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Some Conditions for Certain Two Families of Analytic Functions Associated with Touchard Polynomials

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Abstract. This paper examines necessary and sufficient conditions for a series with Touchard polynomials coefficients and a linear operator defined by using coefficients of Touchard polynomials to be in certain families of analytic functions. Furthermore, we estimate certain inclusion relations between some families. Finally, we give a necessary and sufficient condition for a special integral operator to be in the certain family. Special cases for the families of starlike and convex functions are also considered.

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Key Words and Phrases: Analytic, univalent, Touchard polynomials, Poisson distribution, Bell polynomials, geometric functions

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1. Preliminaries

A fascinating and contemporary area of study is the use of special functions in geometric function theory. It's widely used in many fields, including engineering, technology and mathematics. Unexpectedly, L. de Branges [1] solved the well-known Bieberbach conjecture using the generalized hypergeometric function. Numerous types of special functions have analytical and geometric features covered in a large body of literature, particularly the generalized Gaussian hypergeometric functions ([2–4]).

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A series of polynomials with important applications in number theory, probability theory, and combinatorics are called Touchard polynomials (also called the exponential polynomials (see [5]) or Bell polynomials (see [6]). These polynomials, which bear the name Jacques Touchard [7], are closely related to Bell numbers, which tally the number of ways in which a set can be divided.

A set of t elements can be divided into non-empty subsets in as many ways as possible, with Y distinct labels applied to each subset. This is represented by the expression $TP_t(Y)$. Hence, the Touchard polynomials match the Bell numbers, which count a sets total number of partitions when $Y = 1$. If Y is a random variable with a Poisson distribution and an expected value v, its $\varepsilon-$th moment can be expressed as $E(Y_s) = TP(s, v)$; this gives the following form:

$$TP(s,v) = e^s \sum_{\varepsilon=0}^{\infty} \frac{\varepsilon^v s^\varepsilon}{\varepsilon!} z^\varepsilon$$

Jacques Touchard examined these polynomials for solving both linear and nonlinear integral equations and expanded upon the Bell polynomials to examine a range of permutation inventory issues where the cycles have particular characteristics. In addition, he investigated and introduced a family of related polynomials, recurrence relations, linkages to the other known polynomials, and an exponential generating function (see [5] and [8]). Since it is often difficult to solve integral equations analytically, we must often find approximate solutions. In this situation, the "Touchard polynomials method" is used to solve the linear "Volterra integro-differential equation". The Touchard polynomials method has been applied to solve linear and nonlinear Volterra (Fredholm) integral equations.

Touchard polynomials are crucial for the development of generating functions, particularly exponential generating functions, which allow for the simplification and analysis of sums of exponential terms. doing integral representations, factorial sums, and helping to solve differential equations and recurrence relations.

Their usefulness in various domains makes them an important tool in applied and theoretical mathematics, particularly when it comes to the analysis of special functions, sequences, and series.

The result of the second force is presented using the coefficients of Touchard polynomials as below (see [9]):

$$\digamma_s^v(z) = z + \sum_{\varepsilon=2}^{\infty} \frac{(\varepsilon-1)^v s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!} z^\varepsilon, \quad z \in \Delta,$$

where $s > 0$, $v \geq 0$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and the radius of convergence of above series is infinity by ratio test.

Let Π be the family of analytic and univalent functions in Δ, and Υ be the family of functions $B \in \Pi$ of the form:

$$B(z) = z + \sum_{\varepsilon=2}^{\infty} b_\varepsilon z^\varepsilon, \quad z \in \Delta, \tag{1}$$

such that $B(0) = B'(0) - 1 = 0$.

Now, by the convolution product $(*)$, we define the linear operator $\Lambda(s, v, z)B : \Upsilon \to \Upsilon$

$$\Lambda(s, v, z)B = \digamma_s^v(z) * B(z) = z + \sum_{\varepsilon=2}^{\infty} \frac{(\varepsilon - 1)^v s^{\varepsilon-1} e^{-s}}{(\varepsilon - 1)!} b_\varepsilon z^\varepsilon.$$

We examine the following two subfamilies of analytic functions considered by Thulasiram et al. [10].

