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Helical Estimation of Atangana-Baleanu-Caputo Fractionalized Magnetohydrodynamic Second Grade Fluid for Generalized Boundary Conditions under Porous Environment

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Abstract. In the current work, we have estimated the flow of helices of the unsteady fractionalized second grade fluid with MHD effect between uniaxial annular cylinders. The analytical outcomes are evaluated for the rotational and longitudinal velocities and the shear stresses because of fluid circulation and translating between two infinite coaxial circular cylinders, those are turning their axis. Hardly anyone has done this before or has applied the most modern non-integer Atangana Baleanu Caputo fractional time derivatives on the governing equation of second grade fluid with some natural effects. The outcomes are determined by using integral transforms such as Laplace and finite Hankel transforms. The acquired outcomes exhibited in integral and series forms with newly defined special $\mathbf{M}_c^{a,b}(\kappa,t)$ function. The outcome fulfills both the governing equation and all designated account conditions. Additionally, the respective outcome for Newtonian fluid for the same movement is acquired in limited cases. The impact of material parameter α and kinematic viscosity of the α is also discussed. Last, we examined the behavior of distinct parameters on fluid movement along with graphically analogizing second grade and Newtonian fluids.

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1. Introduction

From the last decade of the seventeen century, when Leibniz[1] introduced it as a paradox, also mentioned that it will have applicable outcomes in future. Mathematical analysis branch[2] - fractional calculus has paid a significant contribution in many fields of real life[3]. Ross, B.[4] cited exhaustively the basic ideas and earlier definitions of fractional integro-differentiation of Euler, Laplace, Lacroix, Fourier and of other mathematicians who worked on the topic till 1900. Podlubny *et al.*[5] titled Abel as the father of fractional calculus. Abel introduced the complete conceptions of Riemann Liouville fractional integral and Caputo fractional derivative, which were his extremely interesting discoveries. Due to extraordinary influence in the research, Bertram Ross arranged the first conference on fractional calculus and its applications[6] at the University of New Haven during mid of 1974, and in the same year, Oldham and Spanier were the first who edited the proceedings and published the first monograph on the fractional calculus. Machado *et al.*[2] also presented the detailed analysis of journals, exclusively issues in peer-reviewed journals, books (authored or edited), conferences, separate symposium in conferences, courses, tutorials and plenaries, algorithms / packages, other websites and packages, some papers on computational / numerical procedures for special functions, math keywords and entries in the new MSC2010 and patents on the advancement of fractional calculus since 1974. Caputo and Fabrizio[7] worked on time variables that are applicable for the Laplace transform and on spatial variables that are more convenient with the Fourier transform. Atangana and Baleanu[8] also proposed the latest fractional derivatives, applicable on the fractionalized heat transfer framework. They recommenced the first one on Caputo aspects and the second on Riemann-Liouville perspective a non-local and non-singular kernel[9]. The first one is also famous as Atangana Baleanu Caputo ABC fractional derivative.

Before Atangana Baleanu Caputo ABC fractional differential operator, the previous models were an ordinary model of second grade fluid or fractionalized in the sense of Caputo and Riemann-Liouville with respect to the power-law kernel and Caputo and Fabrizio with respect to the exponential kernel but both have kernel singularity complication because of the constant function does not lead to zero value during differentiation. Subsequently, some scholars investigated and applied them to different models and identified the locality issues of the related kernel, and the derivative could not explain the memory effects. Atangana and Baleanu addressed these issues and succeeded in dealing with the locality issues, they incorporated a non-local and non-singular kernel. Their operator is known as Atangana Baleanu Caputo differential operator, its kernel has stochastic and deterministic properties and is generalized in terms of Mittag-Leffler law[9]. Sania and Atangana[10] worked on the mathematical modeling of the infectious disease dengue, caused by mosquitoes and found that Caputo, Caputo-Fabrizio, and Atangana-Baleanu-Caputo are more effective differential operators than other classical derivatives. Hong-Guang *et al.*[3] collected and reviewed in detail the significance and practice of fractional calculus in the disciplines of physics, dynamical systems, computer science, life science,

NOMENCLATURE **[in SI units]**

Π_1	interior cylinder radius $[m]$	μ	specific gravity
Π_2	exterior cylinder radius $[m]$	ϕ	parameter of porosity
v	translational velocity field $[m/s]$	κ	porous medium permeability
ω	rotational velocity field $[m/s]$	G_p	porosity constant= $\frac{\mu\phi}{k}$
$S_{\rho z}(\rho,t)$	tors. shear stress (time dept.) $[N/s^2]$	ρ	longitudinal component
$S_{\rho\theta}(\rho,t)$	longit. shear stress (time dept.) $[N/s^2]$	α	material / fluid parameter
σ	electrical conductivity of fluid $[S/m]$	β	fractional parameter
Π_0	magnitude of applied magnetic field $[T]$	D_t^β	ABC fractional operator
ϱ	density of fluid $[kg/m^3]$	$\overline{\omega}$	Laplace transform of ω
G_m	magnetic constant= $\frac{\sigma\Pi_o^2}{\varrho}$	$\overline{v}$	Laplace transform of v
η	dynamic viscosity $[Pa\ s]$	w_H	Hankel transform of ω
$\nu=\eta/\varrho$	kinematic viscosity $[m^2/s]$	v_H	Hankel transform of v
$\mathbf{M}_c^{a,b}$	special function $M(\omega,\tau)$		

environmental science, macroeconomic modelings, interdisciplinary materials science, and multidisciplinary engineering. Faraz *et al.*[11] redesigned some models, those are regularly adopted modern day researchers, employ fractional operators to examine an extensive study of preventive steps for COVID-19. Anwar *et al.*[12] provided a more detailed explanation of the technical work introduced by Caputo, Fabrizio, and Atangana on time and spatial variables with real life cases. The most recent notable work of fractional differential operators on fluid dynamics are; Anwar *et al.*[13] analyzed multiparametric fractional operator on the mixture of aluminum and titanium while Asifa *et al.*[14] investigated Prabhakar fractional operator on heat transfer of hybrid nanofluid. Many research scholars are also attracted towards the practical applications of fraction differential operators on magnetohydrodynamics(MHD) and other phenomena[15–18].

