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On a b-chromatic Sum of a Mycielskian of Paths

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Abstract. A b-coloring of a graph G is a proper coloring such that there exists a vertex in each color class that is adjacent to at least one vertex in other color classes. The b-chromatic number of a graph G, denoted by $\varphi(G)$, is the largest integer k such that G has a b-coloring with k colors. The b-chromatic sum of a graph G, denoted by $\varphi'(G)$, is defined as the minimum sum of the colors $c(v)$ of v for all $v \in V$ where c is a b-coloring using $\varphi(G)$ colors. In this work, we improve the bounds on the b-chromatic sum given by Lisna and Sunitha [1]. We give the b-chromatic sum of the Mycielskian of a path $\mu(P_n)$ when $n = 7, 9$ and $n \geq 16$. For the case $10 \leq n \leq 15$, we give bounds on $\varphi'(\mu(P_n))$.

2020 Mathematics Subject Classifications: 05C15, 05C78, 05C76, 05C38, 05C69

Key Words and Phrases: b-coloring, b-chromatic number, b-dominating, b-chromatic sum, path, Mycielskian

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1. Introduction

For a graph $G = (V, E)$ with a proper coloring $c : V \to \{1, \ldots, k\}$, we denote a color class of $C_i = \{v \in V : c(v) = i\}$ for $i = 1, \ldots, k$. For $S \subset V$, we denote $c(S) = \{c(s) : s \in S\}$. A vertex v is a *b-dominating vertex* of color class i if v is adjacent to a vertex from each color class $j \neq i$. A *b-coloring* is a proper coloring where each color has a b-dominating vertex. R. W. Irving and D. F. Manlove [2] introduced the concept of the b-chromatic number of a graph G. The *b-chromatic number* $\varphi(G)$ of a graph G is the largest positive integer k such that G admits a b-coloring. Many research on the

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b-chromatic number and its bounds have been studied [3–5]. A collection of results on the b-chromatic number appeared in [6]. The application of the b-chromatic number appears in data clustering [7]. In comparison to the well-known chromatic number, the idea of chromatic sum was originated in 1989 by Kubicka and Schwenk [8]. The chromatic sum had also gained more attention from its application in the resource allocation problem [9]. Later in 2015, Lisna and Sunitha [10] introduced the b-chromatic sum following the same structure as the chromatic sum. The *b-chromatic sum* of a graph G, denoted by $\varphi'(G)$, is the minimum of $\sum_{v\in V} c(v)$ over a b-coloring c giving the b-chromatic number. Many questions related to the b-chromatic sum are still widely open. The b-chromatic sum of only a few classes of graphs had been investigated. Some examples include paths, cycles, wheel graphs, complete graphs [11, 12].

For a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, the *Mycielskian* or *Mycielski graph* $\mu(G)$ of G is a graph in which the vertex set consists of two copies of the vertices in G and a vertex u, i.e., $V(\mu(G)) = V \cup U \cup \{u\}$ such that $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_n\}$ where u_i is a copy of v_i. For the edge set of $\mu(G)$, we keep the adjacency of the vertices in $V \subset V(\mu(G))$ as in G, then join u to each vertex u_i, for $1 \le i \le n$, and join u_i to each neighbor of v_i in G. In 2011, Massimiliano et. al. [13] discussed the applications of Mycielski graphs in the multiprocessor task scheduling problem.

In 2017, Lisna and Sunitha [1] gave an upper bound on the b-chromatic sum of a Mycielskian path $\mu(P_n)$. They stated that such a bound is the b-chromatic sum of $\mu(P_n)$ for $n \ge 2$; however, we find that such results are a close upper bound on the b-chromatic sum of $\mu(P_n)$ when $n = 7$ or $n \ge 9$. In this work, we improve such results and give the b-chromatic sum of $\mu(P_n)$ when $n = 7, 9$ or $n \ge 16$ and give a lower bound and an upper bound when $10 \le n \le 15$.

2. Preliminaries

In this section, we present definitions and related results for the Mycielskian graph and the b-chromatic sum.

Definition 1. *([14]). Let G be a graph with n vertices, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Mycielskian or Mycielski graph $\mu(G)$ is a graph where $V(\mu(G)) = V(G) \cup \{u, u_1, \ldots, u_n\}$ and $E(\mu(G)) = E(G) \cup \{uu_i : 1 \le i \le n\} \cup \{u_i v : 1 \le i \le n$ and $v \in N_G(v_i)\}$.*

Let $U_n = \{u_i : 1 \le i \le n\}$. We partition $V(\mu(P_n))$ according to the definition of the Mycielskian into $\{V(P_n), U_n, \{u\}\}$ as shown in Figure 1.

