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The Stability of the Generalized Functional Equation

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Abstract. The aim of this paper is to prove the stability (in the sense of Ulam) of the functional equation:

$$f(x) = \alpha(x)f(f_1(x)) + \beta(x)f(f_2(x)),$$

where α and β are given real valued functions defined on a nonempty set S such that $\sup\{|\alpha(x)| : x \in S\} < 1$ and $f_i(x)(i = 1, 2)$ are given mappings.

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1. Introduction and preliminaries

The origin of the stability problem for functional equations traces back to a question posed by Ulam [1]. Hyers [2] made remarkable headway in 1941. In the context of Banach spaces, he derived highly renowned and captivating results regarding the Cauchy functional equation. Later, there are also very many generalizations of Cauchy equations (see [3–5]). For more information concerning various functional equations, see [6–16].

Baker was the first to use the fixed point method to study the Hyers-Ulam stability of functional equations in [17]. In that work, he actually utilized the following variant of Banach's fixed point theorem. For the functional equation

$$f(t) = (\alpha(t) + \beta(t))f(\phi(t)),$$

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if

$$\|g(t) - \{\alpha(t) + \beta(t)\}g(\phi(t))\| \leq \delta,$$

then there exists a unique mapping f satisfying this functional equation, and

$$\|f(t) - g(t)\| \leq \frac{\delta}{1-\lambda}.$$

Sikorska [18] built a direct method (see, e.g., [19, 20]) to improve the approximating constant for the following mappings

$$\|f(x) - af(h(x)) - bf(-h(x))\| \leq \epsilon(x),$$
$$\|f(x) - af(h^n(x)) - bf(h^{n+1}(x))\| \leq \epsilon(x).$$

It improves the research results of Sikorska [18], breaking through the limitation that her method could only handle cases where the intermediate function is odd. The paper proposes a new direct method, which is not only used to solve specific functional equations but also to construct and study a series of functional equations. This provides new research tools and ideas for this field. The fixed point theory is a pivotal method for proving the stability of functional equations[21],widely applied in differential equations[22] and computer science[23], this theory transforms abstract solvability problems into concrete analyses of operator properties, remaining indispensable for establishing existence and uniqueness in functional equation frameworks.

Thoroughly, we explore the dependence relationships and properties of different parameters in generalized functional equations. When studying single-variable abstract equations, we need to analyze the influence of parameters such as α and β on the solutions. Also, expand the parameters from the real number field to the complex number field to reveal the characteristics of the equations more comprehensively.

In this paper, we discuss the Hyers-Ulam stability of the following functional equation

$$\|f(x) - \alpha(x)f(g(x)) - \beta(x)f(h(x))\| \leq \epsilon(x)$$

in Banach spaces.

In fact, the above mentioned inequality represents an even more general form. In Section 2, we introduce certain improvements to the existing approximations. Specifically, we enhance the existing approximation methods, aiming to obtain more accurate and comprehensive results. These improvements contribute to a deeper understanding of the problem and provide more effective tools for subsequent research. In Section 3, we explore several applications of the stability results. We not only demonstrate the practical value of the stability theory in different scenarios but also expand its scope of application, showing its potential in solving real-world problems. For the sake of simplicity, we present our results for functions with values in Banach spaces. However, with some minor additional assumptions, these results can be extended and reformulated in more general Banach space settings. This indicates that our research has the potential to be further generalized, making it applicable to a wider range of mathematical models and real world applications.

2. Main results

In this section, we prove the main theorem of the paper.

Theorem 1. *Suppose that X is a linear normed space and Y is a Banach space. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(x) - \alpha(x)f(g(x)) - \beta(x)f(h(x))\| \leq \epsilon(x) \tag{1}$$

for all $x \in X$. If $\alpha(x)$ and $\beta(x)$ are functions from X to $\mathbb{R}$, and $g, h : X \to X$ are mappings and $\epsilon : X \to [0, \infty)$ is a function such that

$$\sum_{n=0}^{\infty} (\Lambda^n \epsilon)(x) =: \epsilon^*(x) < \infty$$

holds and Λ is a linear operator defined by

$$(\Lambda\delta)(x) := |\alpha(x)|\delta(g(x)) + |\beta(x)|\delta(h(x))$$

for a function $\delta : X \to [0, \infty)$ and $x \in X$, then there exists a unique determined mapping $K : X \to Y$, given by

$$K(x) = \alpha(x)K(g(x)) + \beta(x)K(h(x))$$

such that

$$\|f(x) - K(x)\| \leq \epsilon^*(x) \tag{2}$$

for all $x \in X$.