Special generalized functions in fractional calculus are growing as one of the essential tools to represent the outcomes. Kiryakova[19] conducted the survey and discussed the special functions in depth like Meijer G-function, generalized hypergeometric functions ${}_pF_q$, Wright generalized hypergeometric functions ${}_q\Psi_p$, Fox H-functions, Mittag-Leffler type functions in fractional calculus. Mathai *et al.*[20] in his publication, dedicated an exclusive section on fractional calculus in which he connected fraction differential and fractional integral operators to H-function and examined thoroughly all famous fractional operators. Qureshi[21] investigated the Mass-Spring-Damper mechanical engineering model through the Caputo fractional operator, earned the exact solution and connected it with the Fox H-function. Ali *et al.*[22] contemplated Casson fluid in generalized form with heat transfer, applied Caputo fractional derivative for mathematical modeling, and established the result in the Wright function $\Phi(a,\alpha,\tau)$. Sharma and Jain[23] explained the generalized M-series ${}_{p}^{\alpha}\mathtt{M}_{q}^{\beta}$, its relations with Wright generalized hypergeometric function, Fox H-function, and generalized Mittag-Leffler function, and link to fractional integrals and derivatives. Jamil *et al.*[24] worked on fractionalized MHD Maxwell fluid and show their outcomes in newly defined M-function M_p^q and its relationship with Wright function ${}_q\Psi_p$ is as $M_q^p(z) = t^{b_{q-1}}{}_q\Psi_p(z)$. Their newly prescribed M-function is the simpler form of the Wright function.

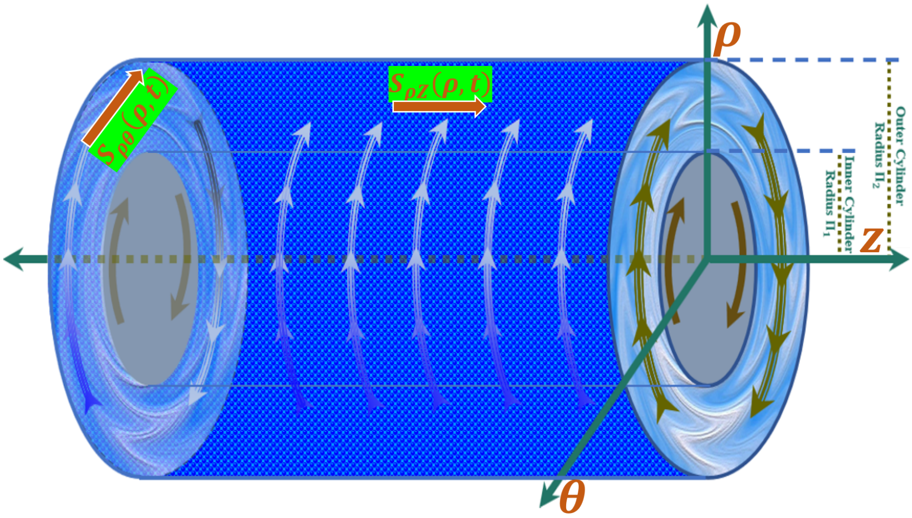

Figure 1: Physical structure of the discussion: Helical motion of fluid in the coaxial circular cylinders

The helical flow of non-Newtonian fluid in the cylindrical zone is utilized regularly in experimental and theoretical research[25, 26]. Bayat *et al.*[27] worked on the human eye and examined the partial vitreous liquefaction flow. In the article, he also explained exclusively the behavior of Newtonian and non-Newtonian fluids by the Weissenberg effect and showed the simulation and its numerical result, such flow produced helices but in opposite directions. Alharbi *et al.*[28] investigated the properties of helically pressure-induced flow in coaxial cylinders, also known as Poiseuille flow, of the Bingham fluids. Bingham fluid is sometimes referred to as concrete in civil engineering and mud during drilling in the ground. Worku *et al.*[29] developed the multi-linear regression modeling of pharmaceutical power having ingredients in bulk quantity which is used to manufacture the solid dosage in the form of tablets and capsules in a cylinder with the rotating helical blade for mixing. Jamil *et al.*[30] and Kamran *et al.*[31] separately considered the helical flow of Maxwell fluid in two coaxial cylindrical region, after employing the integral transforms, they presented their result of shear stress in the more generalization of $M_{\alpha,\beta}^{y}$, and $G_{a,b,c}(.,.)$ functions respectively. More recently, Javaid *et al.*[32] studied the flow of fractionalized Burgers' fluid, after working through some integral transform and modified

Bessel equations, they earned their results of velocity field and shear stress and for validation and comparison of results, they adopted the Gaver-Stehfest s algorithm and Tzou s algorithm.

The influence of two more phenomena are dominant in the literature of fluid dynamics, those are magnetic and cellular (porous) effects. Jamil and Zafarullah[33] wrote some applications of magnetohydrodynamics (MHD) motion of second grade non-Newtonian fluid and employed some integral transforms, demonstrating the result in convolution product form of inverse Laplace transformation and endorsed by graphical behavior. Jamil *et al.*[24] extended their earlier study by taking fractionalized magnetohydrodynamics of Maxwell fluid in the cellular cylinder and validating the result by visual presentation. In a cellular or porous environment, there are many micro cells / pores that can absorb something on the surface of an object. Many examples of porosity are available in modern literature. Liu and Chen[34] characterized porous materials either naturally such as sandstone, human lung and bones, eggshell and limbs of birds, plant leaves and wood, and some marine invertebrates, or synthetically such as porous ceramics, polymer foams, tissue, and absorbent papers, fabric, filtration devices. Cai *et al.*[35] reviewed the vision 2020 of InterPore. Mochalin *et al.*[36] worked numerically on fluid flow through filtrating round hollows passing in spinning porous (cellular) cylinders by applying computational fluid dynamics techniques.

