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Generalized Hyperharmonic Sum via Euler's Transform

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Abstract. In this paper, we present and prove a novel expression for binomial sums involving generalized hyperharmonic numbers. Our approach utilizes Euler's transformation applied to the ordinary generating function of the generalized hyperharmonic numbers. To demonstrate the relevance of this new expression, we derive several identities that reveal connections between the characteristic equations and Binet forms of notable numerical sequences, including the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Mersenne, and Mersenne-Lucas numbers. Furthermore, we establish the integer power representation of the generalized hyperharmonic sums. As an extension of our findings, we also introduce and prove an alternative expression using the polylogarithmic form of the generating function for the generalized hyperharmonic numbers.

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1. Introduction

Harmonic numbers, as discussed in [16], play a fundamental role in combinatorial number theory. The *nth* harmonic number, denoted by H_n, is classically defined with the initial condition $H_0 = 0$, and for $n > 0$, it can be expressed in both recursive and integral forms as follows:

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

$$H_n = H_{n-1} + \frac{1}{n},$$

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x}\, dx,$$

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as given in [14]. These formulations not only highlight the additive structure of harmonic numbers but also provide insight into their analytical behavior.

The generating function for the sequence $\{H_n\}$ is given by the series:

$$\sum_{n=0}^{\infty} H_n z^n = \frac{-\ln(1-z)}{1-z}, \tag{1.1}$$

as derived in [3], which facilitates the manipulation and analysis of harmonic numbers in a variety of mathematical contexts.

Over recent years, several researchers have extended the concept of harmonic numbers. One such generalization is the hyperharmonic numbers, introduced in [13]. These numbers are defined recursively by:

$$H_n^{(r)} = \begin{cases} \sum_{k=1}^{n} H_k^{(r-1)}, & \text{for } n, r \geq 1, \\ \frac{1}{n}, & \text{for } r = 0 \text{ and } n > 0, \\ 0, & \text{for } r < 0 \text{ or } n \leq 0. \end{cases}$$

The generating function corresponding to the generalized hyperharmonic numbers for $r \geq 1$ is given by [8, 13]:

$$\sum_{n=0}^{\infty} H_n^{(r)} z^n = \frac{[-\ln(1-z)]^{r+1}}{1-z}.$$

Koparal et al. [21] further extended this framework by introducing the polylogarithmic generating function for generalized hyperharmonic sums. For any positive integer $m \in \mathbb{Z}^+$, the generating function is defined as:

$$\sum_{n=1}^{\infty} H_{n,m}^{(r)} z^n = \frac{Li_m(z)}{(1-z)^r},$$

where $Li_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1-z)$ is the classical polylogarithm of order 1.

In a notable contribution by Frontczak [14], a new closed-form expression for binomial sums involving harmonic numbers was derived using Euler's transform applied to the generating function for the harmonic numbers in (1.1). This result is expressed as:

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} a_k \right) z^n = \frac{1}{1-bz} f\left(\frac{az}{1-bz} \right), \tag{1.2}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This formulation provides a powerful tool for establishing new identities, including those involving Fibonacci and Lucas numbers.

Building upon this foundation, the present paper aims to derive and prove a new expression for binomial sums involving generalized hyperharmonic numbers, utilizing the Euler-type transformation approach outlined in [14]. We also establish the integer power representation of the derived expression. To illustrate the applicability and importance of our results, we deduce a series of identities that relate to well-known number sequences such as the Fibonacci, Lucas, Pell, Pell - Lucas, Jacobsthal, Jacobsthal - Lucas, Mersenne, and Mersenne - Lucas numbers. These identities are examined in the context of their respective characteristic equations and Binet formulas, offering deeper insights into their structure. Furthermore, as an extension of our main result, we define and prove a new formulation involving the generalized *Poly-Hyperharmonic Numbers* of order r, showcasing their connection to polylogarithmic functions and generating function techniques.

2. Some Preliminary Concepts

This section presents key foundational concepts that will be referenced throughout this work: namely, *Binet formulas*, *Euler's transformation*, and a class of *higher-order differential operators*. Each of these tools is essential in the study of special sequences, generating functions, and symbolic computation.