Theorem 1 provides the maximum number of colors that $\mu(P_n)$ admit a b-coloring. We note that for a graph G and a coloring c to admit a b-chromatic number, there must be at least $\varphi(G)$ b-dominating vertices. Thus, G has at least $\varphi(G)$ vertices with $\varphi(G) - 1$ degree.

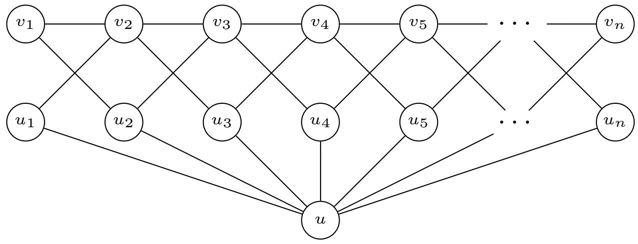

Figure 1: The graph of $\mu(P_n)$

Theorem 1. *([15]). The b-chromatic number of Mycielskian path $\mu(P_n)$ is*

$$\varphi(\mu(P_n)) = \begin{cases} 3 & \text{if } 2 \le n \le 4, \\ 4 & \text{if } 5 \le n \le 7, \\ 5 & \text{if } n \ge 8. \end{cases}$$

Lisna and Sunitha [1] provided the following results. We note that Theorem 3 was originally stated as the exact value of the b-chromatic sum $\varphi'(\mu(P_n))$. However, the proof assumes that a b-coloring c_n yielding the b-chromatic sum of $\mu(P_n)$ must be obtained from a b-coloring c_{n-1} yielding the b-chromatic sum of $\mu(P_{n-1})$ for $n \ge 9$. However, this assumption does not always hold. So, the b-colorings provided in the proof only give an upper bound on $\varphi'(\mu(P_n))$. The results for small values of n, except for $n = 7$, are valid as listed in Theorem 2.

Theorem 2. *([1]). The b-chromatic sum of Mycielskian of path $\mu(P_n)$ is*

$$\varphi'(\mu(P_n)) = \begin{cases} 4 & \text{if } n = 1, \\ 3 + 2(\lceil \frac{n}{2} \rceil + 2\lfloor \frac{n}{2} \rfloor) & \text{if } n = 2, 3, 4, \\ 6 + n + 3\lfloor \frac{n-1}{2} \rfloor + 2\lceil \frac{n-1}{2} \rceil & \text{if } n = 5, 6, \\ 44 & \text{if } n = 8. \end{cases}$$

Theorem 3. *([1]). The b-chromatic sum of Mycielskian of path $\mu(P_n)$ is as follows*

$$\varphi'(\mu(P_n)) \le \begin{cases} 28 & \text{if } n = 7, \\ 48 & \text{if } n = 9, \\ 45 + 4\lceil \frac{n-8}{2} \rceil + 2\lfloor \frac{n-8}{2} \rfloor & \text{if } n \ge 10. \end{cases}$$

For $n \ge 10$, Theorem 3 can be restated as

$$\varphi'(\mu(P_n)) \le \begin{cases} 3n + 21 & \text{if } n \text{ is even}, \\ 3n + 22 & \text{if } n \text{ is odd}. \end{cases}$$

In this work, we improve Theorem 3 by giving the value of the b-chromatic sum in the case of $n = 7, 9$ and $n \ge 16$ and we lower the upper bound given by Lisna and Sunitha in the case when $10 \le n \le 15$. We also give an explicit coloring giving the sum of the colors. To be more precise, we show that $\varphi'(\mu(P_7)) = 27$, $\varphi'(\mu(P_9)) = 46$, $3n + 18 \le \varphi'(\mu(P_n)) \le 3n + 19$ for $10 \le n \le 15$, and $\varphi'(\mu(P_n)) = 3n + 19$ for $n \ge 16$.

3. Main results

In [1], they gave the b-chromatic sum of $\mu(P_8)$ by giving a b-coloring that achieves such value and showed the minimality by counting the number of vertices in each color class. Then they adjusted such coloring to be c_n for $n \geq 9$. However, the adjusted coloring does not yield the b-chromatic sum. In this section, we improve the result of Lisna and Sunitha [1]. We give b-chromatic sum of $\mu(P_n)$ when $n = 7, 9$ and $n \geq 16$, and its bound when $10 \leq n \leq 15$.

We give the b-chromatic sum of $\mu(P_7)$ in Theorem 4. Then we investigate several properties of $\mu(P_n)$ and its colorings that lead to the bound and the exact value of the b-chromatic sum as mentioned.

Theorem 4. *$\varphi'(\mu(P_7)) = 27$.*

Proof. By Theorem 1, we have $\varphi(\mu(P_7)) = 4$. Let $c : V(\mu(P_7)) \to \{1, 2, 3, 4\}$ be a b-coloring giving a b-chromatic number of $\mu(P_7)$.

