-algebra continuous [15]. Furthermore, Hoo in his study on topological MV-algebra use the homomorphism map from a MV-algebra to a quotient MV-algebra (determined by a MV-ideal) to study on topological homomorphisms [14]. Moreover, A. Satirad and A. Iampan use this kind of homomorphism map to study on topological homomorphisms in a topological UP-algebra [16].

This study aimed to establish the topological homomorphism in a topological dual B-algebra and its properties using the natural dual B-homomorphism. Findings of this research expanded some concepts in topology such as properties of a topological spaces and mappings between topological spaces. Since the tdB-homomorphism simultaneously look on the algebraic structure and topological structure of the dual B-algebra, future applications of the study may link to development of logical algebras, algebraic topology or other related fields which studies about mappings on a topological structure having algebraic properties. The topological isomorphism in a topological dual B-algebra was also initiated.

2. Preliminaries

Definition 1. [17] A *dual B-algebra* (or dB-algebra), X is a triple $(X, \circ, 1)$ where X is a nonempty set with a binary operation " $\circ$ " and a constant 1 satisfying the following axioms for all x, y, z in X:

$$\text{(DB1) } x \circ x = 1; \quad \text{(DB2) } 1 \circ x = x; \quad \text{(DB3) } x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z.$$

Example 1. [10] Let $X = \{1, a, b, c\}$ with the binary operation $\cdot$ as defined in the table:

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X, \circ, 1)$ is a dB-algebra.

Definition 2. [10] Let X be a dB-algebra and S a nonempty subset of X. Then S is called a *dual B-subalgebra*, (or dB-subalgebra), of X if S itself is a dB-algebra with binary operation of X on S.

Definition 3. [10] Let X be a dB-algebra and N a nonempty subset of X. Then N is a *normal* subset of X if for any $a \circ b, x \circ y \in N, (a \circ x) \circ (b \circ y) \in N$. A dB-subalgebra S of a dB-algebra X is called a *normal dB-subalgebra* if S is a normal subset of X.

Theorem 1. [10] Let $(X, \circ, 1)$ be a dB-algebra and S a normal dB-subalgebra of X. The relation defined by $x \sim y$ if and only if $x \circ y, y \circ x \in S$ is a *congruence relation* on X for any $x, y \in X$.

Definition 4. [10] Let $(X, \circ, 1)$ be a dB-algebra and S a normal dB-subalgebra of X. Define a *congruence class* $[x]_S$ by $[x]_S = \{y \in X | y \sim x\}$ and define X/S to be the set of all congruence classes of X, that is $X/S = \{[x]_S | x \in X\}$.

Lemma 1. [11]. Let S be a normal dB-subalgebra of a dB-algebra $(X, \circ, 1)$ and $x, y \in X$. Then $[x]_S = [y]_S$ if and only if $x \sim y$.

Theorem 2. [11] Let S be a normal dB-subalgebra of a dB-algebra $(X, \circ, 1)$. Then $(X/S, *, [1]_S)$ with a binary operation $*$ defined by

$$[x]_S * [y]_S = [x * y]_S$$

for all $x, y \in X$ is a dB-algebra. X/S is called the quotient dB-algebra of X by S.

Definition 5. [11] Let $(X, \circ, 1_X)$ and $(Y, *, 1_Y)$ be dB-algebras. A mapping $\Phi : X \to Y$ is called a *dual B-homomorphism* (or dB-homomorphism), from X into Y if $\Phi(x \circ y) = \Phi(x) * \Phi(y)$ for any $x, y \in X$. A dB-homomorphism Φ is called, *dB-monomorphism*, *dB-epimorphism*, or *dB-isomorphism*, if Φ is one-to-one, onto, or a bijection, respectively. The *kernel* of the dB-homomorphism Φ, denoted by $\ker \Phi$, is the set whose elements of X are map to 1_Y.

Theorem 3. [11] Let $\Phi : X \to Y$ be a dB-homomorphism, $(X, \circ, 1_X)$, $(Y, *, 1_Y)$ be dB-algebras, then Φ is a dB-monomorphism if and only if $\ker \Phi = 1_X$.

Theorem 4. [11] Let S be a normal dB-subalgebra of a dB-algebra $(X, \circ, 1)$. Then the mapping $\Phi : X \to (X/S, *, [1]_S)$ given by $\Phi(x) = [x]_S$ for all $x \in X$ is a dB-epimorphism and $\ker \Phi = S$. The mapping is called the *natural dB-homomorphism* from X onto X/S.

Remark 1. [11] Let Φ be a dB-homomorphism from $(X, \circ, 1_X)$ into $(Y, *, 1_Y)$. If Φ is surjective, then $(X/\ker \Phi, \theta, [1]_{\ker\Phi}) \cong Y$.

Definition 6. [18] Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) X and $\varnothing$ are members of τ.

(ii) Each finite intersection of members of τ is also a member of τ.

(iii) Each union of members of τ is also a member of τ .

Definition 7. [18] A couple (X, τ) consisting of a set X and a topology τ in X is called a *topological space*. Elements of topological spaces are called *points*. The members of τ are called the *open sets* or *τ-open sets* of the topological space (X, τ) (or of the topology τ).