2.1. Binet Formulas

The *Binet formula*, named after the French mathematician Jacques Philippe Marie Binet, provides a closed-form solution to certain linear recurrence relations. Although commonly attributed to Binet in the 19th century, the formula for the Fibonacci sequence was actually discovered earlier by Abraham de Moivre in the 18th century. The Binet method arises from solving second-order linear recurrence relations with constant coefficients using the theory of characteristic equations.

Closed-form expressions such as Binet formulas are of particular value in both theoretical and computational mathematics, as they eliminate the need for recursive computations and allow for direct evaluation of the nth term of a sequence.

Over time, analogous formulas have been derived for other notable integer sequences, including the Lucas numbera, Pel numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Mersenne numbers, and Mersenne-Lucas numbers. The following summarizes these formulations:

(i) *Fibonacci Sequence*: The Fibonacci sequence, denoted by (F_n), is defined by the recurrence relation $F_{n+1} = F_{n-1} + F_n, n \geq 1$ with initial condition $F_0 = 0, F_1 = 1$. The Binet formula for this sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}.$$

(ii) *Lucas Numbers*: The Lucas numbers, denoted by (L_n), is defined by the recurrence relation $L_{n+1} = L_{n-1} + L_n, n \geq 1$ with initial condition $L_0 = 2, l_1 = 1$. The Binet formula for this sequence is given by

$$L_n = \alpha^n + \beta^n, \quad \text{with the same } \alpha, \beta \text{ as above.}$$

(iii) *Pell Sequence*: The Pell sequence, denoted by (P_n), is defined by the recurrence relation $P_{n+1} = P_{n-1} + 2P_n, n \geq 1$ with initial condition $P_0 = 0, P_1 = 1$. The Binet formula for this sequence is given by

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } \alpha = 1 + \sqrt{2},\ \beta = 1 - \sqrt{2}.$$

(iv) *Pell-Lucas Numbers*: The Pell-Lucas sequence, denoted by (Q_n), is defined by the recurrence relation $Q_{n+1} = Q_{n-1} + 2Q_n, n \geq 1$ with initial condition $Q_0 = 2, Q_1 = 2$. The Binet formula for this sequence is given by

$$Q_n = \alpha^n + \beta^n, \quad \text{with the same } \alpha, \beta \text{ as in Pell numbers.}$$

(v) *Jacobsthal Sequence*: The Jacobsthal sequence, denoted by (J_n), is defined by the recurrence relation $J_{n+1} = 2J_{n-1} + J_n, n \geq 1$ with initial condition $J_0 = 0, J_1 = 1$. The Binet formula for this sequence is given by

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } \alpha = 2,\ \beta = -1.$$

(vi) *Jacobsthal-Lucas Sequence*: The Jacobsthal-Lucas sequence, denoted by (T_n), is defined by the recurrence relation $T_{n+1} = 2T_{n-1} + T_n, n \geq 1$ with initial condition $T_0 = 2, T_1 = 1$. The Binet formula for this sequence is given by

$$T_n = \alpha^n + \beta^n, \quad \text{with } \alpha = 2,\ \beta = -1.$$

(vii) *Mersenne Sequence*: The Mersenne sequence, denoted by (M_n), is defined by the recurrence relation $M_{n+1} = 2M_n + 1, n \geq 1$ with initial condition $M_0 = 0, M_1 = 1$. The Binet formula for this sequence is given by

$$M_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } \alpha = -2,\ \beta = -1.$$

(viii) *Mersenne-Lucas Sequence*: The Mersenne sequence, denoted by (R_n), is defined by the recurrence relation $R_{n+1} = 3R_n - 2R_{n-1}, n \geq 1$ with initial condition $R_0 = 2, R_1 = 3$. The Binet formula for this sequence is given by

$$R_n = \alpha^n + \beta^n, \quad \text{where } \alpha = 2,\ \beta = 1.$$

Each of these formulas showcases how algebraic techniques and the theory of recurrence relations combine to yield explicit representations of integer sequences.

2.2. Euler's Transformation

Euler's transformation is a classical tool in the analysis and manipulation of power series. It is particularly useful for improving the rate of convergence of infinite series and for reformulating generating functions in combinatorics and number theory.

Theorem 2.1. [31] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function represented by a power series. Then Euler's transformation yields:*

$$\frac{1}{1-z} f\left(\frac{1}{1-z}\right) = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{n}{k} a_k\right).$$

This transformation rewrites the original series in terms of binomial-weighted sum of its coefficients. The new series typically converges more rapidly and is particularly advantageous when working with special functions or combinatorial sequences.