Proof. Let $T : Y^X \to Y^X$ be an operator satisfying $(Tf)(x)) = \alpha(x)f(g(x)) + \beta(x)f(h(x))$ in (1). Then we get

$$\|f(x) - (Tf)(x))\| \leq \epsilon(x). \tag{3}$$

At the same time, we can see that

$$\begin{aligned} &\|(T\xi)(x) - (T\zeta)(x)\| \\ &\leq |\alpha(x)|\|\xi(g(x)) - \zeta(g(x))\| + |\beta(x)|\|\xi(h(x)) - \zeta(h(x))\| \end{aligned} \tag{4}$$

for all $\xi, \zeta \in Y^X$ and $x \in X$.

First, we get by induction that, for all $n \in \mathbb{N}$,

$$\|(T^n f)(x)) - (T^{n+1} f)(x)\| \leq (\Lambda^n \epsilon)(x), x \in X. \tag{5}$$

Obviously, from (3), for the case $n = 0$, (5) holds. Now fix $n \in \mathbb{N}$ and suppose that the inequality (5) is valid. Then, using (4) for all $x \in X$, we have

$$\begin{aligned} &\|(T^{n+1} f)(x) - (T^{n+2} f)(x)\| \\ &\leq |\alpha(x)|\|(T^n f)(g(x)) - (T^{n+1} f)(g(x))\| + |\beta(x)|\|(T^n f)(h(x)) - (T^{n+1} f)(h(x))\| \\ &\leq |\alpha(x)|(\Lambda^n \epsilon)(g(x)) + |\beta(x)|(\Lambda^n \epsilon)(h(x)) = (\Lambda^{n+1} \epsilon)(x). \end{aligned}$$

Thus we complete the proof of (5). For $n, k \in \mathcal{N}, k > 0$,

$$\begin{aligned} &\|(T^n f)(x) - (T^{n+k} f)(x)\| \leq \sum_{i=0}^{k-1} \|(T^n f)(x) - (T^{n+i+1} f)(x)\| \\ &\leq \sum_{i=n}^{n+k-1} (\Lambda^i \epsilon)(x) \leq \epsilon^*(x), x \in X. \end{aligned} \tag{6}$$

From the convergence of the series $\sum(\Lambda^n \epsilon)(x)$, for every $x \in X$, $\{(T^n f)(x)\}_{n\in\mathcal{N}}$ is a Cauchy sequence. Since Y is complete, we can define $\lim_{n\to\infty}(T^n f)(x) := \psi(x)$. Taking $n = 0$ and $k \to \infty$ in (6), we know that (2) holds, and

$$\|(T\psi)(x) - (T^{n+1} f)(x)\| \leq (\Lambda^{n+1})\epsilon(x), n \in \mathcal{N}, x \in X,$$

and thus

$$(T\psi)(x) = \lim_{n\to\infty} (T^{n+1}\psi)(x) = \psi(x), x \in X.$$

In order to prove the uniqueness of ψ, suppose that $\psi_1, \psi_2 \in Y^X$ are two fixed points of T with $\|\psi_i(x) - f(x)\| \leq \epsilon^*(x)$ for all $x \in X, i = 1, 2$. We can easily show that

$$\|\psi_1(x) - \psi_2(x)\| = \|(T^m\psi_1)(x) - (T^m\psi_2)(x)\| \leq 2\sum_{i=m}^{\infty}(\Lambda^i\epsilon)(x), x \in X.$$

Thus $\psi_1(x) = \psi_2(x)$.

Remark 1. *We can easily prove*

$$\left\| f(x) - \sum_{i=0}^{n} \alpha_i(x)(f(f_i(x))) \right\| \leq \epsilon(x)$$

for all $x \in X$. Thus there exists a unique determined mapping $K : X \to Y$, given by

$$K(x) = \sum_{i=0}^{n} \alpha_i(x)(K(f_i(x)), x \in X,$$

such that

$$\|f(x) - K(x)\| \leq \epsilon^*(x)$$

for all $x \in X$.

