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Applications of Strongest Fuzzy Dot Bd-Subalgebras in Bd-Algebras

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Abstract. In 2024, Nakkhasen et al. introduced the concept of fuzzy Bd-subalgebras of Bd-algebras. This paper will present the concept of fuzzy dot Bd-subalgebras in Bd-algebras as a generalization of fuzzy Bd-subalgebras. Subsequently, we evaluate the connections of fuzzy dot Bd-subalgebras in the framework of a homomorphism of Bd-algebras. Finally, we propose the concept of strongest fuzzy dot Bd-subalgebras and explain their characteristics in relation to Bd-subalgebras.

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1. Introduction

The subalgebras, such as BCK/BCI-subalgebras [1], BE-subalgebras [2], and BG-subalgebras [3], are the most frequently examined concepts when examining the characteristics of different ideas in each algebraic structure. Exploring the characteristics of non-empty subsets of an algebra and applying the same operations as that algebra while preserving the structure of the original algebra are key components of the subalgebra notion. In B-algebras, Walendziak [4] investigated the concept of normal subalgebras by showing that the notion of a normal subalgebra is equivalent to the normal subgroup of the derived group.Jun et al. [5] researched d-algebras using the theory of a falling shadow. To accomplish this, they developed the concept of falling d-subalgebras and examined their various characteristics, including how to classify the properties of falling d-subalgebras in d-algebras.In BCK/BCI-algebras, Balami et al. [6] presented the concepts of soft

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BCK/BCI-algebras and soft BCK/BCI-subalgebras and discussed some of their characteristics. Mathematicians also study the features of subalgebras in additional intriguing algebraic structures, such as [7], [8], [9], and [10]. Those who are interested can learn more about these structures.

The concept of the fuzzy set of δ in a non-empty set X is a function δ from X to the closed interval $[0, 1]$ in the real numbers. This concept was introduced by Zadeh [11], and it has since become a fundamental tool in various fields, such as artificial intelligence, control systems, and decision-making processes. Fuzzy sets allow for the representation of uncertain and imprecise information, enabling more flexible and realistic modelling than traditional binary sets. Rosenfeld [12] applied fuzzy sets to establish the concepts of fuzzy subgroups and fuzzy ideals in groups. Subsequently, Kuroki [13] examined the classifications of fuzzy subsemigroups, investigating the features and uses in semigroups. Next, Rezaei and Saeid [14] developed the concept of fuzzy subalgebras into BE-algebras and studied various characterizations of these fuzzy subalgebras. Afterwards, Muhiuddin [15] defined the concept of $(\in, \in \vee q_0^\delta)$-fuzzy subalgebras of BCK/BCI-algebras as a more general type of $(\in, \in \vee q)$-fuzzy subalgebras. Following that, Tacha et al. [16] introduced the concepts of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras of UP-algebras. The researchers also investigated the relationships between length fuzzy UP-subalgebras (mean fuzzy UP-subalgebras) and hyper fuzzy UP-subalgebras of UP-subalgebras. For research related to fuzzy subalgebras, further studies can be found in [17], [18], [19], and [20].

For the general concept of fuzzy subalgebras, another concept that has been continuously studied is the fuzzy dot. Saeid [21] presented the notion of fuzzy dot BCK-subalgebras and fuzzy dot topological BCK-algebras within the framework of BCK-algebras. Following this, Senapati et al. [22] provided definitions and examined related features of fuzzy dot subalgebras, fuzzy normal dot subalgebras, and fuzzy dot ideals of BG-algebras. In the same year, Senapati et al. [23] introduced fuzzy dot subalgebras, fuzzy normal dot subalgebras, and fuzzy dot ideals to the investigation in B-algebras. Later, Dejen [24] gave the idea of fuzzy dot subalgebras in the structure of fuzzy dot d-subalgebras and studied several of its characteristics. In addition, Jiang [25] established the notions of hesitant fuzzy dot subalgebras, hesitant fuzzy normal dot subalgebras, and hesitant fuzzy dot ideals of B-algebras and explored properties related to these concepts of B-algebras.

In 2022, the concept of Bd-algebras is derived from certain properties of d-algebras and B-algebras, introduced by Bantaojai et al. [26], who defined various concepts, one of which is Bd-subalgebras. Thereafter, the notion of fuzzy Bd-subalgebras of Bd-algebras was recently defined by Nakkhasen et al. [27]. In their presentation, they discussed an opportunity of fuzzy multiplications, fuzzy magnified translations, and fuzzy translations to characterize fuzzy Bd-subalgebras in Bd-algebras. To further investigate the general idea of fuzzy Bd-subalgebras, this article will introduce the notion of fuzzy dot Bd-subalgebras, which serves as a generalization of fuzzy Bd-subalgebras in Bd-algebras. In Section 3, we explore certain features of fuzzy dot Bd-subalgebras in Bd-algebras. Moreover, we further examine the relationships of fuzzy dot Bd-subalgebras under a homomorphism of Bd-

algebras. Finally, Section 4 presents the concept of strongest fuzzy dot Bd-subalgebras in Bd-algebras and studies the connections between strongest fuzzy dot Bd-subalgebras and fuzzy dot Bd-subalgebras in Bd-algebras, while characterizing the strongest fuzzy dot Bd-subalgebras via Bd-subalgebras of Bd-algebra.

2. Preliminaries

In this section, we will review the fundamental concepts necessary for use in the following sections.

Let X be a nonempty set. A *fuzzy set* [11] ζ of X is a function from X into the interval $[0,1]$. Let ζ and ξ be any two fuzzy sets of a nonempty set X. Then we denote:

(i) $(\zeta \cap \xi)(x) := \min\{\zeta(x), \xi(x)\}$ for all $x \in X$;

(ii) $(\zeta \cup \xi)(x) := \max\{\zeta(x), \xi(x)\}$ for all $x \in X$.