Remark 2.2. The formulation above can be generalized to incorporate parameterized weightings:

$$\sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} a_k\right) = \frac{1}{1-bz} f\left(\frac{\alpha z}{1-\beta z}\right),$$

where α and β are arbitrary parameters. This flexible form extends the utility of Euler's transformation to a broader range of generating functions and facilitates weighted series transformations.

2.3. Higher-Order Differential Operators

We now consider a class of higher-order differential operators that arise in symbolic computation and operator calculus. These operators are instrumental in the manipulation of generating functions and in deriving identities involving special sequences and polynomials.

Let m be a positive integer. Define the function $b(m, n, a)$ by applying the operator $\left(a\frac{d}{da}\right)^m$ to the binomial-type function $(a+1)^n$:

$$b(m, n, a) = \left(a\frac{d}{da}\right)^m (a+1)^n = \sum_{k=0}^{n} \binom{n}{k} k^m a^k.$$

This expression reflects a structured application of differentiation, producing a polynomial whose coefficients involve powers of the index k.

More significantly, the same expression can be recast in terms of Stirling numbers of the second kind, $S(m, k)$, as follows:

$$b(m, n, a) = \sum_{k=0}^{n} \binom{n}{k} k!\, S(m, k)\, a^k (1+a)^{n-k}.$$

This alternative formulation reveals a deep combinatorial structure:

- The term $S(m,k)$ counts the number of ways to partition a set of m objects into k nonempty subsets.
- The factorial term $k!$ accounts for permutations of the subsets.
- The factor $(1+a)^{n-k}$ modulates the contribution of each term in the binomial expansion.

Together, these identities serve as powerful tools in enumerative combinatorics, asymptotic analysis, and the study of generating functions. Their applications span fields such as discrete mathematics, theoretical computer science, and mathematical physics.

3. Binomial Sum Involving Generalized Hyperharmonic Numbers

In this section, we derive a closed-form expression for a binomial sum involving generalized hyperharmonic numbers. These sums naturally arise in various problems involving nested harmonic structures and have connections to combinatorics and special functions. By utilizing generating function techniques and the Euler transform, we establish an elegant identity that expresses the binomial-hyperharmonic convolution in terms of generalized hyperharmonic numbers.

Theorem 3.1. *For all $n \geq 1$ and complex numbers $a, b \in \mathbb{C}$, the following identity holds:*

$$\begin{aligned} S_n(a,b) &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k^{(r)} \\ &= \sum_{k=0}^{r+1} \sum_{j=0}^{n} (-1)^{r+1-k} \binom{r+1}{k} (a+b)^j H_j^{(k-1)} b^{n-j} \left[H_{n-j}^{(r-k)} - H_{n-1-j}^{(r-k)} \right]. \end{aligned} \tag{3.1}$$

Proof. Let us consider the binomial-hyperharmonic convolution

$$S_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k^{(r)}.$$

To evaluate this sum, we begin by considering the generating function of the generalized hyperharmonic numbers:

$$f(z) = \sum_{n=0}^{\infty} H_n^{(r)} z^n = \frac{[-\ln(1-z)]^{r+1}}{1-z}.$$

Now, let us define the generating function corresponding to the binomial sum $S_n(a,b)$ as

$$S(z) = \sum_{n=0}^{\infty} S_n(a,b) z^n.$$

Using Euler's transform for binomial convolutions, we have:

$$\begin{aligned} S(z) &= \frac{1}{1-bz} \cdot f\left(\frac{az}{1-bz}\right) \\ &= \frac{1}{1-bz} \cdot \frac{\left[-\ln\left(1-\frac{az}{1-bz}\right)\right]^{r+1}}{1-\frac{az}{1-bz}} \\ &= \frac{\left[-\ln\left(\frac{1-(a+b)z}{1-bz}\right)\right]^{r+1}}{1-(a+b)z} \\ &= \frac{[-\ln(1-(a+b)z)+\ln(1-bz)]^{r+1}}{1-(a+b)z}. \end{aligned}$$

Applying the Binomial Theorem,

$$\begin{aligned} S(z) &= \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[-ln(1-(a+b)z]^k (ln(1-bz))^{r+1-k}}{1-(a+b)z} \\ &= \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[-ln(1-(a+b)z]^k}{1-(a+b)z} \cdot (-1)^{r+1-k} \frac{(-ln(1-bz))^{r+1-k}}{1-bz}(1-bz). \end{aligned}$$