A function $B(z)$ of the form (1) is said to be in the subfamily $Q(\delta, \zeta)$, if satisfies the inequality

$$\operatorname{Re}\left(\frac{zB'(z) + \delta z^2 B''(z)}{B(z)}\right) > \zeta, \quad z \in \Delta, \tag{2}$$

where $0 \leq \delta < 1$ and $0 \leq \zeta < 1$.

And be in the subfamily $K(\delta, \zeta)$, if satisfies the inequality

$$\operatorname{Re}\left(\frac{z\left(zB'(z) + \delta z^2 B''(z)\right)'}{zB'(z)}\right) > \zeta, \quad z \in \Delta. \tag{3}$$

Example 1. *[8] If we choosing $\delta = 0$, we get the family $\mathrm{Q}(0, \zeta) \equiv S^*(\zeta)$ (family of starlike function), consists of the functions satisfying the inequality*

$$Re\left(\frac{zB'(z)}{B(z)}\right) > \zeta, \quad z \in \Delta,$$

also we get the subfamily $K(0, \zeta) \equiv \mathrm{K}(\zeta)$ (family of convex function), consists of the functions satisfying the inequality

$$Re\left(1 + \frac{zB''(z)}{B'(z)}\right) > \zeta, \quad z \in \Delta.$$

Numerous authors have determined several necessary and sufficient conditions of different special functions (see [11], [12], [13]-[14]) and different probability distribution series (see [15–19]) for certain families of analytic and univalent functions. Motivated by the works of Ali et al. [20], Murugusundaramoorthy et al. [9] and Soupramanien et al. [21] for Touchard polynomials to be in certain families of analytic functions, in this paper, we determine necessary and sufficient conditions for the function $\digamma_s^v(z)$ to be in the families $Q(\delta, \zeta)$ and $K(\delta, \zeta)$. Furthermore, we estimate certain inclusion relations between the families $\mathcal{R}^\tau(A_1, A_2)$ and $K(\delta, \zeta)$. Finally, we give a necessary and sufficient condition for an integral operator $\mathcal{J}_s^v(z) = \int_0^z \frac{\Lambda(s,v,\varepsilon)}{\varepsilon} d\varepsilon$ to be in the family $K(\delta, \zeta)$.

For function $B \in \Pi$, we will need the following definition and lemma for our investigation.

Definition 1. *The v^{th} moment of the Poisson distribution is defined as*

$$\mu_v' = \sum_{\varepsilon=0}^{\infty} \frac{\varepsilon^v s^\varepsilon}{\varepsilon!} e^{-s}.$$

Lemma 1. *[10] A function $B \in \mathrm{Q}(\delta, \zeta)$ if*

$$\sum_{\varepsilon=2}^{\infty} \left[(\varepsilon + \varepsilon\delta(\varepsilon - 1) - \zeta \right] |b_\varepsilon| \leq 1 - \zeta, \tag{4}$$

and $B \in K(\delta, \zeta)$ if

$$\sum_{\varepsilon=2}^{\infty} \varepsilon \left[(\varepsilon + \varepsilon\delta(\varepsilon - 1) - \zeta \right] |b_\varepsilon| \leq 1 - \zeta. \tag{5}$$

2. Necessary and Sufficient Conditions

In this section, we give necessary and sufficient conditions for the function $\digamma_s^v(z)$ to be in the families $Q(\delta, \zeta)$ and $K(\delta, \zeta)$.

For our next results, we employ the following notations for convenience:

$$\sum_{\varepsilon=2}^{\infty} \frac{s^{\varepsilon-1}}{(\varepsilon-1)!} = e^s - 1, \tag{6}$$

and

$$\sum_{\varepsilon=2}^{\infty} \frac{s^{\varepsilon-1}}{(\varepsilon-q)!} = s^{q-1} e^s, \quad q = 2, 3, 4, \cdots. \tag{7}$$

Theorem 1. *If $s > 0$ and $v \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, then $\digamma_s^v(z) \in \mathrm{Q}(\delta, \zeta)$ if and only if*

$$\begin{cases} \delta\mu_{v+2}' + (\delta+1)\mu_{v+1}' + (1-\zeta)\,\mu_v' & \text{if} \quad v \geq 1 \\ \\ \delta s^2 + (2\delta+1)s + (1-\zeta)\left(1 - e^{-s}\right) & \text{if} \quad v = 0 \end{cases} \leq \zeta. \tag{8}$$