Let $\{r_i \mid i \in \Lambda\}$ be a family of real numbers. Then we denote

$$\bigcap_{i\in\Lambda} r_i = \begin{cases} \min\limits_{i\in\Lambda} r_i & \text{if } \Lambda \text{ is finite,} \\ \inf\limits_{i\in\Lambda} r_i & \text{otherwise.} \end{cases}$$

Let X be a nonempty set and ζ be a fuzzy set of X. The set $\zeta_t := \{x \in X \mid \zeta(x) \geq t\}$, where $t \in [0,1]$ is celled a *level subset* of ζ (see, [27]). Let A be a subset of a nonempty set X. The *characteristic function* (see, [27]) $\mathcal{C}_A$ of A is a fuzzy set of X, defined by for every $x \in X$,

$$\mathcal{C}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *If a, b, c, d are elements in real numbers, then* $\min\{ab, cd\} \geq \min\{a, c\}\min\{b, d\}$.

Proof. Without loss of generality, assume that $a \leq c$. Then $\min\{a, c\} = a$. If $b \leq d$, then $\min\{b, d\} = b$. Thus, $ab \leq cb \leq cd$. It turns out that $\min\{ab, cd\} = ab = \min\{a, c\}\min\{b, d\}$. On the other case, if $d < b$, then $\min\{b, d\} = d$. Since $a \leq c$ and $d < b$, we have $ad \leq cd$ and $ad < ab$. This implies that $\min\{ab, cd\} \geq ad = \min\{a, c\}\min\{b, d\}$. Therefore, $\min\{ab, cd\} \geq \min\{a, c\}\min\{b, d\}$.

Definition 1. *[26] Let X be a nonempty set and $*$ be a binary operation on X. An algebraic structure $(X, *, 0)$ is called a Bd-algebra if it satisfies the following conditions: for each $x, y \in X$,*

(i) $x * 0 = x$;

(ii) if $x * y = 0$ and $y * x = 0$, then $x = y$.

Throughout this study, we denote a Bd-algebra $(X, *, 0)$ by $\mathbf{X}$ the bold letter of its universe set.

Definition 2. *[26] Let* $\mathbf{X}$ *be a Bd-algebra. A nonempty subset A of X is said to be a Bd-subalgebra of* $\mathbf{X}$ *if* $0 \in A$ *and* $x * y \in A$ *for all* $x, y \in A$.

Definition 3. *[27] Let* $\mathbf{X}$ *be a Bd-algebra. A fuzzy set* ζ *of* $\mathbf{X}$ *is called a fuzzy Bd-subalgebra of* $\mathbf{X}$ *if it satisfies the following inequality: for any* $x, y \in X$,

(i) $\zeta(0) \geq \zeta(x)$;

(ii) $\zeta(x * y) \geq \min\{\zeta(x), \zeta(y)\}$.

3. Fuzzy dot Bd-subalgebras of Bd-algebras

In this section, we apply the usual multiplication in real numbers to the fuzzy sets by introducing the notion of fuzzy dot Bd-subalgebras, which is useful as a generalization of fuzzy Bd-subalgebras in the Bd-algebras. Then we examine certain characteristics of fuzzy dot Bd-subalgebras within the Bd-algebras. Subsequently, we investigate the connections of fuzzy dot Bd-subalgebras under a homomorphism of Bd-algebras.

Definition 4. *Let* $\mathbf{X}$ *be a Bd-algebra, and let* ζ *be a fuzzy sets of* $\mathbf{X}$. *Then* ζ *is called a fuzzy dot Bd-subalgebra of* $\mathbf{X}$ *if for every* $x, y \in X$:

(i) $\zeta(0) \geq \zeta(x)$;

(ii) $\zeta(x * y) \geq \zeta(x) \cdot \zeta(y)$, *where "$\cdot$" denotes ordinary multiplication in real numbers.*

Example 1. *Let* $X = \{0, a, b, c\}$ *and* $*$ *be a binary operation on X as defined in the following table:*

$*$	0	a	b	c
0	0	0	a	0
a	a	b	a	a
b	b	b	b	c
c	c	a	a	c

Table 1: The binary operation $*$ on X.

Then $\mathbf{X} := (X, *, 0)$ *is a Bd-algebra. Define a fuzzy set* ζ *of* $\mathbf{X}$ *by*

$$\zeta(0) = 0.80, \zeta(a) = 0.60, \zeta(b) = 0.70, \text{ and } \zeta(c) = 0.70.$$

By calculate routine, we have ζ *is a fuzzy dot Bd-subalgebra of* $\mathbf{X}$.

Proposition 1. *Every fuzzy Bd-subalgebra of a Bd-algebra* $\mathbf{X}$ *is also a fuzzy dot Bd-subalgebra.*

Proof. Let ζ is a fuzzy Bd-subalgebra of a Bd-algebra $\mathbf{X}$. Then $\zeta(0) \geq \zeta(a)$ for all $a \in X$. Now, let $x, y \in X$. If $\zeta(x) \leq \zeta(y)$, then $\min\{\zeta(x), \zeta(y)\} = \zeta(x)$. So, $\zeta(x * y) \geq \min\{\zeta(x), \zeta(y)\} = \zeta(x) \geq \zeta(x) \cdot \zeta(y)$. On the other hand, if $\zeta(x) > \zeta(y)$, then

$\min\{\zeta(x), \zeta(y)\} = \zeta(y)$. Thus, $\zeta(x * y) \geq \min\{\zeta(x), \zeta(y)\} = \zeta(y) \geq \zeta(x) \cdot \zeta(y)$. Hence, ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

Generally, the fuzzy dot Bd-subalgebras need not be a fuzzy Bd-subalgebras in Bd-algebras, as shown in the following example.

