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On Some Applications of Generalized Meijer G - Functions

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Abstract. In recent few years, the researchers are investigating many generalization and extension of special functions because of their applicability in various fields of study especially in mathematical modeling. Keeping in view, some summations involving generalized Meijer G-functions in k-form for different combination of parameters have been investigated in this paper. By using contour integral approach and generalized hypergeometric k-functions we derived some new relations and identities. By taking $k = 1$, we can obtain the classical form of the derived results. Some important applications of generalized Meijer G-functions to the generalized hypergeometric functions have also been considered. The main objective of this paper is to give some useful computational techniques to tackle with the applications related to generalized Meijer G-functions in different mathematical and physical problems, also connect them with some generalized special functions.

2020 Mathematics Subject Classifications: 33C60, 33C20, 33C05, 33C15

Key Words and Phrases: Meijer G-function; Generalized Meijer G - function; Gauss hypergeometric function; Confluent hypergeometric function.

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1. Introduction and Preliminaries

Special functions are widely applicable in different areas of study like pure and applied mathematics, physics, engineering, and mathematical modeling. In the field of special functions the main aim is to derive and investigate the results, which are helpful in solving the problems of different fields. With this aim many researchers are working through different directions. In the last two decades the researchers like Diaz, Pariguan, Mubeen, Habibullah, Kokologiannaki, Krasniqi, Mansour etc. worked on the specific k-symbol and proved number of properties and applications.

Diaz and Pariguan [1], presented the extension of beta and gamma functions respectively, in the form of $k > 0$ as

$$\Gamma_k(o) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{o}{k}-1}}{(o)_{n,k}} \qquad where\ k > 0, o \in \mathbb{C}\backslash kZ^- \tag{1}$$

and

$$\beta_k(o,\omega) = \frac{1}{k}\int_0^1 t^{\frac{o}{k}-1}(1-t)^{\frac{\omega}{k}-1}dt, \qquad \Re(o) > 0, \Re(\omega) > 0 \tag{2}$$

on the basis of extended Pochhammer's symbol

$$(o)_{n,k} = (o)(o+k)(o+2k)...(o+(n-1)k), \qquad \mathsf{n} \geq 1, k > 0.$$

They also discussed the integral form of extended gamma function

$$\Gamma_k(o) = \int_0^\infty t^{\frac{o}{k}-1}\mathsf{e}^{\frac{-t^k}{k}}dt, \qquad \Re(o) > 0, k > 0$$

and the hypergeometric function k-function [1] for all $\epsilon, \wp, \omega \in \mathbb{C}$ and $\omega \neq 0, -1, -2, -3, \cdots,\ |\alpha| < 1$, as

$${}_2F_{1,k}((\omega,\mathsf{k}),(\wp,\mathsf{k});(\epsilon,\mathsf{k});\alpha) = \sum_{\mathsf{m}=0}^{\infty} \frac{(\omega)_{\mathsf{m,k}}(\wp)_{\mathsf{m,k}}}{(\epsilon)_{\mathsf{m,k}}}\frac{\alpha^m}{m!}, \qquad k > 0.$$

After that the researchers [2–11] worked on k-functions, proved many properties and results in k-form. Mubeen *et al.* [12] introduced generalized hypergeometric differential equation also proved twenty four solutions of that equation. Some other functions like Mittag-Leffler function, Fox H-function, Meijer G-function etc. are the functions upon which many researchers [13–25] have worked and gave different properties.
In the last few years, the researchers [26–31] started to see the applicability of the special functions and fractional operators in various fields like Magneto hydrodynamics (MHD) and hybrid nanofluids, entropy analysis in special fluid flows, heat and mass transfer via special functions. Khan *et.al* [26] concentrated on many generalizations of MHD fluids through different fractional operators. They also suggested that many complicated practical life phenomena's can be dealt easily with the help special functions and fractional operators. The Caputo–Fabrizio time-fractional derivatives have been used by [31] to generalize the idea of dusty tetra hybrid nanofluid, also discussed several applications like in

signal processing, diffusion, image processing, damping, and bioengineering etc. of generalized fractional operators.