3. The stability of generalized functional equations in Banach spaces

In the section, we use Theorem 1 to prove the Hyers-Ulam stability of functional inequalities in Banach spaces. The several mathematicians have investigated the problems of Hyers-Ulam stability of functional equations (see [19, 20, 24–35]).

Theorem 2. *Let $\epsilon : X^3 \to [0, \infty)$ be a function with $\epsilon(0,0,0) = 0$ such that there exists an $L < 1$ with*

$$\epsilon(kx, ky, kz) \leq |k| L \epsilon(x, y, z)$$

for all $x, y, z \in X$ and $k \in \mathbb{R}$. Suppose that X is a linear normed space and Y is a Banach space. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &\| f(Ax + By + Cz) + f(Ax) + \gamma(x) f(z) \\ &- \alpha(x) f(Ax + y) - \beta(x) f(x + Cz) - \gamma(x) f(By + z) \| \\ &\leq \epsilon(x, y, z), \end{aligned} \tag{7}$$

for all $x, y, z \in X$, where $\alpha, \beta, \gamma : X \to \mathbb{R}$ are functions and $A, B, C \in \mathbb{R}$ such that $\widetilde{A} := \left|\frac{A+B+C}{A}\right| + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{1}{A}\right| + \left|\alpha\left(\frac{x}{A}\right)\right| \left|\frac{A+1}{A}\right| + \left|\beta\left(\frac{x}{A}\right)\right| \left|\frac{C+1}{A}\right| + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{B+1}{A}\right| < 1$. Then there exists a unique mapping $k : X \to Y$ such that

$$\|f(x) - k(x)\| \leq \frac{1}{1 - \widetilde{A} L} \epsilon(x, x, x)$$

for all $x \in X$.

Proof. Letting $x = y = z$ in (7), we get

$$\begin{aligned} &\| f((A + B + C)x) + f(Ax) + \gamma(x) f(x) \\ &- \alpha(x) f(Ax + x) - \beta(x) f(x + Cx) - \gamma(x) f(Bx + x) \| \\ &\leq \epsilon(x, x, x). \end{aligned}$$

Thus

$$\begin{aligned} &\left\| f\left(\left(\frac{A + B + C}{A}\right) x\right) + f(x) + \gamma\left(\frac{x}{A}\right) f\left(\frac{x}{A}\right) \right. \\ &\left. - \alpha\left(\frac{x}{A}\right) f\left(\frac{(A + 1)x}{A}\right) - \beta\left(\frac{x}{A}\right) f\left(\frac{(C + 1)x}{A}\right) - \gamma\left(\frac{x}{A}\right) f\left(\frac{(B + 1)x}{A}\right) \right\| \\ &\leq \epsilon\left(\frac{x}{A}, \frac{x}{A}, \frac{x}{A}\right) \\ &\leq \left|\frac{1}{A}\right| L \epsilon(x, x, x). \end{aligned}$$

In Remark 1, let $\alpha_1(x) = -\gamma\left(\frac{x}{A}\right), \alpha_2(x) = -1, \alpha_3(x) = \alpha\left(\frac{x}{A}\right), \alpha_4 = \beta\left(\frac{x}{A}\right), \alpha_5(x) = \gamma\left(\frac{x}{A}\right), f_1(x) = \frac{x}{A}, f_2(x) = \frac{A+B+C}{A}, f_3(x) = \frac{(A+1)x}{A}, f_4(x) = \frac{(C+1)x}{A}, f_5(x) = \frac{(B+1)x}{A}$.

Consider $T : Y^X \to Y^X$ given as

$$\begin{aligned} Tf(x) = & -f\left(\left(\frac{A+B+C}{A}\right)x\right) - \gamma\left(\frac{x}{A}\right)f\left(\frac{x}{A}\right) \\ & +\alpha\left(\frac{x}{A}\right)f\left(\frac{(A+1)x}{A}\right) + \beta\left(\frac{x}{A}\right)f\left(\frac{(C+1)x}{A}\right) + \gamma\left(\frac{x}{A}\right)f\left(\frac{(B+1)x}{A}\right) \end{aligned}$$

for all $x \in X, f \in Y^X$.