Theorem 5. [18] Let $\mathcal{B} \subseteq \tau$ be a basis for τ. Then A is open if and only if for each $x \in A$, there exist a U in $\mathcal{B}$ with $x \in U \subseteq A$.

Equivalently, since τ is a basis for itself, then A is open if and only if for each $x \in A$, there exist a U in τ with $x \in U \subseteq A$.

Definition 8. [18] Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X). A map sending open sets to open sets is called an *open map*.

Theorem 6. [18] Let X and Y be topological space and $f : X \to Y$. Then f is open if and only if for each $x \in X$ and neighborhood U of x, there exists a neighborhood W of $f(x)$ in Y such that $W \subseteq f(U)$.

Theorem 7. [18] Let X and Y be topological space and $f : X \to Y$. Then f is continuous if and only if for each $x \in X$ and each neighborhood of W of $f(x)$ in Y, there exists a neighborhood V of x in X such that $f(V) \subseteq W$.

Definition 9. [18] Let $\{Y_\alpha \mid i \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α the topology for Y_α. The Cartersian product topology in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $\langle U_\beta \rangle = \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.

Definition 10. [10] Let X be a dB-algebra. A topology τ on X is called a *dual B-topology* and the couple (X, τ) is called a *dual B-topological space*.

Remark 2. Let (X, τ) be a dual B-topological space. Then A is open if and only if for each $x \in A$, there exist a $U \in \tau$ with $x \in U \subseteq A$.

Definition 11. [10] The triple $(X, \circ, \tau)$ is called a *topological dual B-algebra (or tdB-algebra)* if τ is a dual B-topology and the binary operation $\circ : X \times X \to X$ is continuous where the topology on $X \times X$ is the Cartesian product topology.

Remark 3. [10] Let X be a dB-algebra and nonempty A, $B \subseteq X$. Then,

$$\circ(A \times B) = A \circ B$$

where $A \circ B = \{a \circ b \mid a \in A, b \in B\}$.

Remark 4. Let X be a dB-algebra and nonempty A, $B \subseteq X$. When $B = \{x\}$, then

$$A \circ \{x\} = \{a \circ x \mid a \in A\} = A \circ x.$$

Theorem 8. [10] Let X be a dual B-algebra and τ a dual B-topology. Then $(X, \circ, \tau)$ is a tdB-algebra if and only if for all $x, y \in X$ and $U(x \circ y)$, there exists $U(x)$ and $U(y)$ such that $U(x) \circ U(y) \subseteq U(x \circ y)$.

Example 2. [10] Consider the the dual B-algebra $X = \{1, a, b, c\}$ from Example 1 and let $\tau = \{X, \varnothing, \{1, a\}, \{b, c\}\}$. Then τ is a dual B-topology. Through routine calculations and and with Theorem 8, $(X, \circ, \tau)$ is a tdB-algebra.

Theorem 9. [19] Let $f : X \to Y$ be a function and let $A \subseteq X$ and $C \subseteq Y$. Then

$$\text{(i) } A \subseteq f^{-1}(f(A)), \quad \text{(ii) } f(f^{-1}(B)) \subseteq B, \quad \text{(iii) If } B \subseteq C, f^{-1}(B) \subseteq f^{-1}(C).$$

Theorem 10. [19] Let $f : X \to Y$ be a function and let $\{A_i \mid i \in \mathcal{I}\}$ be a collection of subsets of Y. Then

$$\text{(i) } f^{-1}(\cup_{i\in\mathcal{I}} A_i) = \cup_{i\in\mathcal{I}} f^{-1}(A_i), \text{ and } \quad \text{(ii) } f^{-1}(\cap_{i\in\mathcal{I}} A_i) = \cap_{i\in\mathcal{I}} f^{-1}(A_i).$$

3. Results

In this section, the topology for the quotient dB-algebra was created using the natural dB-homomorphism. The researcher then used the natural dB-homomorphism to determine some tdB-homomorphisms. Furthermore, the researcher established some tdB-isomorphisms.

The first theorem presented a topology for a quotient dB-algebra. Note that a dB-subalgebra S of X should be normal (Definition 3) to form a congruence classes of a dual B-algebra. These congruence classes are exactly the elements of the quotient dB-algebra and were used to define the natural dB-homomorphism (Theorem 4). These concepts were utilized to obtain the elements of the topology for the quotient dB-algebra.