Example 2. *In Example 1, we have the fuzzy set ζ is a fuzzy dot Bd-subalgebra of a Bd-algebra $\mathbf{X}$. However, ζ is not a fuzzy Bd-subalgebra of $\mathbf{X}$, because $\zeta(0 * b) = 0.60 \ngeq 0.70 = \min\{\zeta(0), \zeta(b)\}$.*

Proposition 2. *Let ζ be a fuzzy dot Bd-subalgebra of a Bd-algebra $\mathbf{X}$. If there exists a nonempty subset A of X such that $\sup_{a \in A} \zeta(a) = 1$, then $\zeta(0) = 1$.*

Proof. Assume that X contains a nonempty subset A of X such that $\sup_{a \in A} \zeta(a) = 1$. Since ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$, we have $\zeta(0) \geq \zeta(x)$ for all $x \in X$. Then $1 \geq \zeta(0) \geq \sup_{a \in A} \zeta(a) = 1$. This implies that $\zeta(0) = 1$.

Let ζ be any fuzzy set of a nonempty set X, and m be a positive integer. Define a fuzzy set ζ^m of X by $\zeta^m(x) = (\zeta(x))^m$ for all $x \in X$.

Proposition 3. *Let $\mathbf{X}$ be a Bd-algebra. If ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$, then ζ^m is also a fuzzy dot Bd-subalgebra of $\mathbf{X}$ whenever m is a positive integer.*

Proof. Let ζ be a fuzzy dot Bd-subalgebra of $\mathbf{X}$ and m be a positive integer. For every $x, y \in X$, we have

$$\zeta^m(0) = (\zeta(0))^m \geq (\zeta(x))^m = \zeta^m(x)$$

and

$$\zeta^m(x * y) = (\zeta(x * y))^m \geq (\zeta(x) \cdot \zeta(y))^m = (\zeta(x))^m \cdot (\zeta(y))^m = \zeta^m(x) \cdot \zeta^m(y).$$

Consequently, ζ^m is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

Theorem 1. *Let $\mathbf{X}$ be a Bd-algebra. If ζ and ξ are fuzzy dot Bd-subalgebras of $\mathbf{X}$, then $\zeta \cap \xi$ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.*

Proof. Assume that ζ and ξ are fuzzy dot Bd-subalgebras of $\mathbf{X}$. Let $x, y \in X$. By Lemma 1 and assumption, we have

$$(\zeta \cap \xi)(0) = \min\{\zeta(0), \xi(0)\} \geq \min\{\zeta(x), \xi(x)\} = (\zeta \cap \xi)(x)$$

and

$$\begin{aligned}(\zeta \cap \xi)(x * y) &= \min\{\zeta(x * y), \xi(x * y)\} \\ &\geq \min\{\zeta(x) \cdot \zeta(y), \xi(x) \cdot \xi(y)\} \\ &\geq \min\{\zeta(x), \xi(x)\} \cdot \min\{\zeta(y), \xi(y)\} \\ &= (\zeta \cap \xi)(x) \cdot (\zeta \cap \xi)(y).\end{aligned}$$

Therefore, $\zeta \cap \xi$ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

Corollary 1. *Let* $\{\zeta_i \mid i \in \Lambda\}$ *be a family of fuzzy dot bd-subalgebras of a Bd-algebra* $\mathbf{X}$*. Then* $\bigcap_{i\in\Lambda} \zeta_i$ *is also a fuzzy dot Bd-subalgebra of* $\mathbf{X}$*, where* Λ *is any index set.*

Proof. Let $\zeta := \bigcap_{i\in\Lambda} \zeta_i$. We recall that $\zeta(x) = \bigcap_{i\in\Lambda} \zeta_i(x) = \inf_{i\in\Lambda} \zeta_i(x)$ for all $x \in X$. For every $x, y \in X$, we have $\zeta(0) = \inf_{i\in\Lambda} \zeta_i(0) \geq \inf_{i\in\Lambda} \zeta_i(x) = \zeta(x)$ and

$$\zeta(x * y) = \inf_{i\in\Lambda} \zeta_i(x * y) \geq \inf_{i\in\Lambda} \zeta_i(x) \cdot \zeta_i(y) = \inf_{i\in\Lambda} \zeta_i(x) \cdot \inf_{i\in\Lambda} \zeta_i(y) = \zeta(x) \cdot \zeta(y).$$

Hence, $\bigcap_{i\in\Lambda} \zeta_i$ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

Example 3. *By Example 1, we have the* ζ *is a fuzzy dot Bd-subalgebra of a Bd-algebra* $\mathbf{X}$ *where*

$$\zeta(0) = 0.80, \zeta(a) = 0.60, \zeta(b) = 0.70, \text{ and } \zeta(c) = 0.70.$$

Additionally, we define a fuzzy dot Bd-subalgebra ξ *of* $\mathbf{X}$ *by*

$$\xi(0) = 0.90, \xi(a) = 0.60, \xi(b) = 0.60, \text{ and } \xi(c) = 0.50.$$

We obtain that $(\zeta \cup \xi)(0 * b) = 0.60 \not\geq 0.63 = (\zeta \cup \xi)(0) \cdot (\zeta \cup \xi)(b)$*. This show that* $\zeta \cup \xi$ *is not a fuzzy dot Bd-subalgebra of* $\mathbf{X}$*.*

From Example 3, we conclude that the union of fuzzy dot Bd-subalgebras of Bd-algebras doesn't necessarily have to be a fuzzy dot Bd-subalgebra of Bd-algebras in general.