In this article, we estimate the spiral and linear flow rate of a second grade unsteady fractionalized fluid with MHD phenomenon between two uniaxial cellular cylinders. The analytical results are evaluated for the rotational and longitudinal velocities and the shear stress because of fluid rotation and transformation between two infinite coaxial round porous / cellular cylinders, turning their axes. The outcomes are determined employing the suitable integral transformations, that are effective in circular cylinders, such as Laplace transformation and finite Hankel transformation, by one of the most recent and practicable fractional or non-integer Atangana Baleanu Caputo (ABC) time fractional derivatives. ABC differential operator is popularized and attracted in the last few years by renowned researchers. The acquired outcomes are exhibited in integral and series forms with the convolution product of Laplace inverse transformation and newly prescribed special generalized $\mathbf{M}_c^{a,b}(\kappa, t)$ function which is more simplified pattern of the Wright function. The result meets basic equality and all accounting requirements such as initial and generalized boundary conditions. Only a few researchers hardly ever used such generalized boundary conditions that is why they rarely appeared in research. Furthermore, Additionally, the respective outcome for Newtonian fluid for the same movement is acquired in limited cases. The impact of alpha and kinematic viscosity of the material parameter is also discussed. Last, we examined the behavior of distinct parameters on fluid movement along with graphically analogizing second grade and Newtonian fluids.

2. Advancement in mathematical modeling and its governing equations

The governing equations of second grade fluid flows between two cylinders together with heat and mass transfer effect are:

$$\frac{\partial}{\partial t}w(\rho,t)=\left(\nu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\right)w(\rho,t)-G_m w(\rho,t)-G_p\left(\nu+\alpha\frac{\partial}{\partial t}\right)w(\rho,t);$$

$$\rho\in(\Pi_1,\Pi_2),\quad t>0, \tag{1}$$

$$\frac{\partial}{\partial t}v(\rho,t)=\left(\nu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)v(\rho,t)-G_m v(\rho,t)-G_p\left(\nu+\alpha\frac{\partial}{\partial t}\right)v(\rho,t);$$

$$\rho\in(\Pi_1,\Pi_2),\quad t>0, \tag{2}$$

$$S_{\rho\theta}=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)w(\rho,t), \tag{3}$$

$$S_{\rho z}=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial\rho}v(\rho,t), \tag{4}$$

Here for the fluid, constant density is ϱ, kinematic viscosity is $\nu=\mu/\varrho$ and $\alpha=\alpha_1/\varrho$. Also shear stress are $\tau(w)=S_{\rho\theta}$ and $\tau(v)=S_{\rho z}$.

Now introducing the ABC fractionalized derivatives on the governing equations of an incompressible second grade fluid,

$$\frac{\partial w(\rho,t)}{\partial t}=\left(\nu+\alpha^{\beta}D_t^{\beta}\right)\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\right)w(\rho,t)-G_m w(\rho,t)-G_p\left(\nu+\alpha^{\beta}D_t^{\beta}\right)w(\rho,t); \tag{5}$$
$$\rho\in(\Pi_1,\Pi_2),\quad t>0,$$

$$\frac{\partial v(\rho,t)}{\partial t} = \left(\nu + \alpha^{\beta} D_t^{\beta}\right)\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right) v(\rho,t) - G_m v(\rho,t) - G_p\left(\nu + \alpha^{\beta} D_t^{\beta}\right) v(\rho,t); \quad (6)$$
$$\rho \in (\Pi_1,\Pi_2), \quad t>0,$$

$$S_{\rho\theta}(\rho,t) = \left(\mu + \alpha^{\beta} D_t^{\beta}\right)\left(\frac{\partial}{\partial \rho} - \frac{1}{\rho}\right) w(\rho,t), \quad (7)$$

$$S_{\rho z}(\rho,t) = \left(\mu + \alpha^{\beta} D_t^{\beta}\right)\frac{\partial}{\partial \rho} v(\rho,t), \quad (8)$$

where the α is the second grade parameter and G_m and G_p are magnetic and porous effect dimensionless number. The fractional differential operator D_t^{β} called Atangana Baleanu fractional operator, where β are fractional parameters[8].

With initial condition, the above formula can apply to a real-world problem and will also be very useful for physical problem when employ the Laplace transform. For $\beta \to 1$ the non integers second grade fluid model condenses to the ordinary second grade model.

3. Development of the problem

By consideration fractionalized MHD second grade fluid that is incompressible, initially at rest $t=0$ in annular region of two Π_1 and Π_2 radii circular cylinders as presented in Fig. 1. At $t=0^+$ the cylinders start to rotate or translate around their axis ($\rho = 0$) with the rotational velocity $U_1\,H(t)\,g_1(t)$, $U_2 H(t) g_2(t)$ and the translational velocity $V_1\,H(t)\,g_3(t)$, $V_2\,H(t)\,g_4(t)$, where $g_1(t), g_2(t), g_3(t)$ and $g_4(t)$ are any general functions with the property that they are differentiable and integrable and satisfy $g_1(0) = g_2(0) = g_3(0) = g_4(0) = 0$. The velocity field Δ is of the form

$$\Delta = \Delta(\rho,t) = w(\rho,t)\mathbf{e}_\theta + v(\rho,t)\mathbf{e}_z \quad (9)$$

where $\mathbf{e}_\theta$, $\mathbf{e}_z$ are unit vectors in transverse and z-directions. The appropriate governing equations are given by Eqs. (5 - 8). While the suitable initial and boundary conditions are its velocity being of the form

$$w(\rho,0) = v(\rho,0) = 0; \qquad \rho \in (\Pi_1,\Pi_2), \quad (10)$$

$$S_{\rho\theta}(\rho,0) = S_{\rho z}(\rho,0) = 0; \qquad \rho \in (\Pi_1,\Pi_2), \quad (11)$$

respectively,

$$w(\Pi_1, t) = U_1 H(t) g_1(t), \qquad w(\Pi_2, t) = U_2 H(t) g_2(t), \tag{12}$$

$$v(\Pi_1, t) = V_1 H(t) g_3(t), \qquad v(\Pi_2, t) = V_2 H(t) g_4(t), \quad t, \rho \geq 0, \tag{13}$$

where $H(t)$ is denoted as the Heaviside function and U_1, U_2, V_1, V_2 represents constants.