Proof. To prove that $\digamma_s^v(z) \in Q(\delta, \zeta)$, by virtue of inequality (4), it suffices to show that

$$\sum_{\varepsilon=2}^{\infty} \left[(\varepsilon + \varepsilon\delta(\varepsilon - 1) - \zeta \right] \frac{(\varepsilon-1)^v s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!} \leq \zeta.$$

Now

$$\sum_{\varepsilon=2}^{\infty} \left[(\varepsilon + \varepsilon\delta(\varepsilon - 1) - \zeta \right] \frac{(\varepsilon-1)^v s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!}$$

$$= \sum_{\varepsilon=2}^{\infty} \left(\delta(\varepsilon-1)(\varepsilon-2) + (2\delta+1)(\varepsilon-1) + 1 - \zeta\right) \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!}$$

$$= \sum_{\varepsilon=2}^{\infty} \left(\delta(\varepsilon-1)^{2} + (\delta+1)(\varepsilon-1) + 1 - \zeta\right) \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!}$$

$$= e^{-s}\left[\sum_{\varepsilon=2}^{\infty} \delta \frac{(\varepsilon-1)^{v+2} s^{\varepsilon-1}}{(\varepsilon-1)!} + \sum_{\varepsilon=2}^{\infty} (\delta+1) \frac{(\varepsilon-1)^{v+1} s^{\varepsilon-1}}{(\varepsilon-1)!} + \sum_{\varepsilon=2}^{\infty} (1-\zeta) \frac{(\varepsilon-1)^{v} s^{\varepsilon-1}}{(\varepsilon-1)!}\right]$$

$$= e^{-s}\left[\sum_{\varepsilon=1}^{\infty} \delta \frac{\varepsilon^{v+2} s^{\varepsilon}}{\varepsilon!} + \sum_{\varepsilon=1}^{\infty} (\delta+1) \frac{\varepsilon^{v+1} s^{\varepsilon}}{\varepsilon!} + \sum_{\varepsilon=1}^{\infty} (1-\zeta) \frac{\varepsilon^{v} s^{\varepsilon}}{\varepsilon!}\right]$$

$$= \begin{cases} \delta\mu'_{v+2} + (\delta+1)\mu'_{v+1} + (1-\zeta)\,\mu'_{v} & \text{if } \ v \geq 1 \\ \\ \delta s^{2} + (2\delta+1)s + (1-\zeta)\left(1 - e^{-s}\right) & \text{if } \ v = 0 \end{cases}.$$

But the upper bound for this expression is ζ if and only if (8) holds. Thus the proof is complete

Theorem 2. *If $s > 0$ and $v \in \mathbb{N}_0$, then $\digamma_{s}^{v}(z) \in K(\delta, \zeta)$ if and only if*

$$\begin{cases} \delta\mu'_{v+3} + (2\delta+1)\mu'_{v+2} + (\delta-\zeta+2)\mu'_{v+1} + (1-\zeta)\,\mu'_{v}, & \text{if} \qquad v \geq 1 \\ \\ \delta s^{3} + (5\delta+1)s^{2} + (4\delta-\zeta+3)s + (1-\zeta)\left(1-e^{-s}\right), & \text{if} \qquad v = 0 \end{cases} \leq \zeta. \tag{9}$$

Proof. To prove that $\digamma_{s}^{v}(z) \in K(\delta, \zeta)$, by virtue of inequality (5), it suffices to show that

$$\sum_{\varepsilon=2}^{\infty} \varepsilon\left[(\varepsilon + \varepsilon\delta(\varepsilon-1) - \zeta\right] \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!} \leq \zeta.$$