Theorem 11. *Let S be a normal dB-subalgebra of a tdB-algebra $(X, \circ, \tau)$ and Φ is a natural dB-homomorphism from X to X/S. Then, $\tau_S = \left\{Z \subseteq X/S \middle| \Phi^{-1}(Z) \in \tau\right\}$ is a topology on X/S. Furthermore, Φ is a continuous mapping.*

Proof. Since $\Phi^{-1}(\varnothing) = \varnothing \in \tau$, it implies that $\varnothing \in \tau_S$. Moreover, by Theorem 4, Φ is surjective which implies that $\Phi(X) = X/S$. Hence, by Theorem 9 (i), $X \subseteq \Phi^{-1}(\Phi(X)) = \Phi^{-1}(X/S)$ that is $\Phi^{-1}(X/S) = X \in \tau$. Then $X/S \in \tau_S$. Let Z_1, $Z_2 \in \tau_S$. Then $\Phi^{-1}(Z_1)$, $\Phi^{-1}(Z_2) \in \tau$. By Theorem 10 (ii), it follows that $\Phi^{-1}(Z_1 \cap Z_2) = \Phi^{-1}(Z_1) \cap \Phi^{-1}(Z_2) \in \tau$. Hence, $Z_1 \cap Z_2 \in \tau_S$. Let $\{Z_i | i \in \mathcal{I}\}$ be any family of elements in τ_S. Then $\Phi^{-1}(Z_i) \in \tau$, $\forall i \in \mathcal{I}$. By Theorem 10 (i), $\Phi^{-1}(\cup_{i\in\mathcal{I}} Z_i) = \cup_{i\in\mathcal{I}} \Phi^{-1}(Z_i) \in \tau$. Hence, $\cup_{i\in\mathcal{I}} Z_i \in \tau_S$. Therefore, τ_S is a topology on X/S. Furthermore, for all $O \in \tau_S$, $\Phi^{-1}(O) \in \tau$ which implies that Φ is continuous.

The Example 3 illustrates Theorem 11.

Example 3. Consider the tdB-algebra $(X, \circ, \tau)$ in Example 2. Note that $S = \{1\}$ is a normal dB-subalgebra of X. Also, $[1]_S = \{1\}$, $[a]_S = \{a\}$, $[b]_S = \{b\}$, $[c]_S = \{c\}$. Then $X/S = \{\{1\}, \{a\}, \{b\}, \{c\}\}$. Hence, $\tau_S = \{X/S, \varnothing, \{\{1\}, \{a\}\}, \{\{b\}, \{c\}\}\}$.

By Definition 11, $(X/S, *, \tau_S)$ is a tdB-algebra (see Appendix for the manual verification). In general, it does not necessarily imply that if S is a normal dB-subalgebra, $(X/S, *, \tau_S)$ is a tdB-algebra. Notice that in Example 3, the image of each open set in X is open in X/S. Thus the mapping is open. Hence, the significance of the quotient topology gives another way in determining tdB-algebras. That is, to look for open maps between a tdB-algebra X with its corresponding quotient dual B-algebra. If such map exists, the quotient topology makes the binary operation defined in the quotient dual B-algebra continuous which means that it is always guaranteed that the inverse image of each open set in the quotient topology is open in the topology on $X/S \times X/S$. The next theorem summarizes this discussion.

Theorem 12. *Let $(X, \circ, \tau)$ be a tdB-algebra and $S \subseteq X$. If the natural dB-homomorphism $\Phi : X \to X/S$ is an open map, then $(X/S, *, \tau_S)$ is a tdB-algebra.*

Proof. Suppose Φ is open. Let $[a]_S$, $[b]_S \in X/S$ and W be a neighborhood of $[a]_S * [b]_S$. Then $\Phi^{-1}(W) \in \tau$ since Φ is a continuous mapping by Theorem 11. Note that $\Phi(a \circ b) = [a \circ b]_S = [a]_S * [b]_S \in W$ which implies that $a \circ b \in \Phi^{-1}(W) \in \tau$. Since $(X, \circ, \tau)$ is a tdB-algebra, by Theorem 8, there exist neighborhoods $U(a)$ and $V(b)$ of a and b, respectively, such that $U(a) \circ V(b) \subseteq \Phi^{-1}(W)$. Moreover, $\Phi(U(a))$, $\Phi(V(b)) \in \tau_S$ since Φ is open. Note that since $[a]_S = \Phi(a) \in \Phi(U(a))$ implying that $\Phi(U(a))$ is a neighborhood of $[a]_S$. Similarly, $\Phi(V(b))$ is a neighborhood of $[b]_S$. Now

$$\begin{aligned}
\Phi(U(a)) * \Phi(V(b)) &= \{\Phi(x) * \Phi(y) \mid x \in U(a), y \in V(b)\} \\
&= \{\Phi(x \circ y) \mid x \circ y \in U(a) \circ V(b)\} \\
&= \Phi(U(a) \circ V(b)) \subseteq \Phi(\Phi^{-1}(W)) = W \quad \text{since } \Phi \text{ is onto.}
\end{aligned}$$

By Theorem 8, $(X/S, *, \tau_S)$ is a tdB-algebra.

The next example illustrates that not all image of a subset of a tdB-algebra is open.

Example 4. Consider Example 3. Notice that $\{a, b, c\} \subset X$, however, $\Phi(\{a, b, c\}) = \{\{a\}, \{b\}, \{c\}\}$ is not open in X/S.

The next theorem says about the image of a subset of a tdB-algebra when a particular property that the normal dB-subalgebra should hold and its corresponding implication.