We can define the operator $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$, given by

$$\begin{aligned} \Lambda f(x) = & \ f\left(\left(\frac{A+B+C}{A}\right)x\right) + \left|\gamma\left(\frac{x}{A}\right)\right| f\left(\frac{x}{A}\right) \\ & + \left|\alpha\left(\frac{x}{A}\right)\right| f\left(\frac{(A+1)x}{A}\right) + \left|\beta\left(\frac{x}{A}\right)\right| f\left(\frac{(C+1)x}{A}\right) + \left|\gamma\left(\frac{x}{A}\right)\right| f\left(\frac{(B+1)x}{A}\right) \end{aligned}$$

for all $x \in X$. In particular,

$$\begin{aligned} \Lambda\epsilon(x,x,x) = & \ \epsilon\left(\left(\frac{A+B+C}{A}\right)x, \left(\frac{A+B+C}{A}\right)x, \left(\frac{A+B+C}{A}\right)x\right) \\ & + \left|\gamma\left(\frac{x}{A}\right)\right| \epsilon\left(\frac{x}{A}, \frac{x}{A}, \frac{x}{A}\right) \\ & + \left|\alpha\left(\frac{x}{A}\right)\right| \epsilon\left(\frac{(A+1)x}{A}, \frac{(A+1)x}{A}, \frac{(A+1)x}{A}\right) \\ & + \left|\beta\left(\frac{x}{A}\right)\right| \epsilon\left(\frac{(C+1)x}{A}, \frac{(C+1)x}{A}, \frac{(C+1)x}{A}\right) \\ & + \left|\gamma\left(\frac{x}{A}\right)\right| \epsilon\left(\frac{(B+1)x}{A}, \frac{(B+1)x}{A}, \frac{(B+1)x}{A}\right) \\ & \leq \left|\frac{A+B+C}{A}\right| L\epsilon\left(x,x,x\right) \\ & + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{1}{A}\right| L\epsilon\left(x,x,x\right) \\ & + \left|\alpha\left(\frac{x}{A}\right)\right| \left|\frac{A+1}{A}\right| L\epsilon\left(x,x,x\right) \\ & + \left|\beta\left(\frac{x}{A}\right)\right| \left|\frac{C+1}{A}\right| L\epsilon\left(x,x,x\right) \\ & + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{B+1}{A}\right| L\epsilon\left(x,x,x\right) \\ & \leq \left(\left|\frac{A+B+C}{A}\right| + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{1}{A}\right| + \left|\alpha\left(\frac{x}{A}\right)\right| \left|\frac{A+1}{A}\right| \right. \\ & \left. + \left|\beta\left(\frac{x}{A}\right)\right| \left|\frac{C+1}{A}\right| + \left|\gamma\left(\frac{x}{A}\right)\right| \left|\frac{B+1}{A}\right|\right) L\epsilon\left(x,x,x\right) \\ & = \widetilde{A}L\epsilon\left(x,x,x\right). \end{aligned}$$

Since Λ is linear, we can get

$$\Lambda^n \epsilon(x,x,x) = (\widetilde{A}L)^n \epsilon(x,x,x), \quad x \in X,\ n \in \mathbb{N}_0.$$

Since $\widetilde{A} < 1$, the series $\sum_{n=0}^{\infty} \Lambda^n \epsilon(x,x,x)$ is convergent for every $x \in X$ and

$$\begin{aligned} \epsilon^*(x,x,x) &= \sum_{n=0}^{\infty} \Lambda^n \epsilon(x) = \sum_{n=0}^{\infty} \left(\widetilde{A}L\right)^n \epsilon(x,x,x) \\ &= \frac{1}{1-\widetilde{A}L}\epsilon(x,x,x), \quad x \in X. \end{aligned}$$

By Theorem 1, there exists a mapping $l : X \to Y$ such that

$$\begin{aligned} & l(x) = \lim_{n\to\infty} T^n f(x), \\ & l(x) = -l\left(\left(\frac{A+B+C}{A}\right)x\right) - \gamma\left(\frac{x}{A}\right) l\left(\frac{x}{A}\right) \\ & +\alpha\left(\frac{x}{A}\right) l\left(\frac{(A+1)x}{A}\right) + \beta\left(\frac{x}{A}\right) l\left(\frac{(C+1)x}{A}\right) + \gamma\left(\frac{x}{A}\right) l\left(\frac{(B+1)x}{A}\right), \\ & \|f(x) - k(x)\| \le \frac{1}{1-\widetilde{A}L}\epsilon(x,x,x). \end{aligned}$$

This completes the proof.