We first show that there are at most 7 vertices with color 1. Suppose to the contrary that there exists the b-coloring giving a b-chromatic number with at least 8 vertices of color 1. There are at least three b-dominating vertices of colors other than 1. If $c(u) = 1$, then $1 \notin c(U_7)$. Hence $|C_1| \leq \left\lceil \frac{|V(P_7)|}{2} \right\rceil + 1 = 5$ is a contradiction. Now, we suppose that $c(u) \neq 1$. Since there are at most 4 vertices of color 1 in $V(P_7)$, there are at least 4 vertices of color 1 in U_7. If $|C_1 \cap U_7| = 4$, then $|C_1 \cap V(P_7)| = 4$. It follows that $C_1 = \{v_1, v_3, v_5, v_7, u_1, u_3, u_5, u_7\}$. Hence, the vertex u is the only possible b-dominating vertex of color other than 1, which is not possible. For $j = 1, 2, 3$, if $|C_1 \cap U_7| = 4 + j$, then $|N_{P_7}(C_1 \cap U_7)| \geq |C_1 \cap U_7| \geq 4 + j$. Thus, $|V(P_7) \setminus N_{\mu(P_7)}(C_1 \cap U_7)| \leq 3 - j$ which is not enough to complete the color 1. Therefore $|C_1| \leq 7$.

Next, we show that if $|C_4| = 1$, then $|C_3| \geq 2$. Suppose to the contrary that $|C_3| = 1$. Thus, the vertices with color 3 and 4 are both b-dominating vertices and are adjacent. Since $\mu(P_7)$ is triangle-free, there is no b-dominating vertices for colors 1 and 2, a contradiction. Thus, if $|C_4| = 1$, then $|C_3| \geq 2$.

If $|C_4| = 1$, then

$$\begin{aligned} \sum_{v \in V(\mu(P_7))} c(v) &\geq 4|C_4| + 3|C_3| + 2|C_2| + |C_1| \\ &\geq 4 \cdot 1 + 3 \cdot 2 + 2 \cdot 5 + 1 \cdot 7 \\ &= 27. \end{aligned}$$

If $|C_4| \geq 2$, then

$$\sum_{v \in V(\mu(P_7))} c(v) \geq 4 \cdot 2 + 3 \cdot 1 + 2 \cdot 5 + 1 \cdot 7 = 28.$$

Thus, $\varphi'(\mu(P_7)) \geq 27$.

Next, we give a b-coloring giving the b-chromatic sum of $\mu(P_n)$, as shown in Figure 2. For the rest of the paper, each bold vertex in a figure represents a b-dominating vertex and the number above each vertex represents its assigned color. Define $c_7 : V(\mu(P_7)) \to \{1, 2, 3, 4\}$ by

$$c_7(x) = \begin{cases} 1 & \text{if } x = v_1, v_3, u_1, u_3, u_5, u_6, u_7, \\ 2 & \text{if } x = v_2, v_4, v_7, u_2, u_4, \\ 3 & \text{if } x = v_6, u, \\ 4 & \text{if } x = v_5. \end{cases} \tag{1}$$

We can see that c_7 is a b-coloring and $\varphi'(\mu(P_7)) = 27$.

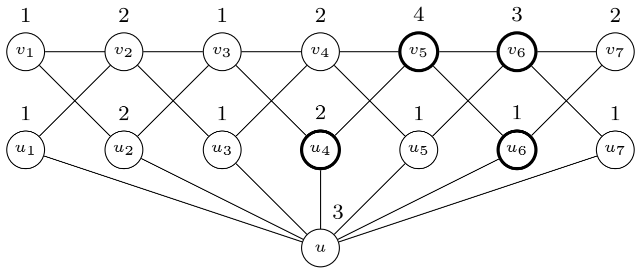

Figure 2: A b-coloring of $\mu(P_7)$ with 4 colors

Lemmas 1–5 give a structure of colors in a b-coloring giving the b-chromatic number. The structure will be used to determine the lower bound of the b-chromatic sum in Theorem 5. The idea of proof of Lemma 1 is extended from the calculation of $\varphi'(\mu(P_8))$ in [1].

Lemma 1. *For a graph $\mu(P_n)$ where $n \geq 8$ with a coloring c_n giving b-chromatic number, the following properties hold:*

(i) the b-dominating vertices are in $V(P_n) \cup \{u\}$,

(ii) each color appears at least twice in $V(P_n) \cup U_n$.

Proof. (1) Let $n \geq 8$. From $\varphi(\mu(P_n)) = 5$, the b-vertices have degree 4 in $\mu(P_n)$. Since u_i has degree 3 in $\mu(P_n)$ for $1 \leq i \leq n$, it follows that the b-vertices must be in $V(P_n) \cup \{u\}$.