Theorem 2. *Let* $\mathbf{X}$ *be a Bd-algebra, and* A *be a nonempty subset of* X*. Then* A *is a Bd-subalgebra of* $\mathbf{X}$ *if and only if* $\mathcal{C}_A$ *is a fuzzy dot Bd-subalgebra of* $\mathbf{X}$*.*

Proof. Assume that A is a Bd-subalgebra of $\mathbf{X}$. Then $0 \in A$. So, $\mathcal{C}_A(0) = 1 \geq \mathcal{C}_A(x)$ for all $x \in X$. Suppose that there exist $a, b \in X$ such that $\mathcal{C}_A(a * b) < \mathcal{C}_A(a) \cdot \mathcal{C}_A(b)$. Thus, $\mathcal{C}_A(a * b) = 0$ and $\mathcal{C}_A(a) \cdot \mathcal{C}_A(b) = 1$; that is, $\mathcal{C}_A(a) = 1$ and $\mathcal{C}_A(b) = 1$. It follows that $a * b \notin A$ and $a, b \in A$. By assumption, we have $a * b \in A$, which is a contradiction. Hence, $\mathcal{C}_A(x * y) \geq \mathcal{C}_A(x) \cdot \mathcal{C}_A(y)$ for all $x, y \in X$. Therefore, $\mathcal{C}_A$ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$. Conversely, assume that $\mathcal{C}_A$ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$. If $0 \notin A$, then $0 = \mathcal{C}_A(0) \geq \mathcal{C}_A(x)$ for all $x \in X$. Also, $\mathcal{C}_A(x) = 0$ for all $x \in X$, implies that $A = \emptyset$. This is a contradiction. So, $0 \in A$. Next, let $x, y \in A$. Then, $\mathcal{C}_A(x * y) \geq \mathcal{C}_A(x) \cdot \mathcal{C}_A(y) = 1$. We obtain that $\mathcal{C}_A(x * y) = 1$; that is, $x * y \in A$. Consequently, A is a Bd-subalgebra of $\mathbf{X}$.

Theorem 3. *Let* $\mathbf{X}$ *be a Bd-algebra, and* ζ *be a fuzzy set of* $\mathbf{X}$*. If a nonempty level subset* ζ_t *is a Bd-subalgebra of* $\mathbf{X}$ *for all* $t \in [0, 1]$*, then* ζ *is a fuzzy dot Bd-subalgebra of* $\mathbf{X}$*.*

Proof. Let $x, y \in X$. Take $\zeta(x) = t'$ for some $t' \in [0, 1]$. Then $x \in \zeta_{t'}$, and so $\zeta_{t'} \neq \emptyset$. By assumption, we have $\zeta_{t'}$ is a Bd-algebra of $\mathbf{X}$. That is, $0 \in \zeta_{t'}$. It follows that $\zeta(0) \geq t' = \zeta(x)$. Next, letting $\zeta(x) \cdot \zeta(y) = s'$ for some $s' \in [0, 1]$. Since $\zeta(x), \zeta(y) \in [0, 1]$, we get $\zeta(x) \geq \zeta(x) \cdot \zeta(y) = s'$ and $\zeta(y) \geq \zeta(x) \cdot \zeta(y) = s'$. Thus, $x, y \in \zeta_{s'}$. By the given assumption, we have $x * y \in \zeta_{s'}$. This implies that $\zeta(x * y) \geq s' = \zeta(x) \cdot \zeta(y)$. Therefore, ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

The converse of Theorem 3 is not always true, as shown in the following example.

Example 4. *In Example 1, we have the fuzzy set ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$, but the level subset $\zeta_{0.70} = \{0, b, c\}$ of ζ is not a Bd-subalgebra of $\mathbf{X}$, since $0 * b = a \notin \zeta_{0.70}$.*

Let $\mathbf{X} := (X, *, 0_X)$ and $\mathbf{Y} := (Y, \circ, 0_Y)$ be Bd-algebras. Let $\omega : X \to Y$ be a mapping of Bd-algebras $\mathbf{X}$ and $\mathbf{Y}$, and let ζ be a fuzzy set of $\mathbf{Y}$. The fuzzy set ζ^ω of $\mathbf{X}$ is defined by $\zeta^\omega(x) = \zeta(\omega(x))$ for all $x \in X$. A function $\omega : X \to Y$ is called a *homomorphism* if $\omega(0_X) = 0_Y$ and $\omega(x * y) = \omega(x) \circ \omega(y)$ for all $x, y \in X$, and a homomorphism ω is called an *epimorphism* if ω is onto.

Theorem 4. *Let $\omega : X \to Y$ be a homomorphism of Bd-algebras $\mathbf{X} := (X, *, 0_X)$ and $\mathbf{Y} := (Y, \circ, 0_Y)$. If ζ is a fuzzy dot Bd-subalgebra of $\mathbf{Y}$, then ζ^ω is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.*

Proof. Assume that ζ is a fuzzy dot Bd-subalgebra of $\mathbf{Y}$. Let $x, y \in X$. Then we have

$$\zeta^\omega(0_X) = \zeta(\omega(0_X)) = \zeta(0_Y) \geq \zeta(\omega(x)) = \zeta^\omega(x)$$

and

$$\zeta^\omega(x * y) = \zeta(\omega(x * y)) = \zeta(\omega(x) \circ \omega(y)) \geq \zeta(\omega(x)) \cdot \zeta(\omega(y)) = \zeta^\omega(x) \cdot \zeta^\omega(y).$$

Thus, ζ^ω is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

By adding specific properties into Theorem 4, the converse of this Theorem will ultimately hold true as delineated below.