Among several special functions, Meijer G-function as a particular instance of Fox H-function is an interesting function due to its generality because many other classical functions are the special cases of this function. Meijer G-function is an essential mathematical tool in studying different applied mathematical models, integral equations, and differential equations. Milgram [32] gave some techniques to tackle the summations involving Meijer G-functions and proved some useful results.

In our current investigations, we prove the generalization of some summations involving Meijer G-function in k-form, $k > 0$, and give some techniques to tackle with applications related to generalized Meijer G-functions. All the obtained results will take the classical forms while choosing $k = 1$.

Generalization of Meijer G-function (*i.e.* Meijer (G, k)-function) [33, 34], for $k > 0$ can be defined as:

$$G^{m,n}_{k,p,q}\left[\begin{matrix} e_p \\ f_q \end{matrix}\middle| z\right] = \frac{1}{2\pi i}\oint_L \frac{\Gamma_k(\mathsf{f}_{1,m} - s)\Gamma_k(k - \mathsf{e}_{1,n} + s)}{\Gamma_k(\mathsf{e}_{n+1,p} - s)\Gamma_k(k - \mathsf{f}_{m+1,q} + s)} z^{s/k} ds, \tag{3}$$

where

$$\Gamma_k(o_{n,p} + s) = \prod_{j=n}^{p} \Gamma_k(o_j + s)$$

is the gamma k-function.

Luke [21] considered the path of contour integral of the form given above for various ranges of z. Here, we consider $|z| \leq \frac{1}{k}, k > 0$ and $p \leq q,\ q \geq 1$. We are keen in the summations involving generalized Meijer G-function as:

$$\sum_{l} \frac{x^l g_l}{l!} G^{m,n}_{k,p,q}\left[\begin{matrix} e_p(l) \\ f_q(l) \end{matrix}\middle| z\right], \tag{4}$$

where $e_p(l)$ and $f_q(l)$ are of the form $\omega \pm lk$ and $\rho \pm lk$ respectively.

The basic idea, is to observe that if g_l, $e_p(l)$ and $f_q(l)$ are such that after replacing the summation and integral in (4), the l dependence take the form of Gauss hypergeometric function ${}_2F_{1,k}$ and then different known results and transformations can be applied on ${}_2F_{1,k}$. After solving ${}_2F_{1,k}$, the order of summation and integration can be interchanged again and we will obtain generalized G-function.

2. Main Result

In this section, we investigate a result which implements the technique discussed above.

Theorem 1. *For* $\Re(h-g-k+\omega) > min(\Re(f_j)),\ j=1,...,m,\ k>0,$ *the following equation holds:*

$$S_k(\frac{1}{k},z)=\frac{\Gamma_k(g)}{\Gamma_k(h-g)}G^{m+1,n+1}_{k,p+2,q+1}\left[\begin{matrix}\omega,e_p,\omega+h-k\\ \omega+h-g-k,f_q\end{matrix}\middle|z\right].$$

Proof. Consider the summation

$$S_k(x,z)=\sum_l \frac{\Gamma_k(g+lk)x^l}{l!\Gamma_k(h+lk)}G^{m,n+1}_{k,p+1,q}\left[\begin{matrix}\omega-lk,e_p\\ f_q\end{matrix}\middle|z\right]. \tag{5}$$

By using the definitions of generalization of Meijer G-function and hypergeometric k-function, we get

$$S_k(x,z)=\frac{\Gamma_k(g)}{\Gamma_k(h)}\frac{1}{2\pi i}\oint_L \frac{\prod_{j=1}^m\Gamma_k(f_j-s)\prod_{j=1}^n\Gamma_k(k-e_j+s)}{\prod_{j=n+1}^p\Gamma_k(e_j-s)\prod_{j=m+1}^q\Gamma_k(k-f_j+s)}\Gamma_k(k-\omega+s)\,{}_2F_{1,k}\left[\begin{matrix}g,k-\omega+s\\ h\end{matrix}\middle|x\right]z^{s/k}ds$$

from equation (3) for $|x|<\frac{1}{k}$.