(2) Suppose there is only one vertex $x \in V(P_n) \cup U_n$ of color 1. Since the b-dominating vertices of colors 2, 3, 4, 5 are in $V(P_n) \cup \{u\}$, at least three of them are in $V(P_n)$. These three vertices must be all adjacent to x. However, x has at most two neighbors in $V(P_n)$, a contradiction.

Lemma 2. *For a graph $\mu(P_n)$ where $n \geq 8$ with a coloring c_n giving the b-chromatic number, the following properties hold:*

(i) $|c_n(V(P_n))| = 5$,

(ii) $|C_{c_n(u)}| \leq \lceil \frac{n}{2} \rceil + 1$.

Proof.

(1) We note that $c_n(u)$ does not appear in U_n. For a vertex in $V(P_n)$ to be a b-dominating vertex, the color $c_n(u)$ must appear in $V(P_n)$. Thus $|c_n(V(P_n))| = 5$.

(2) Since v_i and v_{i+1} in $V(P_n)$ cannot get the same color, there are at most $\lceil \frac{n}{2} \rceil$ vertices in $V(P_n)$ with color $c_n(u)$. Since $c_n(u)$ does not appear in U_n, we have $|C_{c_n(u)}| \leq \lceil \frac{n}{2} \rceil + 1$.

Lemma 3. *For a graph $\mu(P_n)$ where $n \geq 8$ with a coloring c_n giving the b-chromatic number, if there exists $m \in \{1,2,3,4,5\}$ where $|C_m| = 2$, then $|C_i| \geq 3$ for $i \neq m$.*

Proof. By Theorem 1, we have $\varphi(\mu(P_n)) = 5$. Without loss of generality, we suppose $|C_5| = 2$. By Lemma 1, we have $c_n(u) \neq 5$. Hence $5 \notin \{c_n(u)\} \cup \{c_n(u_i), c_n(v_i) : i = 1, n\}$. Let $k \in \{2, \dots, n-1\}$ be such that v_k is the b-dominating vertex of color 5 and $j \in \{1, \dots, n\}$ be such that $5 \in \{c_n(u_j), c_n(v_j)\}$. At least one of v_{k-1} and v_{k+1} is a b-vertex. Suppose that v_{k+1} is a b-dominating vertex of color 4, and $c_n(v_{k-1}) = 3$ (see Figure 3). It follows that $c_n(v_{k+2}) = 3$ and $c_n(\{u_{k-1}, u_{k+1}\}) = c_n(\{u_k, u_{k+2}\}) = \{1, 2\}$. Thus $c_n(u) \in \{3, 4\}$, says $c_n(u) = 3$. So $|C_3| \geq 3$. Since the vertices in U_n are not b-dominating vertices, there exists b-dominating vertices of colors 1 and 2 in $V(P_n)$. Thus $|C_1| \geq 3$ and $|C_2| \geq 3$.

There are 2 vertices of color 5. Since v_{k-1} and u cannot be a b-dominating vertex of color 1 nor 2, we have $j \in \{2, 3, \dots, n-1\}$ and the vertices v_{j-1} and v_{j+1} are the b-dominating vertices of color 1 and 2. It remains 3 b-dominating vertices that have to be adjacent to a vertex of color 4 and they have no common neighbor. So, it requires at least 2 more vertices of color 4. Now $|C_4| \geq 3$. Therefore, $|C_i| \geq 3$ for $i = 1, \dots, 4$. This completes the proof.

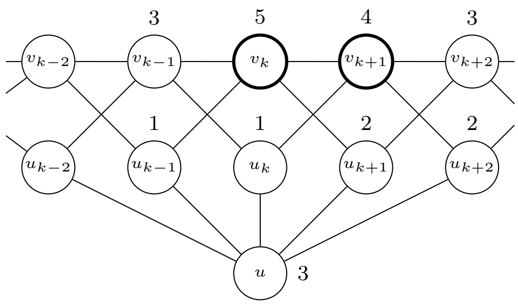

Figure 3: A coloring in Lemma 3

For $n \geq 8$, we consider a b-coloring c_n giving the b-chromatic sum of $\mu(P_n)$. If $|C_5| = 2$, then

$$\sum_{v \in V(\mu(P_n))} c_n(v) \geq 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 3 + 2 \cdot (2n - 7 - |C_1|) + |C_1| = 4n - |C_1| + 17.$$

If $|C_5| \geq 3$, then

$$\sum_{v \in V(\mu(P_n))} c_n(v) \geq 5 \cdot 3 + 4 \cdot 2 + 3 \cdot 3 + 2 \cdot (2n - 7 - |C_1|) + |C_1| = 4n - |C_1| + 18.$$