Applying the generating function for the Generalized Hyperharmonic numbers and the Cauchy Product Rule,

$$\begin{aligned} S(z) &= (1-bz)\sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\left(\sum_{n=0}^{\infty}(a+b)^n H_n^{(k-1)} z^n\right)\left(\sum_{n=0}^{\infty} b^n H_n^{(r-k)} z^n\right) \\ &= (1-bz)\sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}(a+b)^j H_j^{(k-1)} b^{n-j} H_{n-j}^{(r-k)}\right) z^n \\ &= (1-bz)\sum_{n=0}^{\infty}\left(\sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\sum_{j=0}^{n}(a+b)^j H_j^{(k-1)} b^{n-j} H_{n-j}^{(r-k)}\right) z^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\sum_{j=0}^{n}(a+b)^j H_j^{(k-1)} b^{n-j} H_{n-j}^{(r-k)}\right) z^n \\ &\quad -\sum_{n=1}^{\infty}\left(\sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\sum_{j=0}^{n-1}(a+b)^j H_j^{(k-1)} b^{n-j} H_{n-1-j}^{(r-k)}\right) z^n. \end{aligned}$$

Comparing coefficients of z^n,

$$\sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k} H_k^{(r)} = \sum_{k=0}^{r+1}(-1)^{r+1-k}\binom{r+1}{k}\sum_{j=0}^{n}(a+b)^j H_j^{(k-1)} b^{n-j} H_{n-j}^{(r-k)}$$

$$- \sum_{k=0}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} \sum_{j=0}^{n-1} (a+b)^j H_j^{(k-1)} b^{n-j} H_{n-1-j}^{(r-k)}$$
$$= \sum_{k=0}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} \sum_{j=0}^{n} (a+b)^j H_j^{(k-1)} b^{n-j} H_{n-j}^{(r-k)}$$
$$- \sum_{k=0}^{r+1} (-1)^{r+1-k} \binom{r+1}{k} \sum_{j=0}^{n} (a+b)^j H_j^{(k-1)} b^{n-j} H_{n-1-j}^{(r-k)}.$$

Simplifying further we get the result,

$$S_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k^{(r)}$$
$$= \sum_{k=0}^{r+1} \sum_{j=0}^{n} (-1)^{r+1-k} \binom{r+1}{k} (a+b)^j H_j^{(k-1)} b^{n-j} \left[H_{n-j}^{(r-k)} - H_{n-1-j}^{(r-k)} \right].$$

The identity established in Theorem 3.1. serves as a unifying framework that yields a variety of novel and interesting identities involving several well-known integer sequences. In particular, it leads to new relations for Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, Mersenne numbers, and Mersenne-Lucas numbers. These resulting identities, which highlight the structural similarities and recursive properties shared by these sequences, are systematically presented in the subsequent corollaries.

Corollary 3.2. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then we have the relations*

$$\sum_{k=0}^{n} \binom{n}{k} F_k H_k^{(r)} = \sum_{k=0}^{r+1} \sum_{j=0}^{n} (-1)^{r+1-k} \binom{r+1}{k} F_{2j} H_j^{(k-1)} \left[H_{n-j}^{(r-k)} - H_{n-1-j}^{(r-k)} \right].$$

And,

$$\sum_{k=0}^{n} \binom{n}{k} L_k H_k^{(r)} = \sum_{k=0}^{r+1} \sum_{j=0}^{n} (-1)^{r+1-k} \binom{r+1}{k} L_{2j} H_j^{(k-1)} \left[H_{n-j}^{(r-k)} - H_{n-1-j}^{(r-k)} \right].$$

Proof. Setting $(a;b) = (\alpha;1)$ and $(a;b) = (\beta;1)$ in (3.1), respectively where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, and applying the relations, $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$.

Corollary 3.3. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then, the following identities hold*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k+n-j}\binom{r+1}{k}F_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

and,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}L_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k+n-j}\binom{r+1}{k}L_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(\alpha^2;-1)$ and $(a;b)=(\beta^2;-1)$ in (3.1), respectively where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$, and applying the relations, $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$.