Theorem 3. *Suppose that X is a linear normed space, Y is a Banach space and $\theta \ge 0$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(ax+by) + f(x-y) - Af(x) - Bf(y) - Df(-y)\| \le \theta(\|x\| + \|y\|) \tag{8}$$

for all $x, y \in X$. If a, b, A, B and D satisfy $\frac{|A+B|+|D|}{|a+b|} < 1$ and $|A+B+D-2| \neq 0$, then there exists a unique mapping $k : X \to Y$ such that

$$\|f(x) - k(x)\| \le \frac{2\theta\|x\|}{|a+b| - (|A+B| + |D|)}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (8), we get $f(0) = 0$. Letting $y = x$ in (8), we get

$$\left\| f(x) - (A+B) f\left(\frac{1}{a+b}x\right) - Df\left(-\frac{1}{a+b}x\right) \right\| \le \frac{2\theta\|x\|}{|a+b|} \tag{9}$$

for all $x \in X$.

Consider $T : Y^X \to Y^X$ and $\epsilon : X \to \mathbb{R}_+$,

$$Jf(x) = (A+B) f\left(\frac{1}{a+b}x\right) + Df\left(-\frac{1}{a+b}x\right)$$

for all $x \in X, f \in Y^X$ and

$$\epsilon(x) = \frac{2\theta\|x\|}{|a+b|}, \quad x \in X.$$

The inequality (9) becomes

$$\|Jf(x) - f(x)\| \leq \epsilon(x), \forall x \in X.$$

For every $g, h \in Y^X$ and $x \in X$,

$$\begin{aligned}\|Tg(x) - Th(x)\| = &\left\|(A+B)g\left(\frac{1}{a+b}x\right) + Dg\left(-\frac{1}{a+b}x\right)\right.\\ &\left.-(A+B)h\left(\frac{1}{a+b}x\right) - Dh\left(-\frac{1}{a+b}x\right)\right\|\\ &\leq |A+B|\left\|g\left(\frac{1}{a+b}x\right) - h\left(\frac{1}{a+b}x\right)\right\|\\ &+ |D|\left\|g\left(-\frac{1}{a+b}x\right) - h\left(-\frac{1}{a+b}x\right)\right\|.\end{aligned}$$

Thus we can define $f_1(x) = \frac{1}{a+b}x, f_2(x) = -\frac{1}{a=b}x$, $L_1(x) = |A+B|, L_2(x) = |D|$ and the operator $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$, given by

$$\Lambda\eta(x) = |A+B|\eta\left(\frac{1}{a+b}x\right) + |D|\eta\left(-\frac{1}{a+b}x\right)$$

for all $x \in X$. In particular,

$$\begin{aligned}\Lambda\epsilon(x) &= |A+B|\epsilon\left(\frac{1}{a+b}x\right) + |D|\epsilon\left(-\frac{1}{a+b}x\right)\\ &= |A+B|\theta\frac{2\|x\|}{|a+b|^2} + |D|\theta\frac{2\|x\|}{|a+b|^2}\\ &= \left\{\frac{|A+B|+|D|}{|a+b|}\right\}\epsilon(x).\end{aligned}$$

Since Λ is linear, we can get

$$\Lambda^n\varepsilon(x) = \left\{\frac{|A+B|+|D|}{|a+b|}\right\}^n\epsilon(x), \quad x \in X, \ n \in \mathbb{N}_0.$$