Theorem 13. *Let S be open in a tdB-algebra $(X, \circ, \tau)$. Then $\Phi(A)$ is open in X/S for every subset A of X where Φ is a natural dB-homomorphism. In particular, Φ is an open mapping.*

Proof. Let $A \subseteq X$. To show that $\Phi(A)$ is open, it should be proven that $\Phi^{-1}(\Phi(A))$ is open in X/S. Note that $\Phi^{-1}(\Phi(A)) = \{x \in X \mid \Phi(x) \in \Phi(A)\}$. Let $x \in \Phi^{-1}(\Phi(A))$. Then $[x]_S = \Phi(x) \in \Phi(A)$. Hence, $[x]_S = \Phi(a) = [a]_S$, for some $a \in A$. Thus $x \sim^S a$ by Lemma 1. That is, $x \circ a$, $a \circ x \in S$. Since S is open in X and X is a tdB-algebra, it follows that there exist neighborhoods $U_1(x)$ and $U_2(x)$ for x, and $U_1(a)$ and $U_2(a)$ for a such that $U_1(x) \circ U_1(a) \subseteq S$ and $U_2(a) \circ U_2(x) \subseteq S$ (by Theorem 8). Then $(U_1(x) \cap U_2(x)) \circ U_1(a) \subseteq$

$U_1(x) \circ U_1(a) \subseteq S$ and $U_2(a) \circ (U_1(x) \cap U_2(x)) \subseteq U_2(a) \circ U_2(x) \subseteq S$. Since $U_1(x), U_2(x) \in \tau$, it follows that $U_1(x) \cap U_2(x) \in \tau$. Let $y \in U_1(x) \cap U_2(x)$. Then $y \circ a, a \circ y \in S$. Hence, $y \sim^S a$. Thus $\Phi(y) = [y]_S = [a]_S = \Phi(a) \in \Phi(A)$. This implies that $y \in \Phi^{-1}(\Phi(A))$. Hence $U_1(x) \cap U_2(x) \subseteq \Phi^{-1}(\Phi(A))$. By Remark 2, $\Phi^{-1}(\Phi(A))$ is open in X which proves that $\Phi(A)$ is open in X/S. In particular, if A is open, Φ is an open mapping. □

Corollary 1 is a consequence of Theorem 13 and Theorem 12.

Corollary 1. *Let S be open in a tdB-algebra $(X, \circ, \tau)$. Then $(X/S, *, \tau_S)$ is a tdB-algebra.*

Definition 12. Let $(X, \circ, \tau)$ and $(Y, *, \tau^*)$ be *tdB*-algebras. A mapping $\Phi : X \to Y$ is called a *topological dual B-homomorphism* (*tdB*-homomorphism) if

(i) Φ is a *dB*-homomorphism from $(X, \circ, 1_X)$ to $(Y, *, 1_Y)$, and

(ii) Φ is a continuous mapping from (X, τ) to (Y, τ^*).

Example 5. Consider the *tdB*-algebra $(X, \circ, \tau)$ in Example 2. Define a mapping $\Phi : X \to X$ as follows:

$$\Phi(1) = 1, \quad \Phi(a) = a, \quad \Phi(b) = c, \quad \Phi(c) = b.$$

By a program (see Appendix), Φ is a *dB*-homomorphism. By manual computations Φ is a continuous mapping. Hence, Φ is a *tdB*-homomorphism.

Definition 13. A *tdB*-homomorphism Φ is said to be *open* if Φ is an open mapping of the dual B-topological spaces.

Example 6. Consider the same *tdB*-homomorphism in Example 5. By manual computations, Φ is an open mapping.

The next result establishes the *tdB*-homomorphisms and some of its properties with the use of some properties of the natural *dB*-homomorphism.

Theorem 14. *Let $(X, \circ, \tau)$ be a tdB-algebra and $S \subseteq X$. Then the following statements hold:*

(i) *If the the natural dB-homomorphism Φ is open from X onto X/S, then Φ is a tdB-homomorphism; and*

(ii) *If S is open, then the natural dB-homomorphism Φ is an open tdB-homomorphism.*

Proof. Clearly, Φ is a *dB*-homomorphism from X to X/S and by Theorem 11, Φ is continuous. (i) Note that X/S is a *tdB*-algebra by Theorem 12. By Definition 12, it follows that Φ is a *tdB*-homomorphism. (ii) Suppose S is open. Then by Theorem 13 and Corollary 1, Φ is an open mapping and X/S is a *tdB*-algebra, respectively. Hence, Φ is an open *tdB*-homomorphism. □

The next theorem establishes the topological dual B-homomorphism using two natural *dB*-homomorphism.

