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Pseudo-Dual B-Algebra

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Abstract. This paper introduces the structure of a pseudo-dual B-algebra and some of its subsets, specifically pseudo-dual B-subalgebra and pseudo-dual B-filter. Furthermore, it presents some of their properties and gives a characterization of a pseudo-dual B-subalgebra. This study also provides the relationship between a pseudo-dual B-subalgebra and a pseudo-dual B-filter. Moreover, this paper presents a Python program that was used to perform the calculations needed to verify the example of a pseudo-dual B-algebra and to verify the independence of the axioms in a pseudo-dual B-algebra.

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Key Words and Phrases: Dual B-algebra, pseudo-dual B-algebra, pseudo-dual B-subalgebra, pseudo-dual B-filter

1. Introduction

In 1966, Y. Imai and K. Iseki [1] introduced two classes of algebra: BCK-algebra and BCI-algebra. In 2001, G. Georgescu and A. Iorgulescu [2] extended the concept of BCK-algebra to pseudo-BCK-algebra. In 2008, W. Dudek and Y. Jun [3] introduced the notion of pseudo-BCI algebra as an extension of BCI-algebra and investigated some of its properties. In 2002, J.Neggers and H.S. Kim [4] introduced and investigated a new class of algebra which is related to several classes of algebra such as $BCH/BCI/BCK$-algebra called B-algebra. In 2007, A. Walendziak [5] introduced a generalization of B-algebra called BF-algebra and investigated some properties of ideals and normal-ideals in BF-algebra and gave some characterization of them. Simultaneously, H.S. Kim and Y.H. Kim [6] introduced BE-algebra as a generalization of BCK-algebra and studied the filters of BE-algebra. In 2020, H. Al-Malki and D. Al-Kadi [7] introduced the structure of pseudo-BF/BF^*-algebra which is a generalization of BF-algebra together with pseudo-subalgebra, pseudo-ideal, pseudo-normal-ideal, and pseudo-atoms as its subsets. On the

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other hand, in 2013, R. Borzooei and *et al.*, [8] introduced the notion of pseudo BE-algebra which is a generalization of BE-algebra. They also studied the concepts of pseudo-subalgebra, pseudo-filter, and pseudo-upper-set and proved that, under some conditions, pseudo-subalgebra can be a pseudo-filter. Furthermore, in 2019, K. Belleza and J. Vilela [9] introduced and characterized the notion of a dual B-algebra. In 2022, K. Belleza and J. Albaracin [10] introduced the concepts of dual B-subalgebra and dual B-filter.

Motivated by the aforementioned studies, this paper aims to introduce the structure of pseudo-dual B-algebra and some of its subsets, particularly pseudo-dual B-subalgebra and pseudo-dual B-filter. Some of their properties will also be investigated. Furthermore, the relationship between pseudo-dual B-subalgebra and pseudo-dual B-filter will be given. Moreover, this study can contribute to the understanding of how the term "pseudo" is used in algebraic structures, and the results of this study could be used to develop further studies.

2. Preliminaries

Definition 1. [9] A *dual B-algebra* (or dB-algebra) X is a triple $(X, \bullet, 1)$ where X is a nonempty set with a binary operation "$\bullet$" and a constant 1 satisfying the following axioms for all x, y, z in X:

(DB1) $x \bullet x = 1$;

(DB2) $1 \bullet x = x$; and

(DB3) $x \bullet (y \bullet z) = ((y \bullet 1) \bullet x) \bullet z$.

Definition 2. [10] Let X be a dB-algebra and S a nonempty subset of X. Then S is called a *dual B-subalgebra* (or dB-subalgebra) of X if S itself is a dB-algebra with binary operation of X on S.

Definition 3. [10] Let X be a dB-algebra. A subset F of X is called a *dual B-filter* (or dB-filter) if it satisfies the following axioms for all x, y in X:

(i) $1 \in F$; and

(ii) $x \bullet y \in F$ and $x \in F$ imply $y \in F$.