Theorem 5. *Let $\omega : X \to Y$ be an epimorphism of Bd-algebras $\mathbf{X} := (X, *, 0_X)$ and $\mathbf{Y} := (Y, \circ, 0_Y)$. If ζ^ω is a fuzzy dot Bd-subalgebra of $\mathbf{X}$, then ζ is a fuzzy dot Bd-subalgebra of $\mathbf{Y}$.*

Proof. Assume that ζ^ω is a fuzzy dot Bd-subalgebra of $\mathbf{X}$. Let $a, b \in Y$. Then there exist $x, y \in X$ such that $\omega(x) = a$ and $\omega(y) = b$. Thus, we have

$$\zeta(0_Y) = \zeta(\omega(0_X)) = \zeta^\omega(0_X) \geq \zeta^\omega(x) = \zeta(\omega(x)) = \zeta(a)$$

and

$$\zeta(a \circ b) = \zeta(\omega(x) \circ \omega(y))) = \zeta(\omega(x * y)) = \zeta^\omega(x * y) \geq \zeta^\omega(x) \cdot \zeta^\omega(y) = \zeta(\omega(x)) \cdot \zeta(\omega(y)) = \zeta(a) \cdot \zeta(b).$$

Therefore, ζ is a fuzzy dot Bd-subalgebra of $\mathbf{Y}$.

4. Strongest fuzzy dot *Bd*-subalgebras on *Bd*-algebras

In this section, we present some properties of the Cartesian product of fuzzy dot Bd-subalgebras of Bd-algebras. After that, we introduce the concept of strongest fuzzy dot Bd-subalgebras on Bd-algebras and investigate some of its properties and the relationships between strongest fuzzy dot Bd-subalgebras and fuzzy dot Bd-subalgebras in Bd-algebras.

Finally, we characterize the strongest fuzzy dot Bd-subalgebras by Bd-subalgebras of Bd-algebras.

Let $\mathbf{X} := (X, *, 0_X)$ and $\mathbf{Y} := (Y, \circ, 0_Y)$ be Bd-algebras. The mapping $\circledast : (X \times Y) \times (X \times Y) \to X \times Y$ is defined by

$$(x_1, y_1) \circledast (x_2, y_2) = (x_1 * x_2, y_1 \circ y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. We have that $\mathbf{X} \times \mathbf{Y} := (X \times Y, \circledast, (0_X, 0_Y))$ is a Bd-algebra. In particular, if $\mathbf{Y} = \mathbf{X}$, we have $\mathbf{X} \times \mathbf{X} := (X \times X, \circledast, (0_X, 0_X))$ is a Bd-algebra where the binary operation $\circledast$ on $X \times X$ is defined by $(x_1, y_1) \circledast (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$ for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Throughout this section, the Bd-algebra $(X \times X, \circledast, (0_X, 0_X))$ will be replaced by the symbol $\mathbf{X} \times \mathbf{X}$.

Let ζ and ξ be a fuzzy sets of a nonempty set X. The *Cartesian product* [28] $\zeta \times \xi : X \times X \to [0, 1]$ is defined by $(\zeta \times \xi)(x, y) = \zeta(x) \cdot \xi(y)$ for all $x, y \in X$.

Theorem 6. *Let $\mathbf{X}$ be a Bd-algebra. If ζ and ξ are fuzzy dot Bd-subalgebras of $\mathbf{X}$, then $\zeta \times \xi$ is a fuzzy dot Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$.*

Proof. Assume that ζ and ξ are fuzzy dot Bd-subalgebras of $\mathbf{X}$. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Then we have

$$(\zeta \times \xi)(0, 0) = \zeta(0) \cdot \xi(0) \geq \zeta(x_1) \cdot \xi(y_1) = (\zeta \times \xi)(x_1, y_1)$$

and

$$\begin{aligned}(\zeta \times \xi)((x_1, y_1) \circledast (x_2, y_2)) &= (\zeta \times \xi)(x_1 * x_2, y_1 * y_2) \\ &= \zeta(x_1 * x_2) \cdot \xi(y_1 * y_2) \\ &\geq [\zeta(x_1) \cdot \zeta(x_2)] \cdot [\xi(y_1) \cdot \xi(y_2)] \\ &= [\zeta(x_1) \cdot \xi(y_1)] \cdot [\zeta(x_2) \cdot \xi(y_2)] \\ &= (\zeta \times \xi)(x_1, y_1) \cdot (\zeta \times \xi)(x_2, y_2).\end{aligned}$$

Consequently, $\zeta \times \xi$ is a fuzzy dot Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$.

The converse of Theorem 6 is not true, as proved by the following example.

Example 5. *Let $X = \{0, 1, 2\}$ be a set with the binary operation $*$ on X define in the following table:*

$*$	0	1	2
0	0	2	2
1	1	0	2
2	2	1	1

Table 2: The binary operation $*$ on X.

Then $\mathbf{X} := (X, *, 0)$ is a Bd-algebra. Define two fuzzy sets ζ and ξ of $\mathbf{X}$ by

$$\zeta(0) = 0.70, \zeta(1) = 0.70, \zeta(2) = 0.40 \textit{ and } \xi(0) = 0.60, \xi(1) = 0.50, \xi(2) = 0.60.$$

It is not difficult to verify that ξ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$, but ζ is not a fuzzy dot Bd-subalgebra of $\mathbf{X}$. At the same time, $\zeta \times \xi$ is also a fuzzy dot Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$ as shown below. Now, we consider the results of the Cartesian product $\zeta \times \xi$ as follows.