Theorem 15. *Let S and J be normal dB-subalgebras of a tdB-algebra $(X, \circ, \tau)$ such that $S \subset J$. Define a map f from $(X/S, *, \tau_S)$ to $(X/J, *', \tau_J)$ by $f([x]_S) = [x]_J$ for all $[x]_S \in X/S$. If the natural dB-homomorphism Φ from X onto X/S and the natural dB-homomorphism Ψ from X onto X/J are open mappings, then f is a tdB-homomorphism.*

Proof. Claim 1: f is well-defined. Let $[x]_S, [y]_S \in X/S$ such that $[x]_S = [y]_S$. Then $x \sim^S y$ by Lemma 1. It follows that $x \circ y, y \circ x \in S$. Since $S \subseteq J$, $x \circ y, y \circ x \in J$ which implies that $x \sim^J y$. Hence, $[x]_J = [y]_J$. Consequently, $f([x]_S) = f([y]_S)$. Therefore, f is well-defined.
Claim 2: f is dB-homomorphism. Let $[x]_S, [y]_S \in X/S$. Then $f([x]_S * [y]_S) = f([x \circ y]_S) = [x \circ y]_J = [x]_J *' [y]_J = f([x]_S) *' f([y]_S)$. Therefore, f is dB-homomorphism.
Claim 3: f is continuous. Suppose that Φ and Ψ are open mappings. By Theorem 12, X/S and X/J are tdB-algebras. Let U be open in X/J and $[x]_S \in f^{-1}(U)$. Then $f([x]_S) \in f(f^{-1}(U)) \subseteq U$ by Theorem 9 (ii). It follows that $[x]_J = f([x]_S) \in U$. Since U is open, there exists a neighborhood W of $f([x]_S) = [x]_J$ in X/J such that $W \subseteq U$ by Remark 2. Moreover, $[x]_S \in f^{-1}(W)$. Now

$$\begin{aligned} f^{-1}(W) &= \{[x']_S \in X/S \mid f([x']_S) = [x']_J \in W\} \\ &= \{[x']_S \in X/S \mid \Psi(x') = [x']_J \in W\} \\ &= \{[x']_S \in X/S \mid x' \in \Psi^{-1}(W)\} \\ &= \{\Phi(x') \in X/S \mid x' \in \Psi^{-1}(W)\} \\ &= \Phi(\Psi^{-1}(W)) \end{aligned}$$

Since W is open in X/J and Ψ is continuous by Theorem 11, then $\Psi^{-1}(W)$ is open in X. Since Φ is an open mapping, $\Phi(\Psi^{-1}(W)) = f^{-1}(W)$ is open in X/S. Moreover, since $W \subseteq U$, it follows that $f^{-1}(W) \subseteq f^{-1}(U)$ by Theorem 9 (iii). By Remark 2, $f^{-1}(U)$ is open in X/S. Hence, for any $[x]_S \in f^{-1}(U)$, there exists a neighborhood $f^{-1}(W)$ of $[x]_S$ with $[x]_S \in f^{-1}(W) \subseteq f^{-1}(U)$. Therefore, f is continuous.
By *Claim 1,2,3*, f is a tdB-homomorphism. □

The next corollary is a consequence of Theorem 13 and Theorem 15. In Theorem 13, if a normal dB-subalgebra S is open, the natural dB-homomorphism with respect to S is an open mapping which satisfies the hypothesis of Theorem 15.

Corollary 2. *Let S and J be normal dB-subalgebras of a tdB-algebra $(X, \circ, \tau)$ such that $S \subset J$. Define a map f from $(X/S, *, \tau_S)$ to $(X/J, *', \tau_J)$ by $f([x]_S) = [x]_J$ for all $[x]_S \in X/S$. If S is open and the natural dB-homomorphism Ψ from X onto X/J is an open mapping, then f is a tdB-homomorphism.*

The next result is a consequence of Theorem 13 and Corollary 2. Similarly, if a normal dB-subalgebra J is open, the natural dB-homomorphism with respect to J is an open mapping by Theorem 13 which satisfies the hypothesis of Corollary 2.

Corollary 3. *Let S and J be normal dB-subalgebras of a tdB-algebra $(X, \circ, \tau)$ such that $S \subset J$. Define a map f from $(X/S, *, \tau_S)$ to $(X/J, *', \tau_J)$ by $f([x]_S) = [x]_J$ for all $[x]_S \in X/S$. If S and J are open, then f is a tdB-homomorphism.*

For a fixed element s of a tdB-algebra $(X, \circ, \tau)$, define a self-map $f_s : X \to X$ by $f_s(x) = x \circ s$ for all $x \in X$.

Definition 14. A tdB-algebra is said to be *transitive open* if for each $s \in X$, the self-map f_s is open and continuous.

The next lemma will be used to prove additional tdB-homomorphism between transitive open tdB-algebras.

Lemma 2. *Let A be open in a transitive open tdB-algebra $(X, \circ, \tau)$ and $x \in X$. Then the following statements hold:*

(i) *$f_x(A) = A \circ x$ is open in X; and*

(ii) *$f_x^{-1}(A) = \{y \in X \mid y \circ x = f_x(y) \in A\}$ is open in X.*

Proof. Since X is a transitive open tdB-algebra, it follows that f_x is open and continuous. Since A is open in X,

$$\begin{aligned} f_x(A) &= \{y \in X \mid y = f_x(a) \text{ for some } a \in A\} \\ &= \{y \in X \mid y = a \circ x \text{ for some } a \in A\} \\ &= \{a \circ x \mid a \in A\} = A \circ x \quad \text{is open in } X. \end{aligned}$$

and $f_x^{-1}(A) = \{y \in X \mid y \circ x = f_x(y) \in A\}$ is open in X. □

The next theorem obtains a tdB-homomorphism and tdB-isomorphism between two transitive open tdB-algebras with some conditions to hold.