3. Pseudo-Dual *B*-Algebra

Definition 4. A *pseudo-dual B-algebra* (or pseudo-dB-algebra) X is a quadruple $(X, \bullet, *, 1)$ where X is a nonempty set with two binary operations " $\bullet$ " and " $*$ " and a constant 1 satisfying the following axioms for all x, y, z in X:

(PDB1) $x \bullet x = 1$ and $x * x = 1$;

(PDB2) $1 \bullet x = x$ and $1 * x = x$; and

(PDB3) $x \bullet (y * z) = ((y * 1) \bullet x) \bullet z$ and $x * (y \bullet z) = ((y \bullet 1) * x) * z$.

In a pseudo-dB-algebra X, define a binary relation " $\leq$ " by $x \leq y \iff x \bullet y = 1 \iff x * y = 1$, for any $x, y \in X$.

Note: A Python program (see Appendix) was used to perform the calculations needed to verify Example 1, Example 2, Example 3, and Example 4.

Example 1. Consider $(X, \bullet, *, 1)$, where $X = \{1, -1\}$, " $\bullet$ " is the usual multiplication, and " $*$ " is the usual division. Then X is a pseudo-dB-algebra.

In Definition 4, the three axioms (PDB1), (PDB2), and (PDB3) are independent, which ensures that one axiom cannot be derived from the other to avoid redundancy. This is illustrated in Example 2, Example 3, and Example 4, where the axioms (PDB1) and (PDB2) hold but axiom (PDB3) fails, the axioms (PDB1) and (PDB3) hold but axiom (PDB2) fails, and the axioms (PDB2) and (PDB3) hold but axiom (PDB1) fails, respectively. Moreover, Example 2, Example 3, and Example 4 are counterexamples of a pseudo-dB-algebra.

Example 2. Let $X = \{1, a, b\}$. Define the binary operations " $\bullet$ " and " $*$ " on X by the following Cayley tables:

$\bullet$	1	a	b
1	1	a	b
a	a	1	b
b	b	b	1

$*$	1	a	b
1	1	a	b
a	a	1	a
b	b	a	1

Then the axioms (PDB1) and (PDB2) hold. However, (PDB3) fails since $a \bullet (b * b) = a \bullet 1 = a \neq 1 = b \bullet b = (b \bullet a) \bullet b = ((b * 1) \bullet a) \bullet b$.

Example 3. Let $X = \{1, a, b\}$. Define the binary operations " $\bullet$ " and " $*$ " on X by the following Cayley tables:

$\bullet$	1	a	b
1	1	1	1
a	1	1	1
b	1	1	1

$*$	1	a	b
1	1	1	1
a	1	1	a
b	1	a	1

Then the axioms (PDB1) and (PDB3) hold. However, axiom (PDB2) fails since $1 \bullet a = 1 \neq a$.

Example 4. Let $X = \{1, a\}$. Define the binary operations "$\bullet$" and "$*$" on X by $x \bullet y = y$ and $x * y = y$ for all $x, y \in X$. Then the axioms (PDB2) and (PDB3) hold. However, axiom (PDB1) fails since $a \bullet a = a \neq 1$.

Remark 1. In any pseudo-dB-algebra X, if $x \bullet y = x * y$ for all x, y in X, then X is a dB-algebra.

Remark 2. Any two dB-algebras do not necessarily construct a pseudo-dB-algebra. This is illustrated in the next example.

Example 5. Let $X = \{1, a, b, c\}$. Define the binary operations " $\bullet$ " and " $*$ " on X by the following Cayley tables:

$\bullet$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$*$	1	a	b	c
1	1	a	b	c
a	c	1	a	b
b	b	c	1	a
c	a	b	c	1

Then $(X, \bullet, 1)$ and $(X, *, 1)$ are dB-algebras by [9] and [11], respectively. Since $a \bullet (a * c) = a \bullet b = c \neq a = b \bullet c = (c \bullet a) \bullet c = ((a * 1) \bullet a) \bullet c$, $(X \bullet, *, 1)$ is not a pseudo-dB-algebra.

The following lemma provides some properties of a pseudo-dB-algebra.