$$\begin{aligned}
&(\zeta \times \xi)(0,0) = 0.42, (\zeta \times \xi)(0,1) = 0.35, (\zeta \times \xi)(0,2) = 0.42,\\
&(\zeta \times \xi)(1,0) = 0.42, (\zeta \times \xi)(1,1) = 0.35, (\zeta \times \xi)(1,2) = 0.42,\\
&(\zeta \times \xi)(2,0) = 0.24, (\zeta \times \xi)(2,1) = 0.20, (\zeta \times \xi)(2,2) = 0.24.
\end{aligned}$$

We see that $(\zeta \times \xi)(0,0) \geq (\zeta \times \xi)(x,y)$ for all $(x,y) \in X \times X$. Subsequently, below are some computed results.

$$\begin{aligned}
&(\zeta \times \xi)((0,2) \circledast (2,1)) = (\zeta \times \xi)(2,1) = 0.20 > 0.08 = (\zeta \times \xi)(0,2) \cdot (\zeta \times \xi)(2,1),\\
&(\zeta \times \xi)((1,1) \circledast (0,1)) = (\zeta \times \xi)(1,0) = 0.42 > 0.12 = (\zeta \times \xi)(1,1) \cdot (\zeta \times \xi)(0,1),\\
&(\zeta \times \xi)((2,0) \circledast (1,2)) = (\zeta \times \xi)(1,2) = 0.42 > 0.10 = (\zeta \times \xi)(2,0) \cdot (\zeta \times \xi)(1,2),\\
&(\zeta \times \xi)((2,2) \circledast (2,1)) = (\zeta \times \xi)(1,0) = 0.42 > 0.08 = (\zeta \times \xi)(2,2) \cdot (\zeta \times \xi)(1,1).
\end{aligned}$$

By meticulous computations, we have $(\zeta \times \xi)((x_1,y_1) \circledast (x_2,y_2)) \geq (\zeta \times \xi)(x_1,y_1) \cdot (\zeta \times \xi)(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2) \in X \times X$. Consequently, $\zeta \times \xi$ is a fuzzy dot Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$.

Let X be a nonempty set and ζ be any fuzzy set of X. A *fuzzy relation $\mathcal{S}$ on X* [28] is a fuzzy set of $X \times X$. Then the fuzzy relation $\mathcal{S}_\zeta$ on X is called a *fuzzy ζ-product relation on X* [22] if $\mathcal{S}_\zeta(x,y) \geq \zeta(x) \cdot \zeta(y)$ for all $x, y \in X$. Moreover, the *strongest fuzzy ζ-relation $\mathcal{S}_\zeta$ on X* [22] given by $\mathcal{S}_\zeta(x,y) = \zeta(x) \cdot \zeta(y)$ for all $x, y \in X$. For any $t \in [0,1]$, the set $(\mathcal{S}_\zeta)_t := \{(x,y) \mid \mathcal{S}_\zeta(x,y) \geq t\}$ is called a *level subset of $\mathcal{S}_\zeta$*, see [28].

Let A be any subset of a nonempty set X and ζ be a fuzzy set of X. The *characteristic function $\mathcal{C}_\zeta^A$* of $A \times A$ is defined by for every $x, y \in X$,

$$\mathcal{C}_\zeta^A(x,y) := \begin{cases} 1 & \text{if } (x,y) \in A \times A,\\ 0 & \text{otherwise.} \end{cases}$$

Next, the notion of strongest fuzzy dot Bd-subalgebras on Bd-algebras is further introduced as follows.

Definition 5. *Let $\mathbf{X}$ be a Bd-algebra, ζ be a fuzzy set of X, and $\mathcal{S}_\zeta$ be a strongest fuzzy ζ-relation on X. Then $\mathcal{S}_\zeta$ is called a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$ if for every $x_1, x_2, y_1, y_2 \in X$:*

(i) $\mathcal{S}_\zeta(0,0) \geq \mathcal{S}_\zeta(x_1,y_1)$;

(ii) $\mathcal{S}_\zeta(x_1 * x_2, y_1 * y_2) \geq \mathcal{S}_\zeta(x_1,y_1) \cdot \mathcal{S}_\zeta(x_2,y_2)$.

Example 6. *Consider the Bd-algebra* $\mathbf{X} := (X, *, 0)$ *as defined in Example 1. Afterward, we define a fuzzy set* ζ *of* X *by*

$$\zeta(0) = 0.90, \zeta(a) = 0.70, \zeta(b) = 0.80, \zeta(c) = 0.80.$$

Then the strongest fuzzy ζ*-relation* $\mathcal{S}_\zeta$ *on* X *is as follows.*

$$\begin{gathered}
\mathcal{S}_\zeta(0,0) = 0.81, \mathcal{S}_\zeta(0,a) = 0.63, \mathcal{S}_\zeta(0,b) = 0.72, \mathcal{S}_\zeta(0,c) = 0.72, \\
\mathcal{S}_\zeta(a,0) = 0.63, \mathcal{S}_\zeta(a,a) = 0.49, \mathcal{S}_\zeta(a,b) = 0.56, \mathcal{S}_\zeta(a,c) = 0.56, \\
\mathcal{S}_\zeta(b,0) = 0.72, \mathcal{S}_\zeta(b,a) = 0.56, \mathcal{S}_\zeta(b,b) = 0.64, \mathcal{S}_\zeta(b,c) = 0.64, \\
\mathcal{S}_\zeta(c,0) = 0.72, \mathcal{S}_\zeta(c,a) = 0.56, \mathcal{S}_\zeta(c,b) = 0.56, \mathcal{S}_\zeta(c,c) = 0.64.
\end{gathered}$$