To evaluate the summation at $x=\frac{1}{k}$, we use the relation

$$\lim_{x\to(\frac{1}{k})^-}\ {}_2F_{1,k}\left[\begin{matrix}\omega,\eta\\ \rho\end{matrix}\middle|x\right]=\frac{\Gamma_k(\rho)\Gamma_k(\rho-\omega-\eta)}{\Gamma_k(\rho-\omega)\Gamma_k(\rho-\eta)},\qquad \Re(\rho-\omega-\eta)>0.$$

Thus, we obtain

$$S_k(\frac{1}{k},z)=\frac{\Gamma_k(g)}{\Gamma_k(h-g)}G^{m+1,n+1}_{k,p+2,q+1}\left[\begin{matrix}\omega,e_p,\omega+h-k\\ \omega+h-g-k,f_q\end{matrix}\middle|z\right] \tag{6}$$

for $\Re(h-g-k+\omega) > min(\Re(f_j)),\ j=1,...,m,$ in other case, we obtain residues of $\Gamma_k(h-g-k+\omega-s)$ for the poles which lies outside the contour L.

Corollary 1. *For* $|x|<\frac{1}{k}$, *the transformation [13]*

$${}_2F_{1,k}(g,k-\omega+s;h;x)=(1-kx)^{\frac{h-g-k+\omega-s}{k}}\ {}_2F_{1,k}\left[\begin{matrix}h-g,h-k+\omega-s\\ h\end{matrix}\middle|x\right] \tag{7}$$

can be used in (5) to get

$$S_k(x,z)=\frac{\Gamma_k(g)(1-kx)^{\frac{h-g-k+\omega}{k}}}{\Gamma_k(h-g)}\sum_l\frac{\Gamma_k(h-g+lk)x^l}{\Gamma_k(h+lk)l!}G^{m+1,n+1}_{k,p+2,q+1}\left[\begin{matrix}\omega,e_p,\omega+h-k\\ \omega+h-k+lk,f_q\end{matrix}\middle|\frac{z}{1-kx}\right]. \tag{8}$$

Remark:
Similar and even more difficult results can also be obtained by using ${}_{p+1}F_{q,k}$ for known arguments.

3. some variation formulas involving generalized Meijer G-functions

In this section, we consider some summations involving generalized Meijer G-functions for $x = \pm\frac{1}{k}$ and use some transformations to obtain the some useful results.

Theorem 2. *For $0 \le n < p+1$, $p+2 \le q$, $0 \le m \le q$ and $\frac{(\eta-g-k+\omega)}{2}$ which is enclosed by contour for $f_j, j = 1, ..., m$, the following holds:*

$$\frac{1}{\Gamma_k(g)} \sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega - lk, e_p, \eta + lk \\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{2^{\frac{g}{k}}} G^{m+2,n+1}_{k,p+4,q+2}\left[\begin{matrix}\omega, e_p, \frac{(\omega+\eta-k)}{2}, \frac{(\eta+\omega)}{2}, \eta - g \\ \frac{(\eta+\omega-g-k)}{2}, \frac{(\eta+\omega-g)}{2}, f_q\end{matrix}\middle| z\right]. \quad (9)$$

Proof. Consider the left hand side as

$$\frac{1}{\Gamma_k(g)} \sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega - lk, e_p, \eta + lk \\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{\Gamma_k(g)} \sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} \frac{1}{2\pi\iota} \oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j - s) \prod_{j=1}^n \Gamma_k(k - e_j + s)}{\prod_{j=n+1}^p \Gamma_k(e_j - s) \prod_{j=m+1}^q \Gamma_k(k - f_j + s)} z^{\frac{s}{k}} ds$$

$$= \frac{1}{2\pi\iota} \oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j - s) \prod_{j=1}^n \Gamma_k(k - e_j + s)}{\prod_{j=n+1}^p \Gamma_k(e_j - s) \prod_{j=m+1}^q \Gamma_k(k - f_j + s)} \sum_l \frac{(k-\omega+s)_{l,k}\Gamma_k(k-\omega+s)(g)_{l,k}(\frac{1}{k})^l}{(\eta - s)_{l,k}\Gamma_k(\eta - s) l!} z^{\frac{s}{k}} ds$$