Now

$$\begin{aligned} & \sum_{\varepsilon=2}^{\infty} \varepsilon\left[(\varepsilon + \varepsilon\delta(\varepsilon-1) - \zeta\right] \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!} \\ = & \sum_{\varepsilon=2}^{\infty} [\delta(\varepsilon-1)(\varepsilon-2)(\varepsilon-3) + (5\delta+1)(\varepsilon-1)(\varepsilon-2) \\ & + (4\delta-\zeta+3)(\varepsilon-1) + 1 - \zeta] \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!} \end{aligned}$$

$$= \sum_{\varepsilon=2}^{\infty} \left[\delta(\varepsilon-1)^{3} + (2\delta+1)(\varepsilon-1)^{2} + (\delta-\zeta+2)(\varepsilon-1) + 1 - \zeta\right] \frac{(\varepsilon-1)^{v} s^{\varepsilon-1} e^{-s}}{(\varepsilon-1)!}$$

$$= \quad e^{-s}\left[\sum_{\varepsilon=2}^{\infty}\delta\frac{(\varepsilon-1)^{v+3}s^{\varepsilon-1}}{(\varepsilon-1)!}+\sum_{\varepsilon=2}^{\infty}(2\delta+1)\frac{(\varepsilon-1)^{v+2}s^{\varepsilon-1}}{(\varepsilon-1)!}\right.$$
$$\left.+\sum_{\varepsilon=2}^{\infty}(\delta-\zeta+2)\frac{(\varepsilon-1)^{v+1}s^{\varepsilon-1}}{(\varepsilon-1)!}+\sum_{\varepsilon=2}^{\infty}(1-\zeta)\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}}{(\varepsilon-1)!}\right]$$

$$= e^{-s}\left[\sum_{\varepsilon=1}^{\infty}\delta\frac{\varepsilon^{v+3}s^{\varepsilon}}{\varepsilon!}+\sum_{\varepsilon=1}^{\infty}(2\delta+1)\frac{\varepsilon^{v+2}s^{\varepsilon}}{\varepsilon!}+\sum_{\varepsilon=1}^{\infty}(\delta-\zeta+2)\frac{\varepsilon^{v+1}s^{\varepsilon}}{\varepsilon!}+\sum_{\varepsilon=1}^{\infty}(1-\zeta)\frac{\varepsilon^{v}s^{\varepsilon}}{\varepsilon!}\right]$$

$$= \begin{cases} \delta\mu'_{v+3}+(2\delta+1)\mu'_{v+2}+(\delta-\zeta+2)\mu'_{v+1}+(1-\zeta)\mu'_{v}, & \text{if} \quad v\geq 1 \\ \delta(s^3+3s^2+s)+(2\delta+1)(s^2+s)+(\delta-\zeta+2)s+(1-\zeta)(1-e^{-s}), & \text{if} \quad v=0 \end{cases}$$

$$\equiv \begin{cases} \delta\mu'_{v+3}+(2\delta+1)\mu'_{v+2}+(\delta-\zeta+2)\mu'_{v+1}+(1-\zeta)\mu'_{v}, & \text{if} \quad v\geq 1 \\ \delta s^3+(5\delta+1)s^2+(4\delta-\zeta+3)s+(1-\zeta)(1-e^{-s}), & \text{if} \quad v=0. \end{cases}$$

But the upper bound for this expression is ζ if and only if (9) holds. Thus the proof is complete

3. Inclusion Properties

A function $B\in\Upsilon$ is said to be in the family $\mathcal{R}^{\tau}(A_1,A_2)$, ($\tau\in\mathbb{C}\backslash\{0\}.-1\leq A_2 < A_1\leq 1$) if it satisfies the inequality

$$\left|\frac{B'(z)-1}{(A_1-A_2)\tau-A_2[B'(z)-1]}\right| < 1 \quad (z\in\Delta).$$

The family $\mathcal{R}^{\tau}(A_1,A_2)$ was introduced earlier by Dixit and Pal [22].

It is of interest to note that if $\tau=1$, $A_1=\gamma$ and $A_2=-\gamma(0<\gamma\leq 1)$ we obtain the subfamily of functions $B\in\Upsilon$ satisfying the inequality

$$\left|\frac{B'(z)-1}{B'(z)+1}\right| < \gamma, \quad (z\in\Delta)$$

which was studied by Caplinger and Causey [23] and Padmanabhan [24].

Lemma 2. *[22] If $B\in\mathcal{R}^{\tau}(A_1,A_2)$ is of the form (1), then*

$$|b_{\varepsilon}|\leq (A_1-A_2)\frac{|\tau|}{\varepsilon}, \quad \varepsilon\in\mathbb{N}\backslash\{1\}.$$

The result is sharp.