It turns out that $\mathcal{S}_\zeta(0,0) \geq \mathcal{S}_\zeta(x,y)$ *for all* $x, y \in X$*. A select few of results are calculated below.*

$$\begin{gathered}
\mathcal{S}_\zeta(a * 0, b * a) = \mathcal{S}_\zeta(a,b) = 0.56 > 0.35 = \mathcal{S}_\zeta(a,b) \cdot \mathcal{S}_\zeta(0,a), \\
\mathcal{S}_\zeta(c * a, 0 * b) = \mathcal{S}_\zeta(a,a) = 0.49 > 0.40 = \mathcal{S}_\zeta(c,0) \cdot \mathcal{S}_\zeta(a,b), \\
\mathcal{S}_\zeta(b * c, 0 * c) = \mathcal{S}_\zeta(c,0) = 0.72 > 0.46 = \mathcal{S}_\zeta(b,0) \cdot \mathcal{S}_\zeta(c,c).
\end{gathered}$$

By routine calculations, we obtain $\mathcal{S}_\zeta(x_1 * x_2, y_1 * y_2) \geq \mathcal{S}_\zeta(x_1, y_1) \cdot \mathcal{S}_\zeta(x_2, y_2)$ *for all* $x_1, x_2, y_1, y_2 \in X$*. Therefore,* $\mathcal{S}_\zeta$ *is a strongest fuzzy dot Bd-subalgebra on* $\mathbf{X}$*.*

Theorem 7. *Let* $\mathbf{X}$ *be a Bd-algebra,* ζ *be a fuzzy set of* $\mathbf{X}$*, and* $\mathcal{S}_\zeta$ *be a strongest fuzzy* ζ*-relation on* $\mathbf{X}$*. Then* ζ *is a fuzzy dot Bd-subalgebra of* $\mathbf{X}$ *if and only if* $\mathcal{S}_\zeta$ *is a strongest fuzzy dot Bd-subalgebra on* $\mathbf{X}$*.*

Proof. Assume that ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$. Let $x_1, x_2, y_1, y_2 \in X$. Then we have

$$\mathcal{S}_\zeta(0,0) = \zeta(0) \cdot \zeta(0) \geq \zeta(x_1) \cdot \zeta(y_1) = \mathcal{S}_\zeta(x_1, y_1)$$

and

$$\begin{aligned}
\mathcal{S}_\zeta(x_1 * x_2, y_1 * y_2) &= \zeta(x_1 * x_2) \cdot \zeta(y_1 * y_2) \\
&\geq [\zeta(x_1) \cdot \zeta(x_2)] \cdot [\zeta(y_1) \cdot \zeta(y_2)] \\
&= [\zeta(x_1) \cdot \zeta(y_1)] \cdot [\zeta(x_2) \cdot \zeta(y_2)] \\
&= \mathcal{S}_\zeta(x_1, y_1) \cdot \mathcal{S}_\zeta(y_1, y_2).
\end{aligned}$$

Therefore, $\mathcal{S}_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$.

Conversely, assume that $\mathcal{S}_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$. Let $x, y \in X$. We consider

$$(\zeta(0))^2 = \zeta(0) \cdot \zeta(0) = \mathcal{S}_\zeta(0,0) \geq \mathcal{S}_\zeta(x,x) = \zeta(x) \cdot \zeta(x) = (\zeta(x))^2$$

and

$$(\zeta(x * y))^2 = \zeta(x * y) \cdot \zeta(x * y)$$

$$
\begin{aligned}
&= \mathcal{S}_\zeta(x * y, x * y) \\
&\geq \mathcal{S}_\zeta(x, x) \cdot \mathcal{S}_\zeta(y, y) \\
&= [\zeta(x) \cdot \zeta(x)] \cdot [\zeta(y) \cdot \zeta(y)] \\
&= [\zeta(x) \cdot \zeta(y)] \cdot [\zeta(x) \cdot \zeta(y)] \\
&= (\zeta(x) \cdot \zeta(y))^2.
\end{aligned}
$$

Since $\zeta(0), \zeta(x), \zeta(x * y), \zeta(x) \cdot \zeta(y) \geq 0$, we have $\zeta(0) \geq \zeta(x)$ and $\zeta(x * y) \geq \zeta(x) \cdot \zeta(y)$. Consequently, ζ is a fuzzy dot Bd-subalgebra of $\mathbf{X}$.

Theorem 8. *Let $\mathbf{X}$ be a Bd-algebra and $\mathcal{S}_\zeta$ be a strongest fuzzy ζ-relation on $\mathbf{X}$, where ζ is a fuzzy set of $\mathbf{X}$. If a nonempty level subset $(\mathcal{S}_\zeta)_t$ is a Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$ for all $t \in [0, 1]$, then $\mathcal{S}_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$.*

Proof. Let $x_1, x_2, y_1, y_2 \in X$. Choose $t' = \mathcal{S}_\zeta(x_1, y_1)$ for some $t' \in [0, 1]$. Then $(x_1, y_1) \in (\mathcal{S}_\zeta)_{t'}$; that is, $(\mathcal{S}_\zeta)_{t'} \neq \emptyset$. By assumption, we have $(\mathcal{S}_\zeta)_{t'}$ is a Bd-subalgebra of $\mathbf{X}\times\mathbf{X}$. This implies that $(0, 0) \in (\mathcal{S}_\zeta)_{t'}$, and then $\mathcal{S}_\zeta(0, 0) \geq t' = \mathcal{S}_\zeta(x_1, y_1)$. Next, take $s' = \mathcal{S}_\zeta(x_1, y_1)\cdot\mathcal{S}_\zeta(x_2, y_2)$ for some $s' \in [0, 1]$. So, we have $\mathcal{S}_\zeta(x_1, y_1) \geq \mathcal{S}_\zeta(x_1, y_1)\cdot\mathcal{S}_\zeta(x_2, y_2) = s'$ and $\mathcal{S}_\zeta(x_2, y_2) \geq \mathcal{S}_\zeta(x_1, y_1) \cdot \mathcal{S}_\zeta(x_2, y_2) = s'$. It turns out that $(x_1, y_1), (x_2, y_2) \in (\mathcal{S}_\zeta)_{s'}$. By the hypothesis, we get $(x_1 * x_2, y_1 * y_2) = (x_1, y_1) \circledast (x_2, y_2) \in (\mathcal{S}_\zeta)_{s'}$. Thus, $\mathcal{S}_\zeta(x_1 * x_2, y_1 * y_2) \geq s' = \mathcal{S}_\zeta(x_1, y_1) \cdot \mathcal{S}_\zeta(x_2, y_2)$. Therefore, $\mathcal{S}_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$.