Lemma 1. *In a pseudo-dB-algebra X, the following properties hold for any x, y, z in X:*

(i) *If $1 \leq x$, then $x = 1$;*

(ii) *$x \bullet 1 = 1$ if and only if $x * 1 = 1$;*

(iii) *$(x \bullet 1) * (x \bullet y) = y$ and $(x * 1) \bullet (x * y) = y$;*

(iv) *$(x \bullet y) * 1 = y * x$ and $(x * y) \bullet 1 = y \bullet x$;*

(v) *If $z \bullet x = z \bullet y$ (or $z * x = z * y$), then $x = y$;*

(vi) *If $x \bullet y = 1$ (or $x * y = 1$), then $x = y$;*

(vii) *If $x \bullet y = 1$, then $(x \bullet z) * (y \bullet z) = 1$;*

(viii) *If $x * y = 1$, then $(x * z) \bullet (y * z) = 1$;*

(ix) *$x = (x \bullet 1) * 1$ and $x = (x * 1) \bullet 1$; and*

(x) *If $x \bullet 1 = y \bullet 1$ (or $x * 1 = y * 1$), then $x = y$.*

Proof. Let X be a pseudo-dB-algebra and $x, y, z \in X$.

(i) Suppose $1 \leq x$. Then $1 \bullet x = 1 * x = 1$. By (PDB2), $x = 1 \bullet x = 1 * x$. Hence, $x = 1$.

(ii) Suppose $x \bullet 1 = 1$. By (PDB1), (PDB3), and (PDB2), $x * 1 = x * (x \bullet x) = [(x \bullet 1) * x] * x = (1 * x) * x = x * x = 1$. For the converse, suppose $x * 1 = 1$. By (PDB1), (PDB3), and (PDB2), $x \bullet 1 = x \bullet (x * x) = [(x * 1) \bullet x] \bullet x = (1 \bullet x) \bullet x = x \bullet x = 1$. Hence, $x \bullet 1 = 1$ if and only if $x * 1 = 1$.

(iii) By (PDB3), (PDB1), and (PDB2), $(x \bullet 1) * (x \bullet y) = [(x \bullet 1) * (x \bullet 1)] * y = 1 * y = y$ and $(x * 1) \bullet (x * y) = [(x * 1) \bullet (x * 1)] \bullet y = 1 \bullet y = y$.

(iv) By (PDB1) and (PDB3), and (iii), $(x \bullet y) * 1 = (x \bullet y) * (x \bullet x) = [(x \bullet 1) * (x \bullet y)] * x = y * x$ and $(x * y) \bullet 1 = (x * y) \bullet (x * x) = [(x * 1) \bullet (x * y)] \bullet x = y \bullet x$.

(v) Suppose $z \bullet x = z \bullet y$. Then $(z \bullet 1) * (z \bullet x) = (z \bullet 1) * (z \bullet y)$ implies $x = y$ by (iii). Suppose also that $z * x = z * y$. Then $(z * 1) \bullet (z * x) = (z * 1) \bullet (z * y)$ implies $x = y$ by (iii).

(vi) Suppose $x \bullet y = 1$. By (PDB1), $x \bullet y = 1 = x \bullet x$. Hence, $x \bullet y = x \bullet x$ implies $y = x$ by (v). By a similar way, if $x * y = 1$, then $x * y = x * x$, which implies $y = x$.

(vii) Suppose $x \bullet y = 1$ Then $x = y$ by (vi). Hence, $(x \bullet z) * (y \bullet z) = (x \bullet z) * (x \bullet z) = 1$ by (PDB1).

(viii) Suppose $x * y = 1$. Then $x = y$ by (vi). Hence, $(x * z) \bullet (y * z) = (x * z) \bullet (x * z) = 1$ by (PDB1).

(ix) By (PDB1), (PDB2), and (PDB3), $1 = x \bullet x = 1 * (x \bullet x) = [(x \bullet 1) * 1] * x$. Hence, $x = (x \bullet 1) * 1$ by (vi). By a similar way, $1 = x * x = 1 \bullet (x * x) = [(x * 1) \bullet 1] \bullet x$, which implies $x = (x * 1) \bullet 1$.