$$= \frac{1}{2\pi\iota} \oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j - s) \prod_{j=1}^n \Gamma_k(k - e_j + s)\Gamma_k(k-\omega+s)}{\prod_{j=n+1}^p \Gamma_k(e_j - s) \prod_{j=m+1}^q \Gamma_k(k - f_j + s)\Gamma_k(\eta - s)}$$
$$\times \frac{\Gamma_k(\eta - s)\Gamma_k(\eta - s - k - g + \omega - s)}{\Gamma_k(\eta - s - g)\Gamma_k(\eta - s - k - s + \omega)} z^{\frac{s}{k}} ds.$$

After doing simple mathematical calculations we finally obtain

$$\frac{1}{\Gamma_k(g)} \sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega - lk, e_p, \eta + lk \\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{2^{\frac{g}{k}}} G^{m+2,n+1}_{k,p+4,q+2}\left[\begin{matrix}\omega, e_p, \frac{(\omega+\eta-k)}{2}, \frac{(\eta+\omega)}{2}, \eta - g \\ \frac{(\eta+\omega-g-k)}{2}, \frac{(\eta+\omega-g)}{2}, f_q\end{matrix}\middle| z\right].$$

Corollary 2. *For $\eta = g+\omega$, the quadratic transformation $x \to \frac{-4x}{(1-kx)^2}$ of the function ${}_2F_{1,k}$ in equation (9) can be used, as for $x = \frac{-1}{k}$, we get*

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{-1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega-lk, e_p, g+\omega+lk\\ f_q\end{matrix}\middle| z\right] = \frac{\Gamma_k(\frac{k}{2})}{2^{\frac{g}{k}}\Gamma_k\frac{(g+k)}{2}} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega, e_p, \frac{g}{2}+\omega\\ f_q\end{matrix}\middle| z\right]. \tag{10}$$

Theorem 3. *For $0 \le n < p+1$, $p+2 \le q$, $0 \le m \le q$ and $\frac{(\eta-g-k+\omega)}{2}$ which is enclosed by contour for $f_j, j = 1, ..., m$, the following holds:*

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega+lk, e_p, \eta-lk\\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{2^{\frac{g}{k}}} G^{m+3,n+1}_{k,p+5,q+3}\left[\begin{matrix}\omega, e_p, \frac{(\omega+\eta-k)}{2}, \frac{(\omega+\eta)}{2}, \eta, \omega-g\\ \omega, \frac{(\omega+\eta-g-k)}{2}, \frac{(\omega+\eta-g)}{2}, f_q\end{matrix}\middle| z\right]. \tag{11}$$

Proof. Considering the left hand side as

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega+lk, e_p, \eta-lk\\ f_q\end{matrix}\middle| z\right]$$
$$= \sum_l \frac{(g)_{l,k}(\frac{1}{k})^l}{l!}\frac{1}{2\pi\iota}\oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j-s)\Gamma_k(k-\omega-lk+s)\prod_{j=2}^{n+1}\Gamma_k(k-e_j+s)}{\prod_{j=n+2}^{p+1}\Gamma_k(e_j-s)\Gamma_k(\eta-lk-s)\prod_{j=m+1}^{q}\Gamma_k(k-f_j+s)} z^{\frac{s}{k}} ds.$$

Now, by interchanging the order of summation and integration and then applying

$$(k-\omega)_{-n,k} = \frac{(-1)^n}{(\omega)_{n,k}},$$

we obtain

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{-1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega-lk, e_p, g+\omega+lk\\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{2\pi\iota}\oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j-s)\prod_{j=1}^{n}\Gamma_k(k-e_j+s)}{\prod_{j=n+1}^{p}\Gamma_k(e_j-s)\prod_{j=m+1}^{q}\Gamma_k(k-f_j+s)} \times \sum_l \frac{(g)_{l,k}\Gamma_k(k-\omega-lk+s)(\frac{1}{k})^l}{\Gamma_k(\eta-lk-s)l!} z^{\frac{s}{k}} ds.$$