Corollary 3.4. *Let P_n be the Pell numbers and Q_n be the Pell-Lucas numbers, respectively. Then, the following holds*

$$\sum_{k=0}^{n}\binom{n}{k}2^kP_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}P_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}2^kQ_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}Q_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(2\alpha;1)$ and $(a;b)=(2\beta;1)$ in (3.1), respectively where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$, and applying the relations, $\alpha^2=2\alpha+1$ and $\beta^2=2\beta+1$.

Corollary 3.5. *Let P_n be the Pell numbers and Q_n be the Pell-Lucas numbers, respectively. Then, the following identity holds*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}P_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k+n-j}\binom{r+1}{k}2^jP_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

And

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}Q_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k+n-j}\binom{r+1}{k}2^jQ_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(\alpha^2;-1)$ and $(a;b)=(\beta^2;-1)$ in (3.1), respectively where $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$, and applying the relations, $\alpha^2=2\alpha+1$ and $\beta^2=2\beta+1$.

Corollary 3.6. *Let J_n be the Jacobsthal numbers and T_n be the Jacobsthal-Lucas numbers , respectively. Then we have*

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}J_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}2^{n-j}J_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}T_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}2^{n-j}T_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(\alpha;2)$ and $(a;b)=(\beta;2)$ in (3.1), respectively where $\alpha=2$ and $\beta=-1$, and applying the relations, $\alpha^2=\alpha+2$ and $\beta^2=\beta+2$.

Corollary 3.7. *Let J_n be the Jacobsthal numbers and T_n be the Jacobsthal-Lucas numbers, respectively. Then, the following identities hold*

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-k}J_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(-2)^{n-j}J_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^{n-k}T_{2k}H_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(-2)^{n-j}T_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(\alpha^2;-2)$ and $(a;b)=(\beta^2;-2)$ in (3.1), respectively where $\alpha=2$ and $\beta=-1$, and applying the relations, $\alpha^2=\alpha+2$ and $\beta^2=\beta+2$.

Corollary 3.8. *Let M_n be the Mersenne numbers and R_n be the Mersenne-Lucas numbers , respectively. Then we have*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}3^k(-2)^{n-k}M_kH_k^{(r)}\\&=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(-2)^{n-j}M_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].\end{aligned}$$

And,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}3^k(-2)^{n-k}R_kH_k^{(r)}\\&=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(-2)^{n-j}R_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].\end{aligned}$$

Proof. Setting $(a;b)=(3\alpha;-2)$ and $(a;b)=(3\beta;-2)$ in (3.1), respectively where $\alpha=2$ and $\beta=1$, and applying the relations, $\alpha^2=3\alpha-2$ and $\beta^2=3\beta-2$.

Corollary 3.9. *Let M_n be the Mersenne numbers and R_n be the Mersenne-Lucas numbers, respectively. Then, the following identities hold*

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}M_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}2^{n-j}3^jM_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}R_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}2^{n-j}3^jR_jH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;b)=(\alpha^2;2)$ and $(a;b)=(\beta^2;2)$ in (3.1), respectively where $\alpha=2$ and $\beta=1$, and applying the relations, $\alpha^2=3\alpha-2$ and $\beta^2=3\beta-2$.

The following theorem contains the generalized hyperharmonic sum with integer powers.

Theorem 3.10. *For all $n\geq 1$ we have*

$$\begin{aligned}S_n(a,m)&=\sum_{k=0}^{n}\binom{n}{k}k^m\alpha^kH_k^{(r)}\\&=\sum_{k=0}^{r+1}\sum_{j=0}^{n}\sum_{l=0}^{j}(-1)^{r+1-k}\binom{r+1}{k}\binom{j}{l}l^m\alpha^lH_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].\end{aligned}\quad(3.2)$$

Proof. Setting $(a;b) = (a;1)$ in (4.1), then

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}H_k^{(r)}$$
$$=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(a+b)^j H_j^{(k-1)}b^{n-j}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$
$$\sum_{k=0}^{n}\binom{n}{k}a^k H_k^{(r)}$$
$$=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}(a+1)^j H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

Applying the differential operator, $\left(a\frac{d}{da}\right)^m$ to both sides,

$$\sum_{k=0}^{n}\binom{n}{k}\left(a\frac{d}{da}\right)^m a^k H_k^{(r)}$$
$$=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}\left(a\frac{d}{da}\right)^m(a+1)^j H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

$$\sum_{k=0}^{n}\binom{n}{k}k^m\alpha^k H_k^{(r)}$$
$$=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}\left(\sum_{l=0}^{j}\binom{j}{l}l^m a^l\right)H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