Lemma 4. *For $n \geq 8$, let c_n be a coloring giving the b-chromatic number of $\mu(P_n)$. If there exists m such that $|C_m| = 2$, then $c(u) \neq m$ and $|C_{c(u)}| \geq 4$.*

Proof. Let m be such that $|C_m| = 2$. From Lemma 1, we have $c(u) \neq m$. Without loss of generality, we suppose $|C_5| = 2$ and $c(u) = 3$. By Lemma 3, we have $|C_k| \geq 3$ for $k \neq 5$. Next, we suppose to the contrary that $|C_3| = 3$. We see that $3 \notin c_n(U_n)$. Let v_i, v_j be such that $c_n(v_i) = c_n(v_j) = 3$. We have that $N_{P_n}(\{v_i, v_j\}) = \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$ consists of b-dominating vertices of colors 1,2,4 and 5, says $c(v_{j-1}) = 5$. Since $|C_5| = 2$, either $c_n(u) = 5$ or v_{i+1} and v_{j-1} are adjacent. In either case, we need 3 vertices of color 5 to complete the neighbors of the b-vertices, a contradiction.

We note that, by using Lemmas 1–4, a lower bound of $\varphi'(\mu(P_8))$ can be obtained from a configuration where $|C_5| = 2, |C_4| = |C_3| = 3$. From Lemmas 2 and 4, $c_8(u) \in \{1, 2\}$ and $4 \leq |C_{c_8(u)}| \leq \lceil \frac{n}{2} \rceil + 1 = 5$. So, $|C_2| = 4$ and $|C_1| = 5$ give the lowest sum. Hence,

$$\sum_{v \in V(\mu(P_8))} c_8(v) \geq 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 44.$$

Combined with the b-coloring provided by [1], it follows that $\varphi'(\mu(P_8)) = 44$, which aligns with the result in [1].

For $n \geq 9$, Lemma 5 will give a better lower bound on the b-chromatic sum (if not exact) than using only Lemmas 1–4.

Lemma 5. *For $n \geq 9$, let c_n be a coloring giving the b-chromatic number of $\mu(P_n)$. There are at most $n - 1$ vertices of color 1.*

Proof. If $u \in C_1$, then $|C_1| \leq 1 + \lceil \frac{n}{2} \rceil \leq n - 1$. Without loss of generality, we assume that $u \in C_2$. Let v_k be a b-dominating vertex of color 3 where $2 \leq k \leq n - 1$. There is only one vertex in $N_{\mu(P_n)}(v_k) = \{v_{k-1}, v_{k+1}, u_{k-1}, u_{k+1}\}$ with color 1.

We consider two cases of positions of 1's up to left-right reflection as in Figure 4. The X marked in all figures mentioned in this proof means that the color of such vertex cannot be 1. Let $l = k - 2$ and $r = n - k$. We have l and r columns that possibly contain 1 on the left and right of v_k, respectively.

Case 1. $c_n(v_{k-1}) = 1$ or $c_n(v_{k+1}) = 1$.

Without loss of generality, we suppose that $c_n(v_{k+1}) = 1$ as shown in Figure 4 (the left figure). The left side of v_k contains at most l of 1's when l is even, and $l + 1$ of 1's when l is odd. The right side of v_k contains at most r of 1's. Hence $|C_1| \leq (l + 1) + r = n - 1$.

Case 2. $c_n(u_{k-1}) = 1$ or $c_n(u_{k+1}) = 1$.

Without loss of generality, we suppose that $c_n(u_{k+1}) = 1$ as shown in Figure 4 (the right figure). There are at most $l + 1$ and r of 1's on the left and right of v_k, respectively. In this case, it is possible that $c_n(u_k) = 1$. Hence $|C_1| \leq (l + 1) + r + 1 = n$. Suppose to a contrary that $|C_1| = n$. It implies that there are exactly $l + 1$ of 1's on the left of v_k. Thus l is odd and the positions of 1's must appear as in Figure 5. It follows that there are no b-dominating vertex of color 4 and 5 in these columns. Thus, the b-dominating vertices of 4 and 5 are on the right side of v_k. Since there are exactly r vertices of color 1 on the right of v_k, there is only one vertex that can possibly be the b-dominating vertex of both colors 4 and 5, as shown in Figure 6. This leads to a contradiction.

Therefore $|C_1| \leq n-1$.