(x) Suppose $x \bullet 1 = y \bullet 1$. By (PDB1), (PDB2),(PDB3), and (ix) we have $1 = x \bullet x = 1*(x \bullet x) = [(x \bullet 1) * 1]*x = [(y \bullet 1) * 1]*x = y*x$. Hence, $y = x$ by (vi). By a similar way, if $x*1 = y*1$, then $1 = x*x = 1 \bullet (x*x) = [(x * 1) \bullet 1] \bullet x = [(y * 1) \bullet 1] \bullet x = y \bullet x$, which implies $y = x$. □

The following proposition shows that the relation " $\leq$ " on a pseudo-dB-algebra X is an equivalence relation.

Proposition 1. *Let X be a pseudo-dB-algebra. Then the relation " $\leq$ " on a set X is an equivalence relation.*

Proof. Let X be a pseudo-dB-algebra and $x, y, z \in X$. Note that $x \bullet x = x * x = 1$ by (PDB1). Hence, $x \leq x$ and so " $\leq$ " is reflexive. Suppose $x \leq y$. Then $x \bullet y = 1$ and $x * y = 1$. By Lemma 1(vi), $x = y$. Thus, $y \leq x$ and so " $\leq$ " is symmetric. Moreover, suppose $x \leq y$ and $y \leq z$. Since we have shown that $x \leq y$ implies $x = y$, $x \leq z$. Hence, " $\leq$ " is transitive. Therefore, " $\leq$ " is an equivalence relation. □

Theorem 1. *Let $X = (X, \bullet, *, 1)$ be any algebra of type $(2, 2, 0)$. If X is a pseudo-dB-algebra, then for any x, y, z in X,*

(i) $x \leq x$;

(ii) $x = (x \bullet 1) * 1$ *and* $x = (x * 1) \bullet 1$; *and*

(iii) $(x * y) \bullet (x * z) = y \bullet z$ *and* $(x \bullet y) * (x \bullet z) = y * z$.

Proof. Suppose X is a pseudo-dB-algebra and and $x, y, z \in X$. Then $x \bullet x = 1$ and $x * x = 1$ by (PDB1). Hence, $x \leq x$. Moreover, by Lemma 1(ix), $x = (x \bullet 1) * 1$ and $x = (x * 1) \bullet 1$. By (PDB3) and Lemma 1(iii), $(x * y) \bullet (x * z) = [(x * 1) \bullet (x * y)] \bullet z = y \bullet z$ and $(x \bullet y) * (x \bullet z) = [(x \bullet 1) * (x \bullet y)] * z = y * z$. Therefore, X satisfies (i), (ii), and (iii). □

The next Proposition gives a condition for the converse of Theorem 1 to hold.

Proposition 2. *Let $X = (X, \bullet, *, 1)$ be any algebra of type $(2, 2, 0)$. For any x, y, z in X, if X satisfies $x \bullet (y * z) = ((y * 1) \bullet x) \bullet z$, $x * (y \bullet z) = ((y \bullet 1) * x) * z$, and (i), (ii), (iii) of Theorem 1, then X is a pseudo-dB-algebra.*

Proof. It remains to show (PDB2). By (iii), (i), and (ii), $1 \bullet x = (x * 1) \bullet (x * x) = (x * 1) \bullet 1 = x$ and $1 * x = (x \bullet 1) * (x \bullet x) = (x \bullet 1) * 1 = x$. Thus, X satisfies (PDB2). Therefore, X is a pseudo-dB-algebra. □

4. Pseudo-Dual *B*-Subalgebra and Pseudo-Dual *B*-Filter

In this section, X is a pseudo-dB-algebra, unless otherwise is stated.

Definition 5. Let S be a nonempty subset of X. Then S is called a *pseudo-dual B-subalgebra* (or pseudo-dB-subalgebra) of X if S itself is a pseudo-dB-algebra with the binary operations of X on S.

Remark 3. If S is a pseudo-dB-subalgebra of X, then $1 \in S$.

The following theorem is a characterization of a pseudo-dB-subalgebra.