Theorem 16. *Let X and Y be transitive open tdB-algebras and g a dB-homomorphism from $(X, \circ, 1_X)$ to $(Y, *, 1_Y)$. Then the following statements hold:*

(i) *If for each neighborhood U of 1_Y in Y, there exists a neighborhood of V of 1_X in X such that $g(V) \subseteq U$, then g is a continuous mapping. That is, g is a tdB-homomorphism; and*

(ii) *If for each neighborhood of V of 1_X in X, there exists a neighborhood U of 1_Y in Y such that $Y \subseteq g(V)$, then g is an open mapping.*

Proof. (i) Assume that A is open in Y. If $A \cap \mathrm{Im}(g) = \varnothing$, then $g^{-1}(A) = \varnothing$, which is open in X. Suppose $A \cap \mathrm{Im}(g) \neq \varnothing$ and $x \in g^{-1}(A)$. Then $g(x) =\in A \cap \mathrm{Im}(g)$. Since Y is transitive open and by Lemma 2 (ii), $f_{g(x)}^{-1}(A)$ is open in Y. Let $B = f_{g(x)}^{-1}(A)$. Note that by DB2, $g(x) = 1_Y * g(x) = f_{g(x)}(1_Y) \in A$. Hence, $1_Y \in B$. That is, B is a neighborhood of 1_Y. By hypothesis, there exists a neighborhood V of 1_X in X such that $g(V) \subseteq B$. Note that $V \circ x$ is open in X by Lemma 2 (i). By DB2, $x = 1_X \circ x \in V \circ x$. Note that for any $v \in B$, $v * g(x) \in B * g(x)$. Also, $v * g(x) = f_{g(x)}(v) \in A$. Then $B * g(x) \subseteq A$. Now, $g(V \circ x) = g(V) * g(x) \subseteq B * g(x) \subseteq A$. Thus, $x \in V \circ x \subseteq g^{-1}(g(V \circ x)) \subseteq g^{-1}(A)$. Therefore, for all $x \in g^{-1}(A)$ there exist a neighborhood $V \circ x$ of x in X such that

$V \circ x \subseteq g^{-1}(A)$. By Remark 2, $g^{-1}(A)$ is open in X. Therefore, g is continuous.
(ii) Assume that A is open in X. Let $y \in g(A)$. Then $y = g(x)$ for some $x \in A$. Note that $f_x^{-1}(A)$ is open in X by Lemma 2 (ii). Now by DB2, $x = 1_X \circ x = f_x(1_X) \in A$. Hence, $1_X \in f_x^{-1}(A)$. Let $B = f_x^{-1}(A)$. By hypothesis, there exist a neighborhood U of 1_Y in Y such that $U \subseteq g(B)$. Note that $U * y$ is open in Y by Lemma 2 (i). By DB2, $y = 1_Y * y \in U * y$. Let $u \in B$. Then $u \circ x \in B \circ x$, $u \circ x = f_x(u) \in A$. Hence, $B \circ x \subseteq A$. Then, $g(B \circ x) \subseteq g(A)$. Now, $U * y = U * g(x) \subseteq g(B) * g(x) = g(B \circ x) \subseteq g(A)$. Therefore, for all $y \in g(A)$, there exists a neighborhood $U * y$ of y in Y such that $U * y \subseteq g(A)$. By Remark 2, $g(A)$ is open in Y. Hence, g is an open mapping. □

Definition 15. Let $(X, \circ, \tau)$ and $(Y, *, \tau^*)$ be *tdB*-algebras. A mapping $\Phi : X \to Y$ is called a *topological dual B-isomorphism* (or *tdB- isomorphism*) if

(i) Φ is a *dB*-isomorphism from $(X, \circ, 1_X)$ to $(Y, *, 1_Y)$, and

(ii) Φ is continuous and open from (X, τ) to (Y, τ^*).

In Definition 15, if the map from X to Y satisfies the two conditions, then X is topologically isomorphic with Y. That is, X is isomorphic with respect to the underlying algebraic structures and at the same time homeomorphic with respect to the underlying topological structures. Hence, isomorphic *tdB*-algebras means that there exists a bijective map that preserves the underlying algebraic structures. Moreover, homeomorphic *tdB*-algebras means that there exists a map that is continuous and open (given that the map is bijective) between the topological spaces. We note that the relationship of open maps and continuous maps from the topological spaces is independent. Even if the map is bijective and continuous, open map is not guaranteed. Thus, in Definition 15 a *tdB*-homomorphism that is bijective is not necessarily a *tdB*-isomorphism. But a *tdB*-isomorphism map is a *tdB*-homomorphism map.

Example 7. Consider Example 5 and Example 6, Φ is a *dB*-isomorphism, and Φ is a continuous and open mapping from (X, τ) to (X, τ). Hence, Φ is a *tdB*-isomorphism.

The next corollary is an immediate result from Theorem 16.