Making use of Lemma 2, we prove the following theorem.

Theorem 3. *Let $s>0, v\in\mathbb{N}_0$ and $B\in\mathcal{R}^{\tau}(A_1,A_2)$. Then $\Lambda(s,v,z)B\in K(\delta,\zeta)$ if*

$$(A_1-A_2)\left|\tau\right|\begin{cases}\delta\mu'_{v+2}+(\delta+1)\mu'_{v+1}+(1-\zeta)\,\mu'_{v} & \textit{if} \quad v\geq 1\\ \delta s^2+(2\delta+1)s+(1-\zeta)\left(1-e^{-s}\right) & \textit{if} \quad v=0\end{cases}\leq\zeta.$$

Proof. Let $B\in\mathcal{R}^{\tau}(A_1,A_2)$. By inequality (5), it suffices to show that

$$\sum_{\varepsilon=2}^{\infty}\varepsilon\left[(\varepsilon+\varepsilon\delta(\varepsilon-1)-\zeta\right]\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}e^{-s}}{(\varepsilon-1)!}\left|b_{\varepsilon}\right|\leq\zeta.$$

Since $B\in\mathcal{R}^{\tau}(A_1,A_2)$, then by Lemma *2*, we have

$$\begin{aligned}
&\sum_{\varepsilon=2}^{\infty}\varepsilon\left[(\varepsilon+\varepsilon\delta(\varepsilon-1)-\zeta\right]\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}e^{-s}}{(\varepsilon-1)!}\left|b_{\varepsilon}\right|\\
&\leq (A_1-A_2)\left|\tau\right|\sum_{\varepsilon=2}^{\infty}\left[(\varepsilon+\varepsilon\delta(\varepsilon-1)-\zeta\right]\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}e^{-s}}{(\varepsilon-1)!}\\
&=(A_1-A_2)\left|\tau\right|\sum_{\varepsilon=2}^{\infty}\left(\delta(\varepsilon-1)(\varepsilon-2)+(2\delta+1)(\varepsilon-1)+1-\zeta\right)\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}e^{-s}}{(\varepsilon-1)!}\\
&=(A_1-A_2)\left|\tau\right|\sum_{\varepsilon=2}^{\infty}\left(\delta(\varepsilon-1)^2+(\delta+1)(\varepsilon-1)+1-\zeta\right)\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}e^{-s}}{(\varepsilon-1)!}\\
&=(A_1-A_2)\left|\tau\right|e^{-s}\left[\sum_{\varepsilon=2}^{\infty}\delta\frac{(\varepsilon-1)^{v+2}s^{\varepsilon-1}}{(\varepsilon-1)!}+\sum_{\varepsilon=2}^{\infty}(\delta+1)\frac{(\varepsilon-1)^{v+1}s^{\varepsilon-1}}{(\varepsilon-1)!}\right.\\
&\quad\left.+\sum_{\varepsilon=2}^{\infty}(1-\zeta)\frac{(\varepsilon-1)^{v}s^{\varepsilon-1}}{(\varepsilon-1)!}\right]\\
&=(A_1-A_2)\left|\tau\right|e^{-s}\left[\sum_{\varepsilon=1}^{\infty}\delta\frac{\varepsilon^{v+2}s^{\varepsilon}}{\varepsilon!}+\sum_{\varepsilon=1}^{\infty}(\delta+1)\frac{\varepsilon^{v+1}s^{\varepsilon}}{\varepsilon!}+\sum_{\varepsilon=1}^{\infty}(1-\zeta)\frac{\varepsilon^{v}s^{\varepsilon}}{\varepsilon!}\right]\\
&=(A_1-A_2)\left|\tau\right|\begin{cases}\delta\mu'_{v+2}+(\delta+1)\mu'_{v+1}+(1-\zeta)\,\mu'_{v} & \text{if} \quad v\geq 1\\ \delta s^2+(2\delta+1)s+(1-\zeta)\left(1-e^{-s}\right) & \text{if} \quad v=0\end{cases}\leq\zeta.
\end{aligned}$$