Theorem 2. *Let S be a nonempty subset of X. Then S is a pseudo-dB-subalgebra of X if and only if for all x, y in S, $x \bullet y \in S$ and $x * y \in S$.*

Proof. Let S be a nonempty subset of X with $x \bullet y \in S$ and $x * y \in S$ for all x, y in S. Note that S satisfies (PDB1), (PDB2), and (PDB3) with $1 = x \bullet x = x * x \in S$. Thus, S is itself a pseudo-dB-algebra. The converse follows immediately by definition of a binary operator. □

Proposition 3. *Let S be a pseudo-dB-subalgebra of X. Then for all x, y in S:*

(i) *If $x * y \in S$, then $y \bullet x \in S$; and*

(ii) *If $x \bullet y \in S$, then $y * x \in S$.*

Proof. For $x, y \in S$, let $x * y \in S$ and $x \bullet y \in S$. By Lemma 1(iv),$(x * y) \bullet 1 = y \bullet x$ and $(x \bullet y) * 1 = y * x$. Since $x * y, x \bullet y, 1 \in S$, $(x * y) \bullet 1 \in S$ and $(x \bullet y) * 1 \in S$. Hence, $y \bullet x \in S$ and $y * x \in S$. □

Proposition 4. *Let S be a pseudo-dB-subalgebra of X. If T is a pseudo-dB-subalgebra of S, then T is a pseudo-dB-subalgebra of X as well.*

Proof. Suppose S is a pseudo-dB-subalgebra of X. Suppose further that T is a pseudo-dB-subalgebra of S. Then $1 \in T$ by Remark 3. Hence, $T \neq \varnothing$. Let $x, y \in T$. Since T is a pseudo-dB-subalgebra of S, $x \bullet y, x * y \in T$ for all $x, y \in T$. Therefore, T is a pseudo-dB-subalgebra of X. □

The next theorem shows that the intersection of a nonempty collection of pseudo-dB-subalgebra of X is a pseudo-dB-subalgebra.

Theorem 3. *Let $\{S_\alpha : \alpha \in \mathcal{I}\}$ be a nonempty collection of pseudo-dB-subalgebra of X. Then $\bigcap_{\alpha\in\mathcal{I}} S_\alpha$ is also a pseudo-dB-subalgebra of X.*

Proof. Since S_α is a pseudo-dB-subalgebra for each α, $1 \in S_\alpha$ for all $\alpha \in \mathcal{I}$. Hence, $1 \in \bigcap_{\alpha\in\mathcal{I}} S_\alpha$ and $\bigcap_{\alpha\in\mathcal{I}} S_\alpha \neq \varnothing$. Let $x, y \in \bigcap_{\alpha\in\mathcal{I}} S_\alpha$. Then $x, y \in S_\alpha$ for all $\alpha \in \mathcal{I}$. Since S_α is a pseudo-dB-subalgebra for each α, $x \bullet y, x * y \in S_\alpha$ for all $\alpha \in \mathcal{I}$. Thus, $x \bullet y, x * y \in \bigcap_{\alpha\in\mathcal{I}} S_\alpha$. Therefore, $\bigcap_{\alpha\in\mathcal{I}} S_\alpha$ is a pseudo-dB-subalgebra of X. □

Definition 6. Let F be a nonempty subset of X. Then F is called a *pseudo-dual B-filter* (or pseudo-dB-filter) of X if it satisfies the following axioms for all x, y in X:

(PDBF1) $1 \in F$; and

(PDBF2) $x \bullet y \in F$, $x * y \in F$, and $x \in F$ imply $y \in F$.

Lemma 2. *If F is a pseudo-dB-filter of X, then for all x, y, z in X,*

(i) *If $x \leq y$ and $x \in F$, then $y \in F$; and*

(ii) *If $x \leq (y \bullet z)$, $x \leq (y * z)$, and $x, y \in F$, then $z \in F$.*

Proof. Let F be a pseudo-dB-filter of X and $x, y, z \in X$.

(i) Suppose $x \leq y$ and $x \in F$. Since $x \leq y$, $x \bullet y = 1$ and $x * y = 1$. By (PDBF1), $1 \in F$. Hence, $x \bullet y \in F$ and $x * y \in F$. Thus, $y \in F$ by (PDBF2).