Applying Legendre Duplication k-formula for $z = \frac{\omega-g-k+\eta-2s}{2}$ and $z = \frac{\omega+\eta}{2} - s$, we get

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{-1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\omega-lk, e_p, g+\omega+lk\\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{1}{2^{\frac{g}{k}}2\pi\iota}\oint_L \frac{\prod_{j=1}^m \Gamma_k(f_j-s)\prod_{j=2}^{n+1}\Gamma_k(k-e_j+s)}{\prod_{j=n+2}^{p+1}\Gamma_k(e_j-s)\prod_{j=m+1}^{q}\Gamma_k(k-f_j+s)}$$

$$\times \frac{\Gamma_k(\omega-s)\Gamma_k(\frac{\omega-g+\eta}{2}-s)\Gamma_k(k-\omega-s)\Gamma_k(\frac{\omega-g-k+\eta}{2}-s)}{\Gamma_k(\eta-s)\Gamma_k(\omega-s-g)\Gamma_k(\frac{\omega+\eta}{2}-s)\Gamma_k(\frac{\omega+\eta-k}{2}-s)} z^{\frac{s}{k}} ds$$
$$= \frac{1}{2^{\frac{g}{k}}} G^{m+3,n+1}_{k,p+5,q+3}\left[\begin{matrix}\omega, e_p, \frac{\omega-k+\eta}{2}, \frac{\omega+\eta}{2}, \eta, \omega-g \\ \omega, \frac{\omega-k+\eta-g}{2}, \frac{\omega+\eta-g}{2}, f_q\end{matrix}\middle| z\right].$$

Corollary 3. *If we choose $\omega=\eta+g$ in Theorem (3), then*

$$\frac{1}{\Gamma_k(g)}\sum_l \frac{\Gamma_k(g+lk)(\frac{-1}{k})^l}{l!} G^{m,n+1}_{k,p+2,q}\left[\begin{matrix}\eta+g+lk, e_p, \eta-lk \\ f_q\end{matrix}\middle| z\right]$$
$$= \frac{\Gamma_k(\frac{k}{2})}{2^{\frac{g}{k}}\Gamma_k\frac{(g+k)}{2}} G^{m+1,n+1}_{k,p+3,q+1}\left[\begin{matrix}\eta+g, e_p, \eta+\frac{g}{2}, \eta \\ \eta+g, f_q\end{matrix}\middle| z\right],$$

where m,n,p and q are identical to classical Meijer G-function.

By keeping in view the above variants, we consider another variant as

Theorem 4. *For $0\le m\le q$, $0\le n\le p\le q$, $\Re(h-\eta+\omega)>1$, $k>0$ the following holds:*

$$\sum_l \frac{(\frac{1}{k})^l}{\Gamma_k(h+lk)l!} G^{m+1,n+1}_{k,p+1,q+1}\left[\begin{matrix}\omega-lk, e_p \\ \eta+lk, f_q\end{matrix}\middle| z\right]$$
$$= \Gamma_k(h-\eta+\omega-k) G^{m+1,n+1}_{k,p+2,q+2}\left[\begin{matrix}\omega, e_p, \omega+h-k \\ \eta, f_q, \eta+h+K\end{matrix}\middle| z\right].$$

Proof. Consider the left side as

$$\sum_l \frac{(\frac{1}{k})^l}{\Gamma_k(h+lk)l!} G^{m+1,n+1}_{k,p+1,q+1}\left[\begin{matrix}\omega-lk, e_p \\ \eta+lk, f_q\end{matrix}\middle| z\right]$$
$$= \sum_l \frac{(\frac{1}{k})^l}{\Gamma_k(h+lk)l!}\frac{1}{2\pi\iota}\oint_L \frac{\prod_{j=1}^{m+1}\Gamma_k(f_j-s)\prod_{j=1}^{n+1}\Gamma_k(k-e_j+s)}{\prod_{j=n+2}^{p+1}\Gamma_k(e_j-s)\prod_{j=m+2}^{q+1}\Gamma_k(k-f_j+s)} z^{\frac{s}{k}} ds.$$