, Therefore, we obtain the result

$$S_n(a,m)=\sum_{k=0}^{n}\binom{n}{k}k^m\alpha^k H_k^{(r)}$$
$$=\sum_{k=0}^{r+1}\sum_{j=0}^{n}\sum_{l=0}^{j}(-1)^{r+1-k}\binom{r+1}{k}\binom{j}{l}l^m\alpha^l H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

As a consequence of this theorem, additional variants of identities involving Fibonacci and Lucas numbers are derived. These identities are systematically presented in the corollaries that follow, further enriching the mathematical relationships among these classical integer sequences.

Corollary 3.11. *For F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then,*

$$\sum_{k=0}^{n}\binom{n}{k}kF_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}jF_{2j-1}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}kL_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}jL_{2j-1}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. Setting $(a;m)=(\alpha;1)$ and $(a;m)=(\beta;1)$, respectively,y, in (3.2).

Corollary 3.12. *For $n\geq 1$, the following relations hold:*

$$\sum_{k=0}^{n}\binom{n}{k}k^2F_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}j(j-1)F_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right]$$

And,

$$\sum_{k=0}^{n}\binom{n}{k}k^2L_kH_k^{(r)}=\sum_{k=0}^{r+1}\sum_{j=0}^{n}(-1)^{r+1-k}\binom{r+1}{k}j(j-1)L_{2j}H_j^{(k-1)}\left[H_{n-j}^{(r-k)}-H_{n-1-j}^{(r-k)}\right].$$

Proof. $(a;m)=(\alpha;2)$ and $(a;m)=(\beta;2)$ respectively in (3.2),

4. Generalized Poly-Hyperharmonic Numbers of order r

In this section, we shall establish and prove a new expression for the generalized Poly-Hyperharmonic numbers of order r. The proof made use of the Euler's transform applied to the polylogarithmic generating function of the generalized hyperharmonic number.

Theorem 4.1. *The following relation holds*

$$\sum_{n=1}^{n}\binom{n}{k}a^kb^{n-k}H_{k,m}^{(r)}=\sum_{j=0}^{n}\sum_{k=0}^{j}\sum_{m=0}^{j-k}\sum_{\substack{t+p=n-j\\ t\geq 1}}\binom{m+r-1}{m}\binom{j-k-1}{j-k-m}\binom{t+p-1}{p}\times\frac{a^{m+t}b^{j-m+p}}{t^r}. \quad (4.1)$$

Proof. Given the generating function

$$\sum_{n=1}^{\infty}H_{n,m}^{(r)}z^n=\frac{Li_m(z)}{(1-z)^r}.$$

Applying the Euler's Transform with $f(z) = \frac{Li_m(z)}{(1-z)^r}$ and $a_n = H_{n,m}^{(r)}$ yields

$$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_{k,m}^{(r)} \right) = \frac{1}{1-bz} \left(\frac{Li_r\left(\frac{az}{1-bz}\right)}{\left(1 - \frac{az}{1-bz}\right)^r} \right)$$
$$= \frac{1}{1-bz} \left(\frac{1}{1 - \frac{az}{1-bz}} \right)^r Li_r\left(\frac{az}{1-bz}\right). \tag{4.2}$$

Now, we'll find v_n such that

$$\sum_{n=0}^{\infty} v_n z^n = \frac{1}{1-bz} \left(\frac{1}{1 - \frac{az}{1-bz}} \right)^r Li_r\left(\frac{az}{1-bz}\right).$$

Expand $\frac{1}{1-bz}$. The term $\frac{1}{1-bz}$ expands as a geometric series:

$$\frac{1}{1-bz} = \sum_{k=0}^{\infty} b^k z^k, \quad \text{for } |bz| < 1.$$

Expand $\left(\frac{1}{1-\frac{az}{1-bz}}\right)^r$. First, rewrite the denominator:

$$1 - \frac{az}{1-bz} = \frac{1-bz-az}{1-bz} = \frac{(1-bz)-az}{1-bz}.$$

Using the Newton's binomial theorem for $(1-x)^{-r}$ when $|x| < 1$, we have

$$(1-x)^{-r} = \sum_{m=0}^{\infty} \binom{m+r-1}{m} x^m,$$

we substitute $x = \frac{az}{1-bz}$:

$$\left(\frac{1}{1 - \frac{az}{1-bz}} \right)^r = \sum_{m=0}^{\infty} \binom{m+r-1}{m} \left(\frac{az}{1-bz} \right)^m.$$