(ii) Suppose $x \leq (y \bullet z)$, $x \leq (y * z)$, and $x, y \in F$. By (i), $y \bullet z, y * z \in F$. Since $y \bullet z, y * z, y \in F$, $z \in F$ by (PDBF2). □

Theorem 4. *Let F be a subset of X containing 1. Then F is a pseudo-dB-filter of X if and only if for all x, y, z in X, if $x \leq (y \bullet z)$, $x \leq (y * z)$, and $x, y \in F$, then $z \in F$.*

Proof. Let F be a subset of X containing 1. Suppose F is a pseudo-dB-filter of X. Then for all x, y, z in X, if $x \leq (y \bullet z)$, $x \leq (y * z)$, and $x, y \in F$, then $z \in F$ by Lemma 2 (ii). Conversely, $1 \in F$ by assumption. Hence, (PDBF1) holds. Let $x, x \bullet y, x * y \in F$ and $y \in X$. By (PDB1), $(x \bullet y) \bullet (x \bullet y) = 1, (x \bullet y) * (x \bullet y) = 1, (x * y) \bullet (x * y) = 1$, and $(x * y) * (x * y) = 1$. Hence, $(x \bullet y) \leq (x \bullet y)$ and $(x * y) \leq (x * y)$. Since $x, x \bullet y, x * y \in F$, $y \in F$. Thus, (PDBF2) holds. Therefore, F is a pseudo-dB-filter of X. □

Proposition 5. *Let F be a pseudo-dB-filter of X. If G is a pseudo-dB-filter of F, then G is a pseudo-dB-filter of X as well.*

Proof. Suppose G is a pseudo-dB-filter of F. Then $1 \in G$ by (PDBF1). Let $x, x \bullet y, x * y \in G$ for any $y \in X$. Note that $x, x \bullet y, x * y \in G \subseteq F$. Hence, $x, x \bullet y, x * y \in F$. Since F is a pseudo-dB-filter of X, $y \in F$ by (PDBF2). Now, $y \in F$ implies $y \in G$ since G is a pseudo-dB-filter of F. Therefore, G is a pseudo-dB-filter of X as well. □

The next theorem shows that the intersection of a nonempty collection of pseudo-dB-filter of X is a pseudo-dB-filter.

Theorem 5. *Let $\{F_\alpha : \alpha \in \mathcal{I}\}$ be a nonempty collection of pseudo-dB-filter of X. Then $\bigcap_{\alpha \in \mathcal{I}} F_\alpha$ is also a pseudo-dB-filter of X.*

Proof. Let $\{F_\alpha : \alpha \in \mathcal{I}\}$ be a nonempty collection of pseudo-dB-filter of X. Since F_α is a pseudo-dB-filter for each α, $1 \in F_\alpha$ for all $\alpha \in \mathcal{I}$. Hence, $1 \in \bigcap_{\alpha \in \mathcal{I}} F_\alpha$ and $\bigcap_{\alpha \in \mathcal{I}} F_\alpha \neq \varnothing$. Suppose $x, x \bullet y, x * y \in \bigcap_{\alpha \in \mathcal{I}} F_\alpha$ and $y \in X$. Then $x, x \bullet y, x * y \in F_\alpha$ for all $\alpha \in \mathcal{I}$. Since F_α is a pseudo-dB-filter for each α, $y \in F_\alpha$ for all $\alpha \in \mathcal{I}$. Thus, $y \in \bigcap_{\alpha \in \mathcal{I}} F_\alpha$. Therefore, $\bigcap_{\alpha \in \mathcal{I}} F_\alpha$ is a pseudo-dB-filter of X. □

The following theorem presents the relationship between a pseudo-dB-filter and a pseudo-dB-subalgebra.

Theorem 6. *Any pseudo-dB-filter of X is a pseudo-dB-subalgebra.*

Proof. Suppose F is a pseudo-dB-filter of X and let $x, y \in F$. By (PDBF1), $1 \in F$. Since $1 \in F$ and F is a pseudo-dB-filter, $1 \bullet (x \bullet y) \in F$ and $1 * (x * y) \in F$ imply $x \bullet y, x * y \in F$. □

The following theorem provides a condition for a pseudo-dB-subalgebra to be a pseudo-dB-filter.