Now, interchanging the order of summation and integral also doing some calculations, we have

$$\sum_l \frac{(\frac{1}{k})^l}{\Gamma_k(h+lk)l!} G^{m+1,n+1}_{k,p+1,q+1}\left[\begin{matrix}\omega-lk, e_p \\ \eta+lk, f_q\end{matrix}\middle| z\right]$$
$$= \frac{\Gamma_k(h+\omega-\eta-k)}{2\pi\iota}\oint_L \frac{\prod_{j=1}^{m+1}\Gamma_k(f_j-s)\prod_{j=1}^{n+1}\Gamma_k(k-e_j+s)}{\prod_{j=n+2}^{p+2}\Gamma_k(k-e_j-s)\prod_{j=m+2}^{q+1}\Gamma_k(k-f_j+s)}$$

$$\times \frac{\Gamma_k(\eta - s)\Gamma_k(k - \omega + s)}{\Gamma_k(k - k + h - \eta + s)\Gamma_k(h - \eta + s)\Gamma_k(h - k + \omega - s)} z^{\frac{s}{k}} ds.$$

Hence, we finally obtain

$$\sum_l \frac{(\frac{1}{k})^l}{\Gamma(h + lk)l!} G^{m+1,n+1}_{k,p+1,q+1} \left[\begin{matrix}\omega - lk, e_p \\ \eta + lk, f_q\end{matrix}\middle| z\right]$$
$$= \Gamma_k(h - \eta + \omega - k) G^{m+1,n+1}_{k,p+2,q+2} \left[\begin{matrix}\omega, e_p, h + \omega - k \\ \eta, f_q, h + \eta + k\end{matrix}\middle| z\right].$$

4. Special Cases and Applications

In this section, some results are given for special functions which can be identified as generalized Meijer G-functions. The results that are derived are useful in numerical computations. Here, we consider generalized hypergeometric functions in summation and then after some mathematical calculations, we derive the results in k-form as follows:

Theorem 5. *For $\Re(f_q - \omega - lk - e_p) > 0$, $k > 0$ the following holds:*

$$\sum_l \frac{\Gamma_k(g + lk)(\frac{1}{k})^l}{l!} \, {}_{p+1}F_{q,k} \left[\begin{matrix}\omega + lk, e_p \\ f_q\end{matrix}\middle| y\right]$$
$$= \frac{(-y)^{\frac{-g}{k}} \Gamma_k(f_q)\Gamma_k(g)\Gamma_k(e_p - g)}{\Gamma_k(e_p)\Gamma_k(f_q - g)} \, {}_{p+1}F_{q,k} \left[\begin{matrix}\omega - g, e_p - g \\ f_q - g\end{matrix}\middle| y\right]. \quad (12)$$

Proof. As

$$G^{1,p+1}_{k,p+1,q+1} \left[\begin{matrix}k - \omega - lk, k - e_p \\ 0, k - f_q\end{matrix}\middle| -y\right] = \frac{\Gamma_k(\omega + lk)\Gamma_k(e_p)}{\Gamma_k(f_q)} \, {}_{p+1}F_{q,k} \left[\begin{matrix}\omega + lk, e_p \\ f_q\end{matrix}\middle| y\right].$$

So

$$\sum_l \frac{\Gamma_k(g + lk)(\frac{1}{k})^l}{l!} \, {}_{p+1}F_{q,k} \left[\begin{matrix}\omega + lk, e_p \\ f_q\end{matrix}\middle| y\right]$$
$$= \Gamma_k(g) \sum_l \frac{(g)_{l,k}\Gamma_k(f_q)(\frac{1}{k})^l}{\Gamma_k(e_p)\Gamma_k(\omega + lk)l!} G^{1,p+1}_{k,p+1,q+1} \left[\begin{matrix}k - \omega - lk, k - e_p \\ 0, k - f_q\end{matrix}\middle| -y\right]$$
$$= \frac{\Gamma_k(f_q)\Gamma_k(g)}{\Gamma_k(e_p)} G^{1,p+1}_{k,p+1,q+1} \left[\begin{matrix}k - \omega, k - e_p \\ -g, k - f_q\end{matrix}\middle| -y\right].$$