4. An Integral Operator $\mathcal{J}_s^v(z)$

Theorem 4. *Let* $s > 0$ *and* $v \in \mathbb{N}$. *Then*

$$\mathcal{J}_s^v(z) = \int_0^z \frac{\Lambda(s,v,\varepsilon)}{\varepsilon} d\varepsilon$$

is in the family $\mathrm{K}(\delta,\zeta)$ *if and only if the inequality (8) is satisfied.*

Proof. Since

$$\mathcal{J}_s^v(z) = z + \sum_{\varepsilon=2}^{\infty} \frac{(\varepsilon-1)^v s^{\varepsilon-1}}{(\varepsilon-1)!} e^{-s} \frac{z^\varepsilon}{\varepsilon}.$$

By virtue of inequality (5), it suffices to show that

$$\begin{aligned} & \sum_{\varepsilon=2}^{\infty} \varepsilon\left[(\varepsilon+\varepsilon\delta(\varepsilon-1)-\zeta\right] \frac{(\varepsilon-1)^v s^{\varepsilon-1}}{\varepsilon(\varepsilon-1)!} e^{-s} \\ \equiv\ & \sum_{\varepsilon=2}^{\infty} \left[(\varepsilon+\varepsilon\delta(\varepsilon-1)-\zeta\right] \frac{(\varepsilon-1)^v s^{\varepsilon-1}}{(\varepsilon-1)!} e^{-s} \\ \leq\ & \zeta. \end{aligned}$$

We leave out the specifics because the remaining portion of the proof of Theorem *4* is identical to proof of Theorem *1*.

5. Corollaries and Consequences

By fixing the parameter $\delta = 0$ in Theorems *1-4*, we get the following special cases for the function families $S^*(\zeta)$ and $K(\zeta)$.

Corollary 1. *If* $s > 0$ *and* $v \in \mathbb{N}_0$, *then* $\digamma_s^v(z) \in S^*(\zeta)$ *if and only if*

$$\begin{cases} \mu'_{v+1} + (1-\zeta)\,\mu'_v & \text{if } v \geq 1 \\ s + (1-\zeta)\,(1-e^{-s}) & \text{if } v = 0 \end{cases} \leq \zeta. \tag{10}$$

Corollary 2. *If* $s > 0$ *and* $v \in \mathbb{N}_0$, *then* $\digamma_s^v(z) \in K(\zeta)$ *if and only if*

$$\begin{cases} \mu'_{v+2} + (2-\zeta)\mu'_{v+1} + (1-\zeta)\,\mu'_v, & \text{if } \quad v \geq 1 \\ s^2 + (3-\zeta)s + (1-\zeta)\,(1-e^{-s}), & \text{if } \quad v = 0 \end{cases} \leq \zeta.$$

Corollary 3. *Let* $s > 0, v \in \mathbb{N}_0$ *and* $B \in \mathcal{R}^\tau(A_1, A_2)$. *Then* $\Lambda(s,v,z)B \in K(\zeta)$ *if*

$$(A_1 - A_2)\,|\tau| \begin{cases} \mu'_{v+1} + (1-\zeta)\,\mu'_v & \text{if } \quad v \geq 1 \\ s + (1-\zeta)\,(1-e^{-s}) & \text{if } \quad v = 0 \end{cases} \leq \zeta.$$

Corollary 4. *Let* $s > 0$ *and* $v \in \mathbb{N}$. *Then* $\mathcal{J}_s^v(z) = \int_0^z \frac{\Lambda(s,v,\varepsilon)}{\varepsilon} d\varepsilon$ *is in the family* $\mathrm{K}(\zeta)$ *if and only if the inequality (10) is satisfied.*

6. Conclusions

Using the Touchard polynomials, we examine some necessary and sufficient conditions for the functions $\digamma_s^v(z)$, $\Lambda(s,v,z)B$ and integral operator $\mathcal{J}_s^v(z)$, which are defined by the Touchard polynomials to be in the inclusive two subfamilies $Q(\delta,\zeta)$ and $K(\delta,\zeta)$. Additionally, several corollaries are shown by our results. Following this work, the Touchard polynomials may be used to derive new necessary and sufficient conditions for analytic functions in different subfamilies in the unit disk.

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