Since $\frac{|A+B|+|D|}{|a+b|} < 1$, the series $\sum_{n=0}^{\infty} \Lambda^n \epsilon(x)$ is convergent for every $x \in X$ and

$$\begin{aligned}\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda^n \epsilon(x) = \sum_{n=0}^{\infty} \left\{ \frac{|A+B|+|D|}{|a+b|} \right\}^n \epsilon(x) \\ &= \frac{1}{1-\frac{|A+B|+|D|}{|a+b|}} \epsilon(x) \\ &= \frac{1}{1-\frac{|A+B|+|D|}{|a+b|}} \frac{2\theta\|x\|}{|a+b|} \\ &= \frac{2\theta\|x\|}{|a+b|-(|A+B|+|D|)}, \quad x \in X.\end{aligned}$$

By Theorem 1, there exists a mapping $k : X \to Y$ such that

$$\begin{aligned}&k(x) = \lim_{n\to\infty} J^n f(x), \\ &k(x) = (A+B)k\left(\frac{1}{a+b}x\right) + Dk\left(-\frac{1}{a+b}x\right), \\ &\|f(x)-k(x)\| \leq \frac{2\theta\|x\|}{|a+b|-(|A+B|+|D|)}.\end{aligned}$$

This completes the proof.

4. The stability of the generalized function inequations

This section primarily investigates the stability of functional equations, improves existing results through a novel direct method, and explores the dependency relationships among different parameters in generalized functional equations along with their related properties. It also solves functional inequalities and constructs as well as studies inequalities related to functional equations. Park [36] introduced additive ρ-functional inequalities and employed the direct method to establish the Hyers-Ulam stability of these inequalities within Banach spaces. Subsequently, in 2016, Choi et al. [37] investigated additive ρ-functional inequalities in normed spaces. Leveraging the fixed-point method, they demonstrated the Hyers-Ulam stability of two distinct additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\|$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \|\rho(f(x+y)-f(x)-f(y))\|$$

(where $\rho < 1$) in normed spaces. In 2023, Nawaz et al. [38] analyzed the Hyers-Ulam stability of cubic and quartic ρ-functional inequalities in fuzzy matrix spaces. They used the fixed-point method to study the following functional inequalities:

$$\begin{cases} (f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ -\rho(4f(x+\frac{y}{2})+4(f(x-\frac{y}{2})-f(x+y)-f(x-y))-6f(x),r)| \geq \frac{r}{r+\varphi(x,y)}, \\ f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \\ -\rho(8f(x+\frac{y}{2})+8(f(x-\frac{y}{2})-2f(x+y)-2f(x-y))-12f(x)+3f(y),r) \geq \frac{r}{r+\varphi(x,y)} \end{cases}$$

where $\rho \neq 2$ is a real number.

Functional equations play a fundamental role in mathematics and its applications, particularly in areas such as information theory, economics, and decision sciences. One important equation is the generalized Drygas equation:

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),$$

whose solution is called a *Drygas mapping*. The general solution of the above functional equation was given by Ebanks, Kannappan and Sahoo [39] as

$$f(x)=Q(x)+A(x),$$

where A is an additive mapping and Q is a quadratic mapping offering insights into the underlying economic behavior[40].The following theorem explores the solutions and stability properties of the generalized Drygas equation, extending its applications to broader mathematical and practical contexts.

Theorem 4. *Let $\phi : X^2 \to [0,\infty)$ be a function with $\phi(0,0)=0$ such that there exists an $L<1$ with*

$$\phi(ax,ay) \leq |a|L\phi(x,y)$$

for all $x,y \in X$. Suppose that X is a linear normed space, Y is a Banach space, and $\rho : X \to (0,1]$ is a function. Let $f : X \to Y$ be an odd mapping satisfying

$$\begin{aligned} &\|f(ax+by)+f(ax-by)-Af(x)-Bf(y)-Df(-y)\| \\ &\leq \|\rho(x)(f(ax+by)-f(ax)-f(by))\|+\phi(x,y) \end{aligned} \tag{10}$$

for all $x,y \in X$. If L, A and a satisfy $\frac{2|a|L}{|A|}<1$, then there exists a unique additive mapping $k : X \to Y$ such that

$$\|f(x)-k(x)\| \leq \frac{|A|}{|A|-2L|a|}\phi(x,0)$$

for all $x \in X$.