Usually the boundary conditions of fluid dynamics those are taken in the literature are constants t, t_2, t_n, e^{at}, $sin(\omega t)$, $cos(\omega t)$ etc. Only few papers of the fractionalized governing equations with Caputo fractional operator those are less difficult of the present governing equations, are available in the literature with the above boundary conditions and famous generalized functions like Mittag-Leffler $E_{\alpha,\beta}(z)$, $E^{\gamma}_{\alpha,\beta}(z)$, Lorengo $G_{a,b,c}(d,t), R_{a,b}(c,t)$ functions[30, 36–38]. In order to make our problem more genuine and original work, not only we include the magnetic and porosity terms in governing equations but we consider the helical flow of the problem. Furthermore, instead of solving the problem by taking some specific functions on the boundary conditions given above and to facilitate the engineers and applied scientists, we take the generalized functions on the boundary of the cylinders $g_1(t)$, $g_2(t)$, $g_3(t)$ and $g_4(t)$. Employing such type of generalized boundary conditions is not new in the literature, for this we mention here some recent contributions[39, 40].

4. Computational Work

4.1. For velocity field

Recall the initial condition Eq. (10), employ Laplace transform formula for fractional derivative to Eqs. (5) and (6) and boundary condition Eqs.(12) and (13). We find that

$$s\bar{w}(\rho, s) = \left(\nu + \alpha^{\beta} a_0 \frac{s^{\beta}}{s^{\beta} + a_1}\right)\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\right)\bar{w}(\rho, s) - G_m \bar{w}(\rho, s)$$

$$-G_p\left(\nu + \alpha^{\beta} a_0 \frac{s^{\beta}}{s^{\beta} + a_1}\right)\bar{w}(\rho, s); \qquad \rho \in (\Pi_1, \Pi_2), \tag{14}$$

$$s\bar{v}(\rho, s) = \left(\nu + \alpha^{\beta} a_0 \frac{s^{\beta}}{s^{\beta} + a_1}\right)\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)\bar{v}(\rho, s) - G_m \bar{v}(\rho, s)$$

$$-G_p\left(\nu+\alpha^{\beta}a_0\frac{s^{\beta}}{s^{\beta}+a_1}\right)\overline{v}(\rho,s); \qquad \rho\in(\Pi_1,\Pi_2). \tag{15}$$

where the Laplace transform of the ABC fractional derivative D_t^{β} is

$$L(D_t^{\beta}\Psi(\eta,t))=\frac{a_{\circ}s^{\beta}L(\Psi(\eta,t))-s^{\beta-1}\Psi(\eta,0)}{s^{\beta}+a_1},\ \text{ where } a_{\circ}=\frac{1}{1-\beta},\ a_1=\frac{\beta}{1-\beta}.$$

To achieve the conditions, $\bar{w}(\rho,s)$ and $\bar{v}(\rho,s)$ are image functions of $w(\rho,t)$ and $v(\rho,t)$.

$$\bar{w}(\Pi_1,s)=U_1G_1(s), \qquad \bar{w}(\Pi_2,s)=U_2G_2(s), \tag{16}$$

$$\bar{v}(\Pi_1,s)=V_1G_3(s), \qquad \bar{v}(\Pi_2,s)=V_2G_4(s), \tag{17}$$

and we denote the Hankel transformation of $\bar{w}(\rho,s)$ and $\bar{v}(\rho,s)$ in the following:

$$\bar{w}_H(\rho_n,s)=\int_{\Pi_1}^{\Pi_2}\rho\bar{w}(\rho,s)D_1(\rho,\rho_n)d\rho, \tag{18}$$

and

$$\bar{v}_H(\rho_m,s)=\int_{\Pi_1}^{\Pi_2}\rho\bar{v}(\rho,s)D_0(\rho,\rho_m)d\rho, \tag{19}$$

where

$$D_1(\rho,\rho_n)=J_1(\rho\rho_n)Y_1(\Pi_2\rho_n)-J_1(\Pi_2\rho_n)Y_1(\rho\rho_n), \tag{20}$$

$$D_0(\rho,\rho_m)=J_0(\rho\rho_m)Y_0(\Pi_2\rho_m)-J_0(\Pi_2\rho_m)Y_0(\rho\rho_m). \tag{21}$$

The transcendental equation $D_i(\Pi_1,\rho), i=1,2$ has ρ_n and ρ_m positive roots. In the above relations, Bessel functions $J_p(.)$and $Y_p(.)$ are the first and second kind of order p respectively. $\rho D_1(\rho,\rho_n)$ and $\rho D_0(\rho,\rho_m)$ multiply both sides to Eqs. (14) and (15) respectively and think about the conditions of Eq.(16), integrating with respect to ρ from Π_1 to Π_2

$$\int_{\Pi_1}^{\Pi_2} \rho \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \right) \bar{w}(\rho, s) D_1(\rho, \rho_n) d\rho$$
$$= -\rho_n^2 \bar{w}_H(\rho_n, s) \quad + \tfrac{2}{\pi}\left[\tfrac{\bar{w}(\Pi_2) J_1(\Pi_1\rho_n) - \bar{w}(\Pi_1) J_1(\Pi_2\rho_n)}{J_1(\Pi_1\rho_n)} \right], \quad (22)$$

$$\int_{\Pi_1}^{\Pi_2} \rho \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} \right) \bar{v}(\rho, s) D_0(\rho, \rho_m) d\rho$$
$$= -\rho_m^2 \bar{w}_H(\rho_m, s) \quad + \tfrac{2}{\pi}\left[\tfrac{\bar{v}(\Pi_2) J_0(\Pi_1\rho_m) - \bar{v}(\Pi_1) J_0(\Pi_2\rho_m)}{J_0(\Pi_1\rho_m)} \right]. \quad (23)$$

Now, here we applying the finite Hankel transform.

$$\bar{w}_H$$
$$= \tfrac{2}{\pi}\left[\tfrac{U_2 G_2(s) J_1(\Pi_1\rho_n) - U_1 G_1(s) J_1(\Pi_2\rho_n)}{J_1(\Pi_1\rho_n)} \right] \frac{\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1}}{(\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1})\rho_n^2 + G_m + G_p(\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1}) + s}, \quad (24)$$

$$\bar{v}_H$$
$$= \tfrac{2}{\pi}\left[\tfrac{V_2 G_4(s) J_0(\Pi_1\rho_m) - V_1 G_3(s) J_0(\Pi_2\rho_m)}{J_0(\Pi_1\rho_m)} \right] \frac{\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1}}{(\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1})\rho_m^2 + G_m + G_p(\nu + \alpha^\beta a_0 \frac{s^\beta}{s^\beta + a_1}) + s}. \quad (25)$$