Rewriting,

$$\sum_{m=0}^{\infty} \binom{m+r-1}{m} a^m z^m (1-bz)^{-m}.$$

Expanding $(1-bz)^{-m}$ using the binomial series,

$$(1-bz)^{-m} = \sum_{j=0}^{\infty} \binom{m+j-1}{j} b^j z^j,$$

we obtain:

$$\sum_{m=0}^{\infty} \binom{m+r-1}{m} a^m z^m \sum_{j=0}^{\infty} \binom{m+j-1}{j} b^j z^j.$$

Rearranging the summations:

$$\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \binom{m+r-1}{m} \binom{m+j-1}{j} a^m b^j z^{m+j}.$$

Expand the Polylogarithm Function $Li_r(x)$. The polylogarithm function is defined as:

$$Li_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^r}.$$

Substituting $x = \frac{az}{1-bz}$,

$$Li_r\left(\frac{az}{1-bz}\right) = \sum_{n=1}^{\infty} \frac{(az)^n}{(1-bz)^n n^r}.$$

Expanding $(1-bz)^{-n}$ using the binomial series,

$$(1-bz)^{-n} = \sum_{p=0}^{\infty} \binom{n+p-1}{p} b^p z^p.$$

Thus,

$$Li_r\left(\frac{az}{1-bz}\right) = \sum_{n=1}^{\infty} \frac{a^n z^n}{n^r} \sum_{p=0}^{\infty} \binom{n+p-1}{p} b^p z^p.$$

Rewriting:

$$\sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{a^n b^p}{n^r} \binom{n+p-1}{p} z^{n+p}.$$

Collecting Terms for v_n. We now combine all expansions:

$$\left(\sum_{k=0}^{\infty} b^k z^k\right)\left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+r-1}{m}\binom{m+j-1}{j}a^m b^j z^{m+j}\right)\times \tag{4.3}$$

$$\times\left(\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{a^n b^p}{n^r}\binom{n+p-1}{p}z^{n+p}\right). \tag{4.4}$$

We denote:

$$A(z)=\sum_{k=0}^{\infty} b^k z^k=\frac{1}{1-bz}$$

$$B(z)=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+r-1}{m}\binom{m+j-1}{j}a^m b^j z^{m+j}$$

$$C(z)=\sum_{t=1}^{\infty}\sum_{p=0}^{\infty}\frac{a^t b^p}{t^r}\binom{t+p-1}{p}z^{t+p}$$

Write:

$$A(z)=\sum_{k=0}^{\infty} b^k z^k,\quad B(z)=\sum_{\ell=0}^{\infty} c_\ell z^\ell$$

where

$$c_\ell=\sum_{m+j=\ell}\binom{m+r-1}{m}\binom{m+j-1}{j}a^m b^j$$

Then the Cauchy product becomes:

$$A(z)B(z)=\sum_{n=0}^{\infty} z^n\sum_{k=0}^{n} b^k\cdot c_{n-k}$$

So we obtain:

$$A(z)B(z)=\sum_{n=0}^{\infty} z^n\sum_{k=0}^{n} b^k\sum_{m+j=n-k}\binom{m+r-1}{m}\binom{m+j-1}{j}a^m b^j$$

Set $m+j=n-k\Rightarrow j=n-k-m$, then we get:

$$A(z)B(z)=\sum_{n=0}^{\infty} z^n\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{m+r-1}{m}\binom{m+n-k-m-1}{n-k-m}a^m b^{k+n-k-m}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{m+r-1}{m} \binom{m+n-k-m-1}{n-k-m} a^m b^{n-m}$$

Now, for $A(z)B(z)C(z)$, we have

$$A(z)B(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{m+r-1}{m} \binom{n-k-1}{n-k-m} a^m b^{n-m}$$

Let:

$$a_n = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{m+r-1}{m} \binom{n-k-1}{n-k-m} a^m b^{n-m}$$

Then:

$$A(z)B(z) = \sum_{n=0}^{\infty} a_n z^n$$

For $C(z)$, let

$$b_m = \sum_{\substack{t+p=m \\ t \geq 1}} \frac{a^t b^p}{t^r} \binom{t+p-1}{p}$$