Proof. Letting $y=0$ in (10), we get

$$\|f(x)-\frac{2}{A}f(ax)\| \leq \frac{1}{|A|}\phi(x,0), x \in X.$$

Consider the mapping $J : Y^X \to Y^X$ such that

$$J\xi(x) = \frac{2}{A}\xi(ax), x \in X, \xi \in Y^X.$$

Then we get

$$\|Jf(x) - f(x)\| \leq \frac{1}{|A|}\phi(x,0), x \in X.$$

For every $\xi, \mu \in Y^X$,

$$\|Jg(x) - Jh(x)\| = \frac{2}{|A|}\|g(ax) - h(ax)\|.$$

Thus J satisfies the inequality (4) with $f_1(x) = 2ax$ and $L_1(x) = \frac{2}{|A|}$. Next, we define $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ by

$$\Lambda\phi(x,y) = \frac{2}{|A|}\phi(ax,ay) < \frac{2|a|L}{|A|}\phi(x,y), x \in X, \phi \in \mathbb{R}_+^X.$$

Since Λ is linear, we get

$$\Lambda^n\phi(x,y) = \left(\frac{2|a|L}{|A|}\right)^n \phi(x,y)$$

and

$$\varepsilon^*(x,y) = \sum_{n=0}^{\infty}\left(\frac{2|a|L}{|A|}\right)^n \phi(x,y) = \frac{|A|}{|A| - 2L|a|}\phi(x,y).$$

By Theorem 1, there exists a mapping $k : X \to Y$ such that

$$\begin{aligned} k(x) &= \lim_{n\to\infty} J^n f(x), \\ k(x) &= \frac{2}{A}k(ax), \\ \|f(x) - k(x)\| &\leq \frac{|A|}{|A| - 2L|a|}\phi(x,0). \end{aligned}$$

Next, we prove that k satisfies the generalized Drygas equation. Replacing x by $\frac{x}{a}$ and y by $\frac{y}{b}$ in (10), we have

$$\begin{aligned} &\left\|f(x+y) + f(x-y) - Af\left(\frac{x}{a}\right) - Bf\left(\frac{y}{b}\right) - Df\left(-\frac{y}{b}\right)\right\| \\ &\leq \rho(x)\left\|f\left(x+y\right) - f\left(x\right) - f\left(y\right)\right\| + \phi\left(\frac{x}{a}, \frac{y}{b}\right) \end{aligned}$$

for all $x, y \in X$. Then, from the above inequality, we get

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)\|\\&\le \rho(x)\,\|f\,(x+y)-f\,(x)-f\,(y)\|+\phi\left(\frac{x}{a},\frac{y}{b}\right)+\phi\left(\frac{x}{a},0\right)+\phi\left(0,\frac{y}{b}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\|Jf(x+y)+Jf(x-y)-2Jf(x)-Jf(y)-Jf(-y)\|\\&=\frac{2}{|A|}\,\|f\,(a(x+y))+f\,(a(x-y))-2f\,(ax)-f\,(ay)-f\,(-ay)\|\\&\le \frac{2}{|A|}\rho(x)\|f(x+\frac{a}{b}y)-f(x)-f(\frac{a}{b}y)\|+\frac{2}{|A|}\phi(x,y)+\frac{2}{|A|}\phi(x,0)+\frac{2}{|A|}\phi(0,y)\end{aligned}$$

and so

$$\begin{aligned}&\|k(x+y)+k(x-y)-k(x)-k(y)-k(-y)\|\\&=\lim_{n\to\infty}\|J^nf(x+y)+J^nf(x-y)-2J^nf(x)-J^nf(y)-J^nf(-y)\|\\&\le \lim_{n\to\infty}\|J^nf(x+\frac{a}{b}y)-J^nf(x)-J^nf(\frac{a}{b}y)\|\\&+\lim_{n\to\infty}(J^n\phi(x,y)+J^n\phi(x,0)+J^n\phi(0,y))\end{aligned}$$

for all $n \in \mathbb{N}_0$ and $x, y \in X$. Letting $n \to \infty$, we obtain

$$\|k(x+y)+k(x-y)-2k(x)\| \le \rho(x)\|k(x+y)-k(x)-k(y)\| \tag{11}$$

for all $x, y \in X$. Letting $y = x$ in (11), we get

$$\|k(2x)-2k(x)\| \le \rho(x)\|k(2x)-2k(x)\|$$

for all $x \in X$. Moreover, $k(2x) = 2k(x)$ for all $x \in X$.