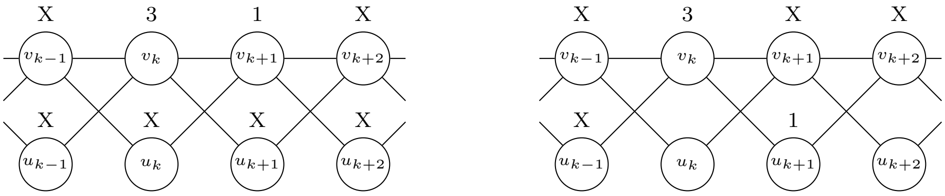

Figure 4: Positions of 1's in each type of b-dominating vertex positioning

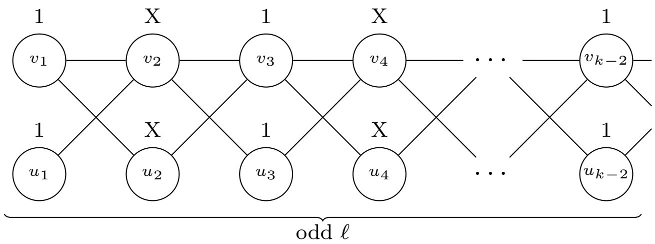

Figure 5: Positions of 1's on the left side of a b-dominating vertex v_k

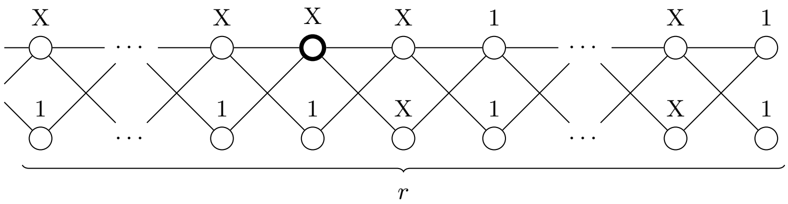

Figure 6: Positions of 1's on the right side of a b-dominating vertex

The following theorem gives a lower bound on $\varphi'(\mu(P_n))$ for $n \geq 9$.

Theorem 5. *For $n \geq 9$,*

$$\varphi'(\mu(P_n)) \geq \begin{cases} 3n+18 & \text{for } 10 \leq n \leq 15, \\ 3n+19 & \text{for } n=9 \text{ or } n \geq 16. \end{cases}$$

Proof. Let $c_n : V(\mu(P_n)) \to \{1,2,3,4,5\}$ be a b-coloring of $\mu(P_n)$ for $n \geq 9$. In this proof, we assume the minimality of the sum of the colors by maximizing the number of vertices with smaller colors and minimizing the number of vertices with larger colors.

Case 1. $c_n(u) \in \{3,4,5\}$.

By Lemmas 1, 3 and 4, we have that $|C_3| \geq 3, |C_4| \geq 3, |C_5| \geq 3$, or $|C_i| \geq 2, |C_j| \geq 3, |C_k| \geq 4$ where $\{i,j,k\} = \{3,4,5\}$. Hence, the minimum of $5|C_5| + 4|C_4| + 3|C_3|$ is

$5 \cdot 2 + 4 \cdot 3 + 3 \cdot 4 = 34$. By Lemma 5, to achieve the minimum sum, we assume the maximality of $|C_1| = n - 1$. Then, we have $|C_2| = n - 7$. Thus,

$$\sum_{w \in V(\mu(P_n))} c_n(w) \geq 34 + 2(n-7) + (n-1) = 3n + 19.$$

Case 2. $c_n(u) = 1$.

By Lemma 2, we have $|C_1| \leq \lceil \frac{n}{2} \rceil + 1$. We assume the maximum value of $|C_1|$ and minimum values of $|C_3|, |C_4|$ and $|C_5|$. We have that $|C_1| = \lceil \frac{n}{2} \rceil + 1, |C_3| = |C_4| = 3$ and $|C_5| = 2$. Hence, $|C_2| = n + \lfloor \frac{n}{2} \rfloor - 8$. It follows that

$$\begin{aligned} \sum_{w \in V(\mu(P_n))} c_n(w) &\geq 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 3 + 2\left(n + \left\lfloor \frac{n}{2} \right\rfloor - 8\right) + \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \\ &= 3n + \left\lfloor \frac{n}{2} \right\rfloor + 16 \\ &> 3n + 19. \end{aligned}$$

Case 3. $c_n(u) = 2$.

We first assume the maximum value of $|C_1|$ and minimum values of $|C_3|, |C_4|$ and $|C_5|$. We have that $|C_1| = n - 1, |C_3| = |C_4| = 3$ and $|C_5| = 2$. It follows that $|C_2| = n - 6$. Hence,

$$\varphi'(\mu(P_n)) \geq 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 3 + 2(n-6) + (n-1) = 3n + 18.$$

Next, we consider two special subcases.