The converse of Theorem 8 is generally not valid, as seen by the following example.

Example 7. *By Example 6, we have $\mathcal{S}_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X} := (X, *, 0)$. Then the level subset $(\mathcal{S}_\zeta)_{0.72} = \{(0, b), (0, c), (b, 0), (c, 0)\}$. We observe that $(\mathcal{S}_\zeta)_{0.72}$ is not a Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$, since $(0, b) \circledast (b, 0) = (a, b) \notin (\mathcal{S}_\zeta)_{0.72}$.*

Theorem 9. *Let $\mathbf{X}$ be a Bd-algebra, A be a nonempty subset of X, and ζ be a fuzzy set of $\mathbf{X}$. Then $A \times A$ is a Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$ if and only if $\mathcal{C}^A_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$.*

Proof. Assume that $A \times A$ is a Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$. Then $(0, 0) \in A \times A$, implies that $\mathcal{C}^A_\zeta(0, 0) = 1 \geq \mathcal{C}^A_\zeta(x, y)$ for all $(x, y) \in X \times X$. Now, suppose that there exist $(a_1, b_1), (a_2, b_2) \in X \times X$ such that $\mathcal{C}^A_\zeta(a_1 * a_2, b_1 * b_2) < \mathcal{C}^A_\zeta(a_1, b_1) \cdot \mathcal{C}^A_\zeta(a_2, b_2)$. We obtain that $\mathcal{C}^A_\zeta(a_1 * a_2, b_1 * b_2) = 0$ and $\mathcal{C}^A_\zeta(a_1, b_1)\cdot\mathcal{C}^A_\zeta(a_2, b_2) = 1$. Since $\mathcal{C}^A_\zeta(a_1, b_1)\cdot\mathcal{C}^A_\zeta(a_2, b_2) = 1$, we have $\mathcal{C}^A_\zeta(a_1, b_1) = 1$ and $\mathcal{C}^A_\zeta(a_2, b_2) = 1$. It follows that $(a_1 * a_2, b_1 * b_2) \notin A \times A$ and $(a_1, b_1), (a_2, b_2) \in A \times A$. By the hypothesis, we have $(a_1 * a_2, b_1 * b_2) = (a_1, b_1) \circledast (a_2, b_2) \in A \times A$. This is a contradiction. Hence, $\mathcal{C}^A_\zeta(x_1 * x_2, y_1 * y_2) \geq \mathcal{C}^A_\zeta(x_1, y_1) \cdot \mathcal{C}^A_\zeta(x_2, y_2)$ for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Therefore, $\mathcal{C}^A_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$.

Conversely, assume that $\mathcal{C}^A_\zeta$ is a strongest fuzzy dot Bd-subalgebra on $\mathbf{X}$. If $(0, 0) \notin A \times A$, then $0 = \mathcal{C}^A_\zeta(0, 0) \geq \mathcal{C}^A_\zeta(x, y)$ for all $(x, y) \in X \times X$. Thus, $\mathcal{C}^A_\zeta(x, y) = 0$ for all $(x, y) \in X \times X$. This means that $A \times A = \emptyset$. This is a contradiction, because A is a nonempty subset of X. Hence, $(0, 0) \in A \times A$. Now, let $(x_1, y_1), (x_2, y_2) \in A \times A$. Then,

$\mathcal{C}_\zeta^A(x_1 * x_2, y_1 * y_2) \geq \mathcal{C}_\zeta^A(x_1, y_1) \cdot \mathcal{C}_\zeta^A(x_2, y_2) = 1$, and so $\mathcal{C}_\zeta^A(x_1 * x_2, y_1 * y_2) = 1$. This implies that $(x_1, y_1) \circledast (x_2, y_2) = (x_1 * x_2, y_1 * y_2) \in A \times A$. Consequently, $A \times A$ is a Bd-subalgebra of $\mathbf{X} \times \mathbf{X}$.

5. Conclusions

In 2024, Nakkhasen et al. [27] applied the concept of fuzzy sets to Bd-algebras, defining the concept of fuzzy Bd-subalgebras. This article presents the notion of fuzzy dot Bd-subalgebras, which provide as a generalization of fuzzy Bd-subalgebras. That means that some of the results obtained from this work will generalize those from [27]. For example, Theorem 1 will be a general implication of Proposition 3.1 in [27]. In Section 3, we studied certain properties of fuzzy dot Bd-subalgebras of the Bd-algebras. Also, the relationships between fuzzy dot Bd-subalgebras under a homomorphism of Bd-algebras were then considered. Subsequently, the notion of strongest fuzzy dot Bd-subalgebras on Bd-algebras introduced in Section 4 and some of its characteristics are examined along with the relationships with fuzzy dot Bd-subalgebras in Bd-algebras. For future work that will extend the knowledge from this article, we will study the properties of the concept of fuzzy dot Bd-ideals on Bd-algebras or may study the properties of fuzzy dot subalgebras on other algebraic structures.

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