Letting $l = x + y, m = x - y$ in (11), we get

$$\left\|k(l)+k(m)-2k\left(\frac{l+m}{2}\right)\right\| \le \rho(x)\left\|k(l)-k\left(\frac{l+m}{2}\right)-k\left(\frac{l-m}{2}\right)\right\| \tag{12}$$

for all $l, m \in X$. By (11) and (12),

$$\|k(l)+k(m)-k(l+m)\| \le \frac{1}{2}(\rho(x))^2\|k(l)+k(m)-k(l+m)\|$$

for all $l, m \in X$. Thus $k(l + m) = k(l) + k(m)$ for all $l, m \in X$. So f is additive.

Theorem 5. *Let $\psi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\psi\,(ax, ay) \le a^2 L\psi(x, y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfying

$$\begin{aligned}&\|f(ax+by)+f(ax-by)-Af(x)-Bf(y)-Df(-y)\|\\ &\leq \|\rho(x)(f(ax+by)+f(ax-by)-2f(ax)-2f(by))\|+\psi(x,y)\end{aligned} \tag{13}$$

for all $x, y \in X$. Assume that L,A and a satisfy $\frac{2a^2L}{|A|} < 1$, and $\rho : X \to (0, 1]$ is a function. Then there exists a unique quadratic mapping $H : X \to Y$ such that

$$\|f(x) - H(x)\| \leq \frac{|A|}{|A| - 2La^2}\psi(x, 0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (13), we get

$$\|f(x) - \frac{2}{A}f(ax)\| \leq \frac{1}{|A|}\psi(x, 0), x \in X.$$

Similar to Theorem 4, we can prove that

$$\begin{aligned}&H(x) = \lim_{n\to\infty} J^n f(x),\\ &H(x) = \frac{2}{A}H(ax),\\ &\|f(x) - H(x)\| \leq \frac{|A|}{|A| - 2La^2}\psi(x, 0).\end{aligned}$$

Moreover,

$$\begin{aligned}&\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\|\\ &\leq \|\rho(x)(f(x+y)+f(x-y)-2f(x)-2f(y))\|\\ &+\psi(x/a,y/b)+\psi(x/a,0)+\psi(0,y/b).\end{aligned}$$

Next, we prove $H(x)$ is quadratic.

By the definition of $H(x)$, we have

$$\begin{aligned}&\|H(x+y)+H(x-y)-2H(x)-2H(y)\|\\ &\leq \|\rho(x)(H(x+y)+H(x-y)-2H(x)-2H(y))\|\end{aligned}$$

for all $x, y \in X$. So $H(x+y)+H(x-y) = 2H(x)+2H(y)$ for all $x, y \in X$. This completes the proof.

5. Conclusion

This paper focuses on the generalized Drygas functional equation and deeply explores its Hyers - Ulam stability in the context from a real vector space to a Banach space. A series of key results have been obtained by using the fixed point method. Regarding the stability of the functional equation, under different conditions (such as the specific restrictions on a, b, A, B, D), the close relationship between the mapping satisfying the relevant inequality and the unique generalized Drygas mapping is clearly given, and the error estimate is accurately provided. This not only deepens the theoretical understanding of the properties of this equation but also provides an important methodological reference and theoretical basis for subsequent studies on the stability of such equations. In the study of the functional inequality, it is successfully proven that there exist unique additive and quadratic mappings under specific conditions, and the corresponding error estimates are obtained. These conclusions are of great significance in theoretical research fields such as mathematical analysis and function approximation. They provide new perspectives and methods for function classification and property characterization. In practical applications, such as in the design of optimization algorithms and signal processing, they provide powerful mathematical tools, which can effectively improve algorithm performance and optimize signal processing effects.

The research results of this paper expand the research boundaries of the stability theory of functional equations and open up new directions for the future development of this field. Follow up research can further explore the properties of the generalized Drygas functional equation and inequality in more complex spaces or under different types of operators. At the same time, it is necessary to strengthen the applied research in interdisciplinary fields, fully explore its potential value, and promote the coordinated development of related fields.

Declarations

Conflict of interest

The authors declare that they have no competing interests.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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