- When $n = 9$, it is not possible that $|C_1| = 8$ since Lemma 4 gives an extra condition that $|C_2| \geq 4$, $|C_3| = |C_4| = 3$ and $|C_5| = 2$ are the lowest possible values. Thus $|C_1| = 7$ and $|C_2| = 4$. Hence,

$$\varphi'(\mu(P_9)) \geq 5 \cdot 2 + 4 \cdot 3 + 3 \cdot 3 + 2 \cdot 4 + 1 \cdot 7 = 46 = 3n + 19.$$

- When $n \geq 16$, we assume $|C_1| = n - 1$ and $|C_4| = 3$ and $|C_5| = 2$. From Lemma 2, we assume $|C_2| = \lceil \frac{n}{2} \rceil + 1$. It follows that

$$|C_3| = (2n+1) - 2 - 3 - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) - (n-1) = \left\lfloor \frac{n}{2} \right\rfloor - 4.$$

  We have

$$\begin{aligned} \sum_{w \in V(\mu(P_n))} c_n(w) &\geq 5 \cdot 2 + 4 \cdot 3 + 3\left(\left\lfloor \frac{n}{2} \right\rfloor - 4\right) + 2\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + (n-1) \\ &\geq 3n + \left\lfloor \frac{n}{2} \right\rfloor + 11 \\ &\geq 3n + 19. \end{aligned}$$

From the three cases, we can conclude that $\varphi'(\mu(P_n)) \geq 3n + 18$ for $10 \leq n \leq 15$, and $\varphi'(\mu(P_n)) \geq 3n + 19$ for $n = 9$ or $n \geq 16$.

The following remark is obtained from the proof of Theorem 5.

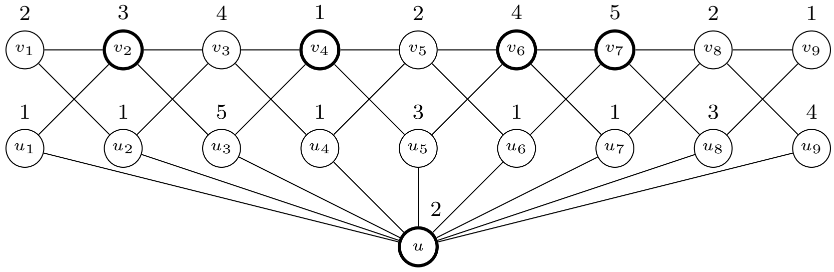

Figure 7: A b-coloring of $\mu(P_9)$

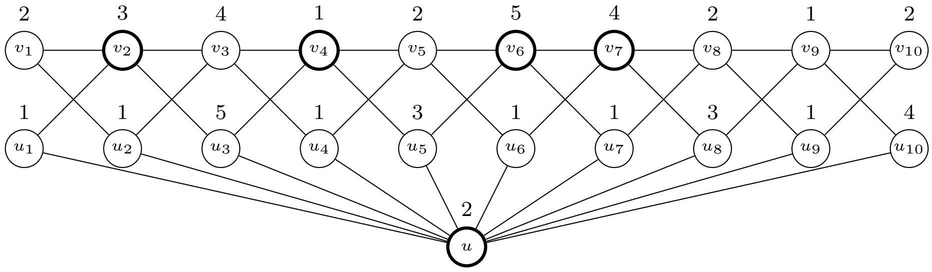

Figure 8: A b-coloring of $\mu(P_{10})$

Remark 1. *For $10 \leq n \leq 15$, if $\varphi'(\mu(P_n)) = 3n+18$, then $c_n(u) = 2$, $|C_1| = n-1, |C_2| = n-6, |C_3| = |C_4| = 3$ and $|C_5| = 2$.*

Theorem 6. *For $n \geq 9$, $\varphi'(\mu(P_n)) \leq 3n + 19$.*

Proof. We define b-colorings c_9, c_{10} and c_{11} of $\mu(P_9)$, $\mu(P_{10})$ and $\mu(P_{11})$ as in Figures 7, 8 and 9, respectively. We see that $\sum_{v\in V(\mu(P_n))} c_n(v) = 3n + 19$ for $9 \leq n \leq 11$. Thus $\varphi'(\mu(P_n)) \leq 3n + 19$ for $9 \leq n \leq 11$.

For an even $n \geq 12$, we define the proper coloring $c_n : V(\mu(P_n)) \to \{1, 2, 3, 4, 5\}$ by

$$c_n(x_i) = \begin{cases} c_{11}(x_{i-1}) & \text{if } x_i \in \{u_i, v_i\} \text{ for } i = 2, \dots, 12, \\ 1 & \text{if } x_i = u_1 \text{ or } x_i \in \{u_i, v_i\} \text{ for even } i \geq 14, \\ 2 & \text{if } x_i = v_1 \text{ or } x_i \in \{u_i, v_i\} \text{ for odd } i \geq 13. \end{cases}$$

For an odd $n \geq 13$, we define the proper coloring $c_n : V(\mu(P_n)) \to \{1, 2, 3, 4, 5\}$ by

$$c_n(x_i) = \begin{cases} c_{11}(x_i) & \text{if } x_i \in \{u_i, v_i\} \text{ for } i = 1, \dots, 11, \\ 2 & \text{if } x_i \in \{u_i, v_i\} \text{ for even } i \geq 12, \\ 1 & \text{if } x_i \in \{u_i, v_i\} \text{ for odd } i \geq 13. \end{cases}$$

It is clear that c_n is a b-coloring and that $\sum\limits_{x\in V(\mu(P_n))} c_n(x) = 3n+19$ for $n \geq 12$. Hence, $\varphi'(\mu(P_n)) \leq 3n + 19$ for $n \geq 12$.