Also

$$G^{1,p+1}_{k,p+1,q+1}\left[\begin{matrix}k-\omega, k-e_p\\ -g, k-f_q\end{matrix}\middle| -y\right] = \frac{\Gamma_k(\omega)\Gamma_k(e_p-g)}{\Gamma_k(f_q-g)}(-y)^{\frac{-g}{k}}\ {}_{p+1}F_{q,k}\left[\begin{matrix}\omega-g, e_p-g\\ f_q-g\end{matrix}\middle| y\right].$$

So

$$\sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!}\ {}_{p+1}F_{q,k}\left[\begin{matrix}\omega+lk, e_p\\ f_q\end{matrix}\middle| y\right] = \frac{\Gamma_k(f_q)\Gamma_k(g)\Gamma_k(e_p-g)}{\Gamma_k(e_p)\Gamma_k(f_q-g)}(-y)^{\frac{-g}{k}}\ {}_{p+1}F_{q,k}\left[\begin{matrix}\omega-g, e_p-g\\ f_q-g\end{matrix}\middle| y\right].$$

Remark:
For $q=0$ and $p=0$ or $p=1$, the above equation can be expressed in the k-form of result [35].

We also derive some results by using generalization of Meijer G-function given as

$$\sum_l \frac{\Gamma_k(g+lk)(\frac{1}{k})^l}{l!}\ {}_pF_{q+1,k}\left[\begin{matrix}e_p\\ f_q, \eta-lk\end{matrix}\middle| y\right]$$
$$= \frac{y\prod_{j=1}^p e_j\Gamma_k(g)\Gamma_k(k-\eta)\Gamma_k(k-g)}{k\eta\Gamma_k(k-\eta-g)\prod_{j=1}^q f_q}\ {}_{p+1}F_{q+2,k}\left[\begin{matrix}k+e_p, k-g\\ k+f_q, k+\eta, 2k\end{matrix}\middle| y\right] \quad (13)$$

and

$$\sum_l \frac{\Gamma_k(g+lk)\Gamma_k(e_p+lk)y^l}{l!\Gamma_k(h+lk)\Gamma_k(f_q+lk)}\ {}_pF_{q,k}\left[\begin{matrix}e_p+lk\\ f_q+lk\end{matrix}\middle| y\right] = \frac{\Gamma_k(g)\Gamma_k(e_p)}{\Gamma_k(h)\Gamma_k(f_q)}\ {}_{p+1}F_{q+1,k}\left[\begin{matrix}e_p, h-g\\ f_q, h\end{matrix}\middle| y\right]. \quad (14)$$

Many other similar results can also be derived for some special functions.
If we take $\phi_k(e;g;y)$ as confluent k-hypergeometric function of second kind, then in term of generalization of Meijer G-function we can write it as

$$\phi_k\left[\begin{matrix}e\\ g\end{matrix}\middle| y\right] = \frac{1}{\Gamma_k(e)\Gamma_k(k+e-g)}G^{2,1}_{k,1,2}\left[\begin{matrix}k-e\\ 0, k-g\end{matrix}\middle| y\right]. \quad (15)$$

After doing some calculations on equation (15), we obtain

$$\sum_l \frac{\Gamma_k(e+lk)\Gamma_k(f+lk)}{l!}\ \phi_k\left[\begin{matrix}e+lk\\ g\end{matrix}\middle| y\right] = \Gamma_k(e)\Gamma_k(f)\phi_k\left[\begin{matrix}e\\ f+g\end{matrix}\middle| y\right]. \quad (16)$$

5. Conclusions

In this paper, we look into the summations involving generalization of Meijer G-functions, hypergeometric functions and confluent hypergeometric functions. By applying different techniques and transformations on summations we obtain the variations of said functions. Throughout this paper if we choose $k=1$ then we will get the classical Meijer's G-function, Hypergeometric function and Confluent Hypergeometric functions and the results obtained will take the classical form.

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