For working $\bar{w}(\rho, s)$ and $\bar{v}(\rho, s)$ employ the inverse Hankel Transform.Hower ever, present a result in more suitable form,we firstly rewrite the Eqs. (24) and (25) in the following equivalent form as follows.

$$\overline{w}_H = \frac{2}{\pi}\frac{1}{\rho_n^2}(1 - \xi_n)\left[\frac{U_2 G_2(s) J_1(\Pi_1\rho_n) - U_1 G_1(s) J_1(\Pi_2\rho_n)}{J_1(\Pi_1\rho_n)} \right] \quad (26)$$

$$\overline{v}_H = \frac{2}{\pi}\frac{1}{\rho_m^2}(1 - \xi_m)\left[\frac{V_2 G_4(s) J_0(\Pi_1\rho_m) - V_1 G_3(s) J_0(\Pi_2\rho_m)}{J_0(\Pi_1\rho_m)} \right], \quad (27)$$

where,

$$\xi_n = \frac{(\nu(s^\beta + a_1) + \alpha^\beta a_0 s^\beta)G_p + (s^\beta + a_1)(G_m + s)}{(\nu(s^\beta + a_1) + \alpha^\beta a_0 s^\beta)(\rho_n^2 + G_p) + (s^\beta + a_1)(G_m + s)},$$

and

$$\xi_m = \frac{(\nu(s^\beta + a_1) + \alpha^\beta a_0 s^\beta)G_p + (s^\beta + a_1)(G_m + s)}{(\nu(s^\beta + a_1) + \alpha^\beta a_0 s^\beta)(\rho_m^2 + G_p) + (s^\beta + a_1)(G_m + s)},$$

and apply the inverse Hankel transform formula

$$\overline{w}(\rho, s) = \frac{\pi^2}{2}\sum_{n=1}^{\infty} \frac{\rho_n^2 J_1^2(\Pi_1\rho_n) D_1(\rho, \rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)} \overline{w}_H(\rho, s), \tag{28}$$

$$\overline{v}(\rho, s) = \frac{\pi^2}{2}\sum_{m=1}^{\infty} \frac{\rho_n^2 J_0^2(\Pi_1\rho_m) D_0(\rho, \rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)} \overline{v}_H(\rho, s), \tag{29}$$

we get $\overline{w}(\rho, s)$ and $\overline{v}(\rho, s)$ in the following equivalent form.

$$\overline{w}(\rho, s) = \frac{U_1 G_1(s)\Pi_1(\Pi_2^2 - \rho^2) + U_2 G_2(s)\Pi_2(\rho^2 - \Pi_1^2)}{(\Pi_2^2 - \Pi_1^2)\rho} - \pi \sum_{n=1}^{\infty} (1 - \xi_n) \frac{J_1(\Pi_1\rho_n) D_1(\rho, \rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times \left[U_2 G_2(s) J_1(\Pi_1\rho_n) - U_1 G_1(s) J_1(\Pi_2\rho_n) \right] \tag{30}$$

$$\bar{v}(\rho, s) = \frac{V_1 G_3(s) \ln(\Pi_2/\rho) + V_2 G_4(s) \ln(\rho/\Pi_1)}{\ln(\Pi_2/\Pi_1)} - \pi \sum_{m=1}^{\infty} (1 - \xi_m) \frac{J_0(\Pi_1\rho_m) D_0(\rho, \rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \left[V_2 G_4(s) J_0(\Pi_1\rho_m) - V_1 G_3(s) J_0(\Pi_2\rho_m) \right] \tag{31}$$

The equivalent forms of the factor of Eqs. (30) and (31) are

$$(1-\xi_n)=\sum_{i=0}^{\infty}(-\rho_n^2)^i\left[\frac{\nu(s^\beta+a_1)+\alpha^\beta a_0 s^\beta}{(\nu(s^\beta+a_1)+\alpha^\beta a_0 s^\beta)G_p+(s^\beta+a_1)(G_m+s)}\right]^i,$$

$$=\sum_{i=0}^{\infty}\frac{(-\rho_n^2)^i}{(\nu s^\beta G_p+\alpha^\beta a_0 s^\beta G_p+G_m s^\beta+s^{\beta+1})^i}\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\nu^{i-j}(\alpha^\beta a_0 s^\beta)^j$$

$$\times\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k s^{\beta(i-j-k)}\left[1+\frac{\nu a_1 G_p+a_1 G_m+a_1 s}{\nu s^\beta G_p+\alpha^\beta a_0 s^\beta G_p+G_m s^\beta+s^{\beta+1}}\right]^{-i},$$

$$=\sum_{i=0}^{\infty}(-\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\nu^{i-j}(\alpha^\beta a_0 s^\beta)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k s^{\beta(i-j-k)}\sum_{l=0}^{\infty}(-1)^l$$

$$\times\frac{(\nu a_1 G_p+a_1 G_m+a_1 s)^l}{(\nu s^\beta G_p+\alpha^\beta a_0 s^\beta G_p+G_m s^\beta+s^{\beta+1})^{l+i}},$$

$$=\sum_{i=0}^{\infty}(-\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\nu^{i-j}(\alpha^\beta a_0)^j s^{\beta j}\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k s^{\beta(i-j-k)}\sum_{l=0}^{\infty}(-a_1)^l$$

$$\times\sum_{h=0}^{l}\frac{l!}{l!(l-h)!}(\nu G_p+G_m)^h s^{l-h}\frac{1}{s^{\beta(l+i)}(s+\nu G_p+\alpha^\beta a_0 G_p+G_m)^{l+i}},$$

$$=\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l$$

$$\times\sum_{h=0}^{l}\frac{l!}{l!(l-h)!}(\nu G_p+G_m)^h\frac{s^{-\beta(k+l)+l-h}}{(s+\nu G_p+\alpha^\beta a_0 G_p+G_m)^{l+i}}. \tag{32}$$

For the functions $\overline{w}(\rho, s)$ and $\overline{v}(\rho, s)$, we possess the inverse Laplace transformation by using the following formula[41].