Therefore, $\varphi'(\mu(P_n)) \leq 3n + 19$ for $n \geq 9$.

The following theorem is a direct result from Theorems 4, 5 and 6.

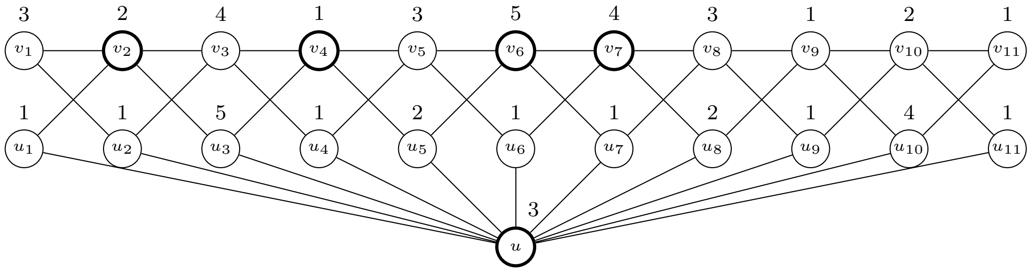

Figure 9: A b-coloring of $\mu(P_{11})$

Theorem 7. *The followings hold:*

- $\varphi'(\mu(P_7) = 27$,
- $3n + 18 \leq \varphi'(\mu(P_n)) \leq 3n + 19$ *for* $10 \leq n \leq 15$,
- $\varphi'(\mu(P_n)) = 3n + 19$ *for* $n = 9$ *or* $n \geq 16$.

4. Conclusion and Discussion

In the work of Lisna and Sunitha [1], they gave the b-coloring giving a b-chromatic sum of $\mu(P_8)$, i.e. $\varphi'(\mu(P_8)) = 44$. Then they extended and adjusted such coloring to a b-coloring of $\mu(P_n)$ for $n \geq 9$. Even though this extended coloring is a b-coloring, the sum of the colors may not be minimum. One of the main reason is because the number of color 1 in the extended coloring is less than the maximum number of color 1 as shown in Lemma 5.

In this work, we give a b-coloring giving a lower b-chromatic sum. We also analyze a lower bound. Our method relies heavily on Lemma 5 by setting $|C_1| = n - 1$ when computing the lower bound. The proof of Lemma 5 also describes valid configurations of color 1. As a result, we give lower and upper bounds on the b-chromatic sum of a Mycielskian path $\mu(P_n)$ for $10 \leq n \leq 15$. We get the exact value of the b-chromatic sum $\varphi'(\mu(P_7)) = 27$ and $\varphi'(\mu(P_n)) = 3n + 19$ when $n = 9$ or $n \geq 16$. Table 1 compares the results given by Lisna and Sunitha [1] and the results in this paper. From the table, we propose the following conjecture.

Conjecture 1. *For* $n \geq 9$, *we have* $\varphi'(\mu(P_n)) = 3n + 19$.

A vast area of research related to the b-chromatic sum still remains open. The b-chromatic sum of only a few specific classes of graphs such as paths, cycles, stars and certain classes of Mycielskian graphs had been studied. One of the intuitive problems is to find a bound on the b-chromatic sum for more generalized classes of graphs, for example, regular graphs and connected graphs. Another possible question is to find an efficient algorithm to compute the b-chromatic sum of a graph.

n	Our results		[1]
	Lower bound	Upper bound	Upper bound
7	27	27	28
8	44	44	44
9	$3n+19$	$3n+19$	$3n+21$
$10, 12, 14$	$3n+18$	$3n+19$	$3n+21$
$11, 13, 15$	$3n+18$	$3n+19$	$3n+22$
even $n \geq 16$	$3n+19$	$3n+19$	$3n+21$
odd $n \geq 17$	$3n+19$	$3n+19$	$3n+22$

Table 1: Bounds on $\varphi'(\mu(P_n))$ for $n \geq 7$

Acknowledgements

We dedicate this work to the memory of Sittichai Chaiyakhot, a young mathematician whose passion, curiosity, and dedication laid the foundation for this project. His commitment to discovery continues to inspire us. Though he is no longer with us, his contribution remains at the heart of this research. We are deeply grateful for his work and honor his memory by delivering his discovery.

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