Theorem 7. *Let F be a pseudo-dB-subalgebra of X. Then F is a pseudo-dB-filter of X if and only if for all $x, y \in X$ if $x \in F$ and $y \notin F$ then $x \bullet y, x * y \notin F$.*

Proof. Let F be a pseudo-dB-subalgebra of X. Suppose $x, y \in X$ and F is a pseudo-dB-filter of X where $x \in F$ and $y \notin F$. If $x \bullet y, x * y \in F$, then $y \in F$ by (PDBF2), which is a contradiction. Hence, $x \bullet y, x * y \notin F$. Conversely, note that $1 \in F$ by Remark 3. Hence, (PDBF1) holds. It remains to show (PDBF2). Suppose that for any $x, y \in X$, $x \bullet y, x * y, x \in F$. If $y \notin F$, then $x \bullet y, x * y \notin F$ by the hypothesis, which is a contradiction. Thus, for any $x, y \in X$, $x \bullet y, x * y, x \in F$ imply $y \in F$. Hence, (PDBF2) holds. Therefore, F is a pseudo-dB-filter. □

5. Conclusion

In this paper, the structure of pseudo-dB-algebra and some of its subsets, particularly pseudo-dB-subalgebra and pseudo-dB-filter, are introduced. It was illustrated that any two dB-algebras do not necessarily construct a pseudo-dB-algebra. This study also presented some properties of pseudo-dB-algebra and its subsets under consideration, and established a characterization of a pseudo-dB-subalgebra. Furthermore, it was shown that any pseudo-dB-filter is a pseudo-dB-subalgebra, and under some condition, a pseudo-dB-subalgebra is a pseudo-dB-filter. Moreover, interested researchers may explore the relationship of the pseudo-dB-algebra with the other pseudo-algebras.

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Appendix

The following Python program was used to verify Example 1. The same script was used to perform the calculations needed in Example 2, Example 3, and Example 4.

```
# Define the elements in the set X
X = [1, '-1']
# Define the bullet operation
def bullet(x, y):
    bullet_table = {
        (1, 1): 1, (1, '-1'): '-1',
        ('-1', 1): '-1', ('-1', '-1'): 1,
    }
    return bullet_table[(x, y)]
# Define the ast operation
def ast(x, y):
    ast_table = {
        (1, 1): 1, (1, '-1'): '-1',
        ('-1', 1): '-1', ('-1', '-1'): 1,
    }
    return ast_table[(x, y)]
# Define the constant 1
constant = 1
# Verify Axiom PDB1: x bullet x = 1 and x ast x = 1
def check_pdb1():
    print("Checking PDB1: x bullet x = 1 and x ast x = 1")
    for x in X:
        bullet_result = bullet(x, x)
        ast_result = ast(x, x)
        print(f"x = {x}: {x} bullet {x} = {bullet_result}, {x} ast {x}
            = {ast_result}")
        if bullet_result != constant or ast_result != constant:
            print("PDB1 fails.")
            return False
    print("PDB1 holds.")
    return True
# Verify Axiom PDB2: 1 bullet x = x and 1 ast x = x
def check_pdb2():
    print("\nChecking PDB2: 1 bullet x = x and 1 ast x = x")
    for x in X:
        bullet_result = bullet(constant, x)
        ast_result = ast(constant, x)
        print(f"x = {x}: 1 bullet {x} = {bullet_result}, 1 ast {x} = {
            ast_result}")
        if bullet_result != x or ast_result != x:
            print("PDB2 fails.")
            return False
    print("PDB2 holds.")
    return True
```

```
# Verify Axiom PDB3
# x bullet (y ast z) = ((y ast 1) bullet x) bullet z
# x ast (y bullet z) = ((y bullet 1) ast x) ast z
def check_pdb3():
    print("\nChecking PDB3:")
    for x in X:
        for y in X:
            for z in X:
                # Check the first part of PDB3
                left1 = bullet(x, ast(y, z))
                right1 = bullet(bullet(ast(y, constant), x), z)
                print(f"x = {x}, y = {y}, z = {z}: x bullet (y ast z) =
                    {left1}, ((y ast 1) bullet x) bullet z = {right1}"
                    )
                if left1 != right1:
                    print("PDB3 fails on first part.")
                    return False
                # Check the second part of PDB3
                left2 = ast(x, bullet(y, z))
                right2 = ast(ast(bullet(y, constant), x), z)
                print(f"x = {x}, y = {y}, z = {z}: x ast (y bullet z) =
                    {left2}, ((y bullet 1) ast x) ast z = {right2}")
                if left2 != right2:
                    print("PDB3 fails on second part.")
                    return False
    print("PDB3 holds.")
    return True
# Main function to check all axioms
def verify_pseudo_dual_b_algebra():
    print("Verifying pseudo-dual B-algebra axioms...\n")
    pdb1 = check_pdb1()
    pdb2 = check_pdb2()
    pdb3 = check_pdb3()
    if pdb1 and pdb2 and pdb3:
        print("\nX is a pseudo-dB-algebra.")
    else:
        print("\nX is not a pseudo-dB-algebra.")
        print(f"PDB1: {pdb1}, PDB2: {pdb2}, PDB3: {pdb3}")
# Run the verification
verify_pseudo_dual_b_algebra()
```