Corollary 4. *Let X and Y be transitive open tdB-algebras and g a dB-isomorphism from $(X, \circ, 1_X)$ to $(Y, *, 1_Y)$. If for each neighborhood U of 1_Y in Y, there exists a neighborhood of V of 1_X in X such that $g(V) \subseteq U$ and if for each neighborhood of V of 1_X in X, there exists a neighborhood U of 1_Y in Y such that $Y \subseteq g(V)$, then g is a tdB-isomorphism.*

In Theorem 14 (ii), a subset S of a *tdB*-algebra should be open to imply that the natural *dB*-homomorphism is an open *tdB*-homomorphism that is, the natural *dB*-homomorphism is an open and continuous map. Now in terms of mapping from a dual B-algebra to a quotient *dB*-algebra, the natural *dB*-homomorphism is surjective by Theorem 4 and in general, it is not a one-to-one mapping. If the natural *dB*-homomorphism is one-to-one, then it is a *tdB*-isomorphism by Definition 15. Hence, with Theorem 14 (ii) such that the natural *dB*-homomorphism is bijective, the next corollary is presented.

Corollary 5. *Let $(X, \circ, \tau)$ be a tdB algebra and S be open in X. If the natural dB-homomorphism Φ is bijective, then Φ is a tdB-isomorphism.*

Theorem 17. *Let $(X, \circ, \tau)$ and $(Y, *, \tau')$ be tdB-algebras and g is an open tdB-homomorphism from $(X, \circ, \tau)$ onto $(Y, *, \tau')$ with $\ker g = S$ and $\{1_Y\}$ is open in Y. If S is a normal dB-subalgebra of X and define $f : X/S \to Y$ by $f([a]_S) = g(a)$, then f is a tdB-isomorphism. That is, X/S is topologically isomorphic to Y.*

Proof. Since g is surjective, by Remark 1, $X/S \cong Y$. That is a map $f : (X, \circ, 1_X) \to (Y, *, 1_Y)$ is a dB-isomorphism. It is left to show that f is continuous and open. Let U be open in Y and $[x]_S \in f^{-1}(U)$. Then $g(x) = f([x]_S) \in f(f^{-1}(U)) \subseteq U$. That is, U is a neighborhood of $g(x)$. Since g is continuous and $y \in U$, it follows from Theorem 7 that there exists a neighborhood V of x such that $g(V) \subseteq U$. Now, g is continuous and $\{1_Y\}$ is open in Y, it follows that $S = \ker g = g^{-1}(\{1_Y\})$ is open in X. Since S is open in X, then by Theorem 13, the natural dB-homomorphism map Φ from $X \to X/S$ is an open mapping. Thus $\Phi(V)$ is open in X/S with $\Phi(x) = [x]_S \in \Phi(V)$. Now, $f(\Phi(V)) = \{f([x']) \mid [x'] \in \Phi(V)\} = \{g(x') \mid x' \in V\} = g(V) \subseteq U$. Hence, $f^{-1}(f(\Phi(V)) \subseteq f^{-1}(U)$ by Theorem 9 (iii). Now, $\Phi(V) = f^{-1}(f(\Phi(V)) \subseteq f^{-1}(U)$ since f is one-to-one. Therefore, for every $[x]_S \in f^{-1}(U)$, there exists a neighborhood $\Phi(V)$ of $[x]_S$ in X/S such that $\Phi(V) \subseteq f^{-1}(U)$. By Remark 2, $f^{-1}(U)$ is open in X/S. Therefore, f is continuous. Let $[x]_S \in X/S$ and U be a neighborhood of $[x]_S$ in X/S. Then $f([x]_S) = g(x) \in Y$. Since $\Phi(x) = [x]_S \in U$, then $x \in \Phi^{-1}(U)$. Since the natural dB-homomorphism Φ is continuous, $\Phi^{-1}(U)$ is open in X which contains x. Let $W = \Phi^{-1}(U)$. Since g is an open mapping, $g(W)$ is open in Y and $f([x]_S) = g(x) \in g(W)$. By Remark 2, there exists a neighborhood G of $f([x]_S)$ in Y with $G \subseteq g(W)$. Let $a \in f^{-1}(G)$. Then $f(a) \in G \subseteq g(W)$. That is, $f(a) = g(a')$ for some $a' \in W$. Since $a' \in W = \Phi^{-1}(U)$, $[a']_S = \Phi(a') \in U$. Thus, $a = f^{-1}(f(a)) = f^{-1}(g(a')) = f^{-1}(f([a']_S)) = [a']_S \in U$ which implies that $f^{-1}(G) \subseteq U$. Hence, $G = f(f^{-1}(G)) \subseteq f(U)$ since f is surjective. By Theorem 6, f is open. Consequently, f is a tdB-isomorphism. □

The next result is a consequence of Theorem 17. With $\ker g = \{1_X\}$, g is one-to-one by Theorem 3 and by Theorem 17, g is open, continuous, and surjective. By Definition 15, g is a tdB-isomorphism.