$$\mathcal{L}^{-1}\left[\frac{s^c}{(s^a-\kappa)^b}\right] = \mathbf{M}_c^{a,b}(\kappa, t); \qquad \mathrm{Re}\,(ac-b) > 0, \quad \mathrm{Re}(s) > 0, \quad \left|\frac{d}{s^a}\right| < 1, \tag{33}$$

where the generalized $\mathbf{M}_c^{a,j}(.,t)$ function is defined by[41]

$$\mathbf{M}_c^{a,b}(\kappa, t) = \sum_{j=0}^{\infty} \frac{\kappa^j \Gamma(c+j) t^{(c+j)a-b-1}}{\Gamma(c)\Gamma(j+1)\Gamma[(c+j)a-b]}. \tag{34}$$

Finally, we employ Laplace inverse transform $\mathcal{L}^{-1}\{\overline{w}(\rho, s)\}$ and $\mathcal{L}^{-1}\{\overline{v}(\rho, s)\}$ which are $w(\rho, t)$ and $v(\rho, t)$ respectively then using the convolution theorem to Eqs. (30) and (31) with

$$\Psi = \sum_{j=0}^{i} \frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j \sum_{k=0}^{\infty} \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^{\infty} (-a_1)^l \sum_{h=0}^{l} \frac{l!}{h!(l-h)!}(\nu G_p + G_m)^h$$

and

$$\Omega = \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p + \alpha^\beta a_0 G_p + G_m), \delta)$$

the resultant expressions of the velocity field are

$$w = \frac{H(t)}{(\Pi_2^2-\Pi_1^2)\rho}\left[U_1 g_1(t)\Pi_1(\Pi_2^2-\rho^2) + U_2 g_2(t)\Pi_2(\rho^2-\Pi_1^2)\right] - \pi H(t)\sum_{n=1}^{\infty} \frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \;\Psi \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n)g_1(t-\delta)]\Omega \; d\delta, \tag{35}$$

$$v = \frac{H(t)}{\ln(\Pi_2/\Pi_1)}\left[V_1 g_3(t)\ln(\Pi_2/\rho) + V_2 g_4(t)\ln(\rho/\Pi_1)\right] - \pi H(t)\sum_{n=1}^{\infty} \frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_m^2)^i \;\Psi \int_0^t [V_2 J_0(\Pi_1\rho_m)g_4(t-\delta) - V_2 J_0(\Pi_2\rho_m)g_3(t-\delta)]\Omega \; d\delta. \tag{36}$$

4.2. For shear stresses

To employ Laplace transform to Eqs. (7) and (8)

$$\bar{S_{\rho\theta}}(\rho, s) = \left(\mu + \alpha^{\beta} a_0 \frac{s^{\beta}}{s^{\beta} + a_1}\right)\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\bar{w}(\rho, s), \tag{37}$$

$$\bar{S_{\rho z}}(\rho, s) = \left(\mu + \alpha^{\beta} a_0 \frac{s^{\beta}}{s^{\beta} + a_1}\right)\frac{\partial}{\partial\rho}\bar{v}(\rho, s), \tag{38}$$

and using the fact that

$$\frac{\partial}{\partial\rho}D_1(\rho, \rho_n) = \rho_n \overline{D}_1(\rho, \rho_n) - \frac{1}{\rho}D_1(\rho, \rho_n) \text{ and } \frac{\partial}{\partial\rho}D_0(\rho, \rho_m) = -\rho_m \overline{D}_0(\rho, \rho_m)$$

where

$$\overline{D}_1(\rho, \rho_n) = J_0(\rho\rho_n)Y_1(\Pi_2\rho_n) - J_1(\Pi_2\rho_n)Y_0(\rho\rho_n),$$

$$\overline{D}_0(\rho, \rho_m) = J_1(\rho\rho_m)Y_0(\Pi_2\rho_m) - J_0(\Pi_2\rho_m)Y_1(\rho\rho_m),$$

and if

$$\widetilde{D}_1(\rho, \rho_n) = \frac{2}{\rho}D_1(\rho, \rho_n) - \rho_n\overline{D}_1(\rho, \rho_n), \;\; \widetilde{D}_0(\rho, \rho_m) = -\rho_m\overline{D}_0(\rho, \rho_m),$$

then, we have

$$\frac{\partial\bar{w}(\rho, s)}{\partial\rho} - \frac{1}{\rho}\bar{w}(\rho, s) = \frac{2U_2G_2(s)\Pi_2\Pi_1^2 - 2U_1G_1(s)\Pi_1\Pi_2^2}{(\Pi_2^2 - \Pi_1^2)\rho^2}$$

$$+\pi\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho, \rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}[U_2G_2(s)J_1(\Pi_1\rho_n) - U_1G_1(s)J_2(\Pi_2\rho_n)]\xi_n \tag{39}$$

$$\frac{\partial\bar{v}(\rho, s)}{\partial\rho} = \frac{V_2g_4(s) - V_1g_3(s)}{\rho\ln(\Pi_2/\Pi_1)}$$

$$+\pi\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}\Big[V_2G_4(s)J_0(\Pi_1\rho_m)-V_1G_3(s)J_0(\Pi_2\rho_m)\Big]\xi_m \tag{40}$$

are obtained from Eqs.(26) and (27) after applying inverse Hankel transformation. Introducing Eqs.(39) and (40) into Eqs. (37) and (38) consequently, employing the inverse Laplace transform then convolution theorem, we gained the shear stress $S_{\rho\theta}(\rho,t)$ and $S_{\rho z}(\rho,t)$ with

$$\Upsilon=\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^{\beta}a_0s^{\beta}}{\nu}\right)^{j}\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}\times(\nu\alpha_1G_p+\alpha_1G_m)^h$$

and

$$\mathfrak{M}=\left(\mu+\alpha^{\beta}a_0\right)\mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p+\alpha^{\beta}a_oG_p+G_m),\delta)$$
$$+\mu a_1\mathbf{M}^{1,l+i}_{-\beta(1+k+l)+l-h}\quad(-(\nu G_p+\alpha^{\beta}a_oG_p+G_m),\delta)$$

then;