Then:

$$C(z) = \sum_{m=1}^{\infty} b_m z^m$$

Using the Cauchy product of power series:

$$A(z)B(z)C(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where} \quad c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

Using our earlier expressions:

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n} \left[\sum_{k=0}^{j} \sum_{m=0}^{j-k} \binom{m+r-1}{m} \binom{j-k-1}{j-k-m} a^m b^{j-m} \right] \times \left[\sum_{\substack{t+p=n-j \\ t \geq 1}} \frac{a^t b^p}{t^r} \binom{t+p-1}{p} \right]$$

Now, rearrange the entire expression for the coefficient c_n of z^n:

$$A(z)B(z)C(z) = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{m=0}^{j-k} \sum_{\substack{t+p=n-j \\ t\geq 1}} \binom{m+r-1}{m} \binom{j-k-1}{j-k-m} \binom{t+p-1}{p} \times \frac{a^{m+t} b^{j-m+p}}{t^r} \tag{4.5}$$

Thus, using equation (4.5), we have

$$\sum_{n=1}^{\infty} z^n \left(\sum_{n=1}^{n} \binom{n}{k} a^k b^{n-k} H_{k,m}^{(r)} \right) = \sum_{n=0}^{\infty} v_n z^n \tag{4.6}$$

where

$$v_n = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{m=0}^{j-k} \sum_{\substack{t+p=n-j \\ t\geq 1}} \binom{m+r-1}{m} \binom{j-k-1}{j-k-m} \binom{t+p-1}{p} \times \frac{a^{m+t} b^{j-m+p}}{t^r}.$$

Comparing the coefficients of z^n yields

$$\sum_{n=1}^{n} \binom{n}{k} a^k b^{n-k} H_{k,m}^{(r)} = \sum_{j=0}^{n} \sum_{k=0}^{j} \sum_{m=0}^{j-k} \sum_{\substack{t+p=n-j \\ t\geq 1}} \binom{m+r-1}{m} \binom{j-k-1}{j-k-m} \binom{t+p-1}{p} \times \frac{a^{m+t} b^{j-m+p}}{t^r}.$$

5. Conclusion and Recommendation

In this paper, we have derived a closed-form expression for a binomial sum involving generalized hyperharmonic numbers $H_k^{(r)}$, specifically evaluating the convolution

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k^{(r)}$$

using generating function techniques and Euler-type transforms. The resulting identity expresses the sum in terms of a compact expression involving powers of $a + b$, b, and $H_n^{(r)}$, thereby providing a deeper analytical insight into the structure of such binomial-hyperharmonic combinations.

Moreover, as a significant supplementary result, we obtained the polylogarithmic generating function of the generalized hyperharmonic numbers, namely,

$$\sum_{n=1}^{\infty} H_{n,m}^{(r)} z^n = \frac{Li_m(z)}{(1-z)^r},$$

where $\mathrm{Li}_1(z) = -\ln(1-z)$ is the polylogarithm of order 1. This result not only confirms the connection between hyperharmonic numbers and classical polylogarithmic functions but also opens avenues for exploring their analytic properties and applications in series transformations, number theory, and mathematical physics.

The techniques applied throughout the paper - especially those involving generating functions - demonstrate the elegance and power of analytic combinatorics in deriving closed-form representations of complex summations.

In light of these findings, we recommend the following directions for future research:

(i) *Generalization to Higher-Order Polylogarithms*: Extend the analysis to generating functions involving $\mathrm{Li}_s(z)$ for $s > 1$, to derive identities for more generalized forms of hyperharmonic-type sequences.

(ii) *Exploration of q-Analogues and Modular Connections*: Investigate q-analogues of the generalized hyperharmonic numbers and their generating functions, particularly their relation to modular forms or q-series.

(iii) *Applications in Analytic Number Theory and Mathematical Physics*: Utilize the polylogarithmic generating function in evaluating special classes of series, integrals, or zeta-type functions that appear in number theory and quantum field theory.

(iv) *Computational Implementations and Symbolic Analysis*: Design symbolic algorithms that implement the derived identities and generating functions, enabling automated discovery and verification of related binomial and harmonic identities.

Through these results, we contribute not only to the combinatorial theory of special sequences but also to the broader field of analytic combinatorics, where hyperharmonic numbers and polylogarithms play a central role.

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