Corollary 6. *Let $(X, \circ, \tau$ and $(Y, *, \tau')$ be tdB-algebras with $\{1_Y\}$ open in Y. If g is an open tdB-homomorphism from $(X, \circ, \tau)$ onto $(Y, *, \tau^{'})$ having $kerg = \{1_X\}$, then g is a tdB-isomorphism. That is, X is topologically isomorphic to Y.*

4. Conclusions

This study provided the existence of the tdB-homomorphism with the use of a program and manual computations. Moreover, this study proved that it is possible to construct a topology for the quotient dual B-algebra using the natural dB-homomorphism in such a way that it will be useful in establishing the tdB-homomorphism and the tdB-isomorphism. Indeed, most of the results were anchored on this topology. This study also introduced the transitive tdB-algebras and some of its properties. These properties were used to obtain a tdB-homomorphism between transitive open tdB-algebras. Future works may explore characterization theorems for the topological dual B-homomorphisms. That is, given an arbitrary mapping between topological dual B-algebras provide a necessary and sufficient condition to conclude that the map is a topological dual B-homomorphism.

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Appendix

Verification that the $(X/S, *, \tau_S)$ in Example 3 is a tdB-algebra

Consider the tdB-algebra in Example 2. That is the dB-algebra $X = \{1, a, b, c\}$ with binary operation "$\circ$" as defined by the Cayley table below,

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

and a topology $\tau = \{X, \varnothing, \{1, a\}, \{b, c\}\}$. Note that $S = \{1\}$ is a normal dB-subalgebra of X. Also, $[1]_S = \{1\},\ [a]_S = \{a\},\ [b]_S = \{b\},\ [c]_S = \{c\}$. Then $X/S = \{[1]_S, [a]_S, [b]_S, [c]_S\}$. Hence, $\tau_S = \{X/S, \varnothing, \{[1]_S, [a]_S\}, \{[b]_S, [c]_S\}\}$. Hence, the cayley table for the operation defined on X/S is described below:

$*$	$[1]_S$	$[a]_S$	$[b]_S$	$[c]_S$
$[1]_S$	$[1]_S$	$[a]_S$	$[b]_S$	$[c]_S$
$[a]_S$	$[a]_S$	$[1]_S$	$[c]_S$	$[b]_S$
$[b]_S$	$[b]_S$	$[c]_S$	$[1]_S$	$[a]_S$
$[c]_S$	$[c]_S$	$[b]_S$	$[a]_S$	$[1]_S$

Now, observed that

$$\begin{aligned} *^{-1}(X/S) &= X/S \times X/S \\ *^{-1}(\varnothing) &= \varnothing \times \varnothing \\ *^{-1}(\{[1]_S, [a]_S\}) &= (\{[1]_S, [a]_S\} \times \{[1]_S, [a]_S\}) \cup (\{[b]_S, [c]_S\} \times \{[b]_S, [c]_S\}) \\ *^{-1}(\{[b]_S, [c]_S\}\}) &= (\{[1]_S, [a]_S\} \times \{[b]_S, [c]_S\}\} \cup (\{[b]_S, [c]_S\} \times \{[1]_S, [a]_S\}) \end{aligned}$$

They are all basic open sets for $X \times X$. Hence, $*$ is continuous, it follows that $(X/S, *, \tau_S)$ is a tdB-algebra.

Python program to verify the *tdB*-homomorphism in Example 5

```
class Algebra:
    def __init__(self, elements, cayley_table):
        self.elements = elements
        self.cayley_table = cayley_table

    def operation(self, a, b):
        index_a = self.elements.index(a)
        index_b = self.elements.index(b)
        return self.cayley_table[index_a][index_b]

def is_homomorphism(h, algebra_A, algebra_B):
    for a in algebra_A.elements:
        for b in algebra_A.elements:
            left_side = h(algebra_A.operation(a, b))
            right_side = algebra_B.operation(h(a), h(b))
            if left_side != right_side:
                return False
    return True

elements_A = ['a', 'b', 'c', 'd']
cayley_table_A = [
    ['a', 'b', 'c', 'd'],
    ['b', 'a', 'd', 'c'],
    ['c', 'd', 'a', 'b'],
    ['d', 'c', 'b', 'a']
]

elements_B = ['a', 'b', 'c', 'd']
cayley_table_B = [
    ['a', 'b', 'c', 'd'],
    ['b', 'a', 'd', 'c'],
    ['c', 'd', 'a', 'b'],
    ['d', 'c', 'b', 'a']
]

A = Algebra(elements_A, cayley_table_A)
B = Algebra(elements_B, cayley_table_B)

def h(element):
    mapping = {
        'a': 'a',
        'b': 'b',
        'c': 'd',
        'd': 'c'
    }
    return mapping[element]
```

```
if is_homomorphism(h, A, B):
    print("h is a homomorphism from A to B.")
else:
    print("h is not a homomorphism from A to B.")
```

Output:

```
PS C:\Users\johar> & C:/Users/johar/OneDrive/Documents/Phyton/python.
    exe "c:/Users/johar/OneDrive/Documents/MS_Thesis_Proposal/Proposal
    - Copy/Test Folder/Homomorphism.py"
h is a homomorphism from A to B.
PS C:\Users\johar>
```