$$S_{\rho\theta}(\rho,t)=\frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}[U_2g_2(t)\Pi_1-U_1g_1(t)\Pi_2][\mu+\alpha^{\beta}a_o]-\frac{\alpha^{\beta}a_0}{(\Pi_2^2-\Pi_1^2)\rho^2}$$

$$\times\int_0^t[U_2\Pi_1g_2(t-\delta)-U_1\Pi_2g_1(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta)d\delta+\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\ \Upsilon\ \int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\ \mathfrak{M}\ d\delta, \tag{41}$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)](\mu+\alpha^{\beta}a_o)-\frac{H(t)\alpha^{\beta}a_0}{\rho\ln(\Pi_2/\Pi_1)}$$

$$\times\int_0^t[V_2g_4(t-\delta)-V_1g_3(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta)d\delta+\pi H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty} (-\nu\rho_n^2)^i \ \Upsilon \ \int_0^t [V_2 J_0(\Pi_1\rho_m) g_4(t-\delta) - V_1 J_0(\Pi_2\rho_m) g_3(t-\delta)] \ \mathfrak{M} \ d\delta. \tag{42}$$

5. Important Limiting cases

5.1. Ordinary Second grade fluid ($\beta \to 1$)

By assuming $\beta \to 1$ in Eqs. (35), (36), (41) and (42), we obtained the outcomes for ordinary second grade fluid.

$$w = \frac{H(t)}{(\Pi_2^2 - \Pi_1^2)\rho}\Big[U_1 g_1(t)\Pi_1(\Pi_2^2 - \rho^2) + U_2 g_2(t)\Pi_2(\rho^2 - \Pi_1^2)\Big] - \pi H(t) \sum_{n=1}^{\infty} \frac{J_1(\Pi_1\rho_n) D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times \sum_{i=0}^{\infty} (-\nu\rho_n^2)^i \sum_{j=0}^{i} \frac{i!}{j!(i-j)!}\left(\frac{\alpha a_0}{\nu}\right)^j \sum_{k=0}^{\infty} \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^{\infty} (-a_1)^l \sum_{h=0}^{l} \frac{l!}{h!(l-h)!} (\nu G_p + G_m)^h$$

$$\times \int_0^t [U_2 J_1(\Pi_1\rho_n) g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n) g_1(t-\delta)] \mathbf{M}^{1,l+i}_{-(k+l)+l-h}(-(\nu G_p + \alpha a_0 G_p + G_m), \delta) d\delta, \tag{43}$$

$$v = \frac{H(t)}{\ln(\Pi_2/\Pi_1)}\Big[V_1 g_3(t)\ln(\Pi_2/\rho) + V_2 g_4(t)\ln(\rho/\Pi_1)\Big] - \pi H(t) \sum_{n=1}^{\infty} \frac{J_0(\Pi_1\rho_m) D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty} (-\nu\rho_m^2)^i \sum_{j=0}^{i} \frac{i!}{j!(i-j)!}\left(\frac{\alpha a_0}{\nu}\right)^j \sum_{k=0}^{\infty} \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^{\infty} (-a_1)^l \sum_{h=0}^{l} \frac{l!}{h!(l-h)!} (\nu G_p + G_m)^h$$

$$\times \int_0^t [V_2 J_0(\Pi_1\rho_m) g_4(t-\delta) - V_2 J_0(\Pi_2\rho_m) g_3(t-\delta)] \mathbf{M}^{1,l+i}_{-(k+l)+l-h}(-(\nu G_p + \alpha a_0 G_p + G_m), \delta) d\delta, \tag{44}$$

$$S_{\rho\theta}(\rho,t)=\frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}[U_2g_2(t)\Pi_1-U_1g_1(t)\Pi_2][\mu+\alpha a_o]-\frac{H(t)\alpha a_0}{(\Pi_2^2-\Pi_1^2)\rho^2}\int_0^t[U_2\Pi_1g_2(t-\delta)$$

$$-U_1\Pi_2g_1(t-\delta)]e^{-a_1t}d\delta+\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i$$

$$\times\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha a_0}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(\nu a_1G_p+a_1G_m)^h$$

$$\times\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\Big[(\mu+\alpha a_0)$$

$$\times\mathbf{M}^{1,l+i}_{-(k+l)+l-h}(-(\nu G_p+\alpha a_oG_p+G_m),\delta)+\mu a_1\mathbf{M}^{1,l+i}_{-(1+k+l)+l-h}(-(\nu G_p+\alpha a_oG_p+G_m),\delta)\Big]d\delta,\tag{45}$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)](\mu+\alpha a_o)-\frac{H(t)\alpha a_0}{\rho\ln(\Pi_2/\Pi_1)}\int_0^t[V_2g_4(t-\delta)-V_1g_3(t-\delta)]$$

$$\times e^{-a_1t}d\delta+\pi H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha a_0}{\nu}\right)^j$$

$$\times\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(\nu a_1G_p+a_1G_m)^h$$

$$\times\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_1J_0(\Pi_2\rho_m)g_3(t-\delta)]\Big[(\mu+\alpha a_0)$$

$$\times\mathbf{M}^{1,l+i}_{-1(k+l)+l-h}(-(\nu G_p+\alpha a_oG_p+G_m),\delta)+\mu a_1\mathbf{M}^{1,l+i}_{-1(1+k+l)+l-h}(-(\nu G_p+\alpha a_oG_p+G_m),\delta)\Big]d\delta.\tag{46}$$

If we take $v(\Pi_1,t)=V_1sin(\Omega_1t)$, and $v(\Pi_2,t)=V_2sin(\Omega_2t)$, then the solutions given by Eqs.(44) and (46) are equivalent to those obtained by [42] in Eqs. (28) and (29).