Output:

```
Checking PDB1: x bullet x = 1 and x ast x = 1
x = 1: 1 bullet 1 = 1, 1 ast 1 = 1
x = -1: -1 bullet -1 = 1, -1 ast -1 = 1
PDB1 holds.
Checking PDB2: 1 bullet x = x and 1 ast x = x
x = 1: 1 bullet 1 = 1, 1 ast 1 = 1
x = -1: 1 bullet -1 = -1, 1 ast -1 = -1
```

```
PDB2 holds.
Checking PDB3:
x = 1, y = 1, z = 1: x bullet (y ast z) = 1, ((y ast 1) bullet x)
    bullet z = 1
x = 1, y = 1, z = 1: x ast (y bullet z) = 1, ((y bullet 1) ast x) ast z
     = 1
x = 1, y = 1, z = −1: x bullet (y ast z) = −1, ((y ast 1) bullet x)
    bullet z = −1
x = 1, y = 1, z = −1: x ast (y bullet z) = −1, ((y bullet 1) ast x) ast
     z = −1
x = 1, y = −1, z = 1: x bullet (y ast z) = −1, ((y ast 1) bullet x)
    bullet z = −1
x = 1, y = −1, z = 1: x ast (y bullet z) = −1, ((y bullet 1) ast x) ast
     z = −1
x = 1, y = −1, z = −1: x bullet (y ast z) = 1, ((y ast 1) bullet x)
    bullet z = 1
x = 1, y = −1, z = −1: x ast (y bullet z) = 1, ((y bullet 1) ast x) ast
     z = 1
x = −1, y = 1, z = 1: x bullet (y ast z) = −1, ((y ast 1) bullet x)
    bullet z = −1
x = −1, y = 1, z = 1: x ast (y bullet z) = −1, ((y bullet 1) ast x) ast
     z = −1
x = −1, y = 1, z = −1: x bullet (y ast z) = 1, ((y ast 1) bullet x)
    bullet z = 1
x = −1, y = 1, z = −1: x ast (y bullet z) = 1, ((y bullet 1) ast x) ast
     z = 1
x = −1, y = −1, z = 1: x bullet (y ast z) = 1, ((y ast 1) bullet x)
    bullet z = 1
x = −1, y = −1, z = 1: x ast (y bullet z) = 1, ((y bullet 1) ast x) ast
     z = 1
x = −1, y = −1, z = −1: x bullet (y ast z) = −1, ((y ast 1) bullet x)
    bullet z = −1
x = −1, y = −1, z = −1: x ast (y bullet z) = −1, ((y bullet 1) ast x)
    ast z = −1
PDB3 holds.
X is a pseudo−dB−algebra.
```