5.2. Newtonian fluid with Porous and Magnetic effect ($\alpha \to 0$)

By assuming $\alpha \to 0$ in Eqs. (35), (36), (41) and (42), we acquire outcomes for Newtonian fluid with porous and magnetic effect.

$$w = \frac{H(t)}{(\Pi_2^2 - \Pi_1^2)\rho}\left[U_1 g_1(t)\Pi_1(\Pi_2^2 - \rho^2) + U_2 g_2(t)\Pi_2(\rho^2 - \Pi_1^2)\right] - \pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-(\nu G_p + G_m),\delta)d\delta, \quad (47)$$

$$v = \frac{H(t)}{\ln(\Pi_2/\Pi_1)}\left[V_1 g_3(t)\ln(\Pi_2/\rho) + V_2 g_4(t)\ln(\rho/\Pi_1)\right] - \pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_m^2)^i \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_m)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-(\nu G_p + G_m),\delta)d\delta, \quad (48)$$

$$S_{\rho\theta}(\rho,t) = \frac{2\Pi_1\Pi_2}{(\Pi_2^2 - \Pi_1^2)\rho^2}H(t)[U_2 g_2(t)\Pi_1 - U_1 g_1(t)\Pi_2]\mu + \pi\mu H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-(\nu G_p + G_m),\delta)d\delta, \quad (49)$$

$$S_{\rho z}(\rho,t) = \frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2 g_4(t) - V_1 g_3(t)]\mu + \pi\mu H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_m)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-(\nu G_p + G_m),\delta)d\delta. \quad (50)$$

5.3. Fractionalized Second grade with Porous effect($G_m \to 0$)

By assuming $G_m \to 0$ in Eqs. (35), (36), (41) and (42), so we acquire the outcomes for second grade with porous effect

$$w = \frac{H(t)}{(\Pi_2^2 - \Pi_1^2)\rho}\Big[U_1 g_1(t)\Pi_1(\Pi_2^2 - \rho^2) + U_2 g_2(t)\Pi_2(\rho^2 - \Pi_1^2)\Big] - \pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(\nu G_p)^h$$

$$\times\int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p + \alpha^\beta a_0 G_p),\delta)d\delta, \tag{51}$$

$$v = \frac{H(t)}{\ln(\Pi_2/\Pi_1)}\Big[V_1 g_3(t)\ln(\Pi_2/\rho) + V_2 g_4(t)\ln(\rho/\Pi_1)\Big] - \pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_m^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(\nu G_p)^h$$

$$\times\int_0^t [V_2 J_0(\Pi_1\rho_m)g_4(t-\delta) - V_2 J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p + \alpha^\beta a_0 G_p),\delta)d\delta, \tag{52}$$

$$S_{\rho\theta}(\rho,t) = \frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2 - \Pi_1^2)\rho^2}[U_2 g_2(t)\Pi_1 - U_1 g_1(t)\Pi_2][\mu + \alpha^\beta a_o] - \frac{H(t)\alpha^\beta a_0}{(\Pi_2^2 - \Pi_1^2)\rho^2}\int_0^t [U_2\Pi_1 g_2(t-\delta)$$

$$-U_1\Pi_2 g_1(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta)d\delta + \pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)}\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i$$

$$\times\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0 s_\beta}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(\nu a_1 G_p)^h$$

$$\times \int_0^t [U_2 J_1(\Pi_1 \rho_n) g_2(t-\delta) - U_1 J_1(\Pi_2 \rho_n) g_1(t-\delta)] \Big[\big(\mu + \alpha^\beta a_0\big)$$

$$\times \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p + \alpha^\beta a_o G_p), \delta) + \mu a_1 \mathbf{M}^{1,l+i}_{-\beta(1+k+l)+l-h}(-(\nu G_p + \alpha^\beta a_o G_p), \delta) \Big] d\delta, \quad (53)$$

$$S_{\rho z}(\rho,t) = \frac{H(t)}{\rho \ln(\Pi_2/\Pi_1)}[V_2 g_4(t) - V_1 g_3(t)](\mu + \alpha^\beta a_o) - \frac{H(t)\alpha^\beta a_0}{\rho \ln(\Pi_2/\Pi_1)} \int_0^t [V_2 g_4(t-\delta)$$

$$-V_1 g_3(t-\delta)] \mathbf{M}^{\beta,1}_0(-a_1,\delta) d\delta + \pi H(t) \sum_{m=1}^\infty \frac{J_0(\Pi_1 \rho_m \widetilde{D}_0(\rho, \rho_m)}{J_0^2(\Pi_1 \rho_m) - J_0^2(\Pi_2 \rho_m)} \sum_{i=0}^\infty (-\nu \rho_n^2)^i$$

$$\times \sum_{j=0}^i \frac{i!}{j!(i-j)!} \left(\frac{\alpha^\beta a_0 s_\beta}{\nu} \right)^j \sum_{k=0}^\infty \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^\infty (-a_1)^l \sum_{h=0}^l \frac{l!}{h!(l-h)!} (\nu a_1 G_p)^h$$

$$\times \int_0^t [V_2 J_0(\Pi_1 \rho_m) g_4(t-\delta) - V_1 J_0(\Pi_2 \rho_m) g_3(t-\delta)] \Big[\big(\mu + \alpha^\beta a_0\big) a_o G_p), \delta)$$

$$\times \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-(\nu G_p + \alpha^\beta + \mu a_1 \mathbf{M}^{1,l+i}_{-\beta(1+k+l)+l-h}(-(\nu G_p + \alpha^\beta a_o G_p), \delta) \Big] d\delta. \quad (54)$$

5.4. Fractionalized Second grade with Magnetic effect ($G_p \to 0$)

By assuming $G_p \to 0$ in Eqs. (35), (36), (41) and (42), we achieved the solutions for second grade fluid with magnetic effect

$$w = \frac{H(t)}{(\Pi_2^2 - \Pi_1^2)\rho} \Big[U_1 g_1(t) \Pi_1 (\Pi_2^2 - \rho^2) + U_2 g_2(t) \Pi_2 (\rho^2 - \Pi_1^2) \Big] - \pi H(t) \sum_{n=1}^\infty \frac{J_1(\Pi_1 \rho_n) D_1(\rho, \rho_n)}{J_1^2(\Pi_1 \rho_n) - J_1^2(\Pi_2 \rho_n)}$$

$$\times \sum_{i=0}^\infty (-\nu \rho_n^2)^i \sum_{j=0}^i \frac{i!}{j!(i-j)!} \left(\frac{\alpha^\beta a_0}{\nu} \right)^j \sum_{k=0}^\infty \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^\infty (-a_1)^l \sum_{h=0}^l \frac{l!}{h!(l-h)!} (G_m)^h$$

$$\times \int_0^t [U_2 J_1(\Pi_1\rho_n) g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n) g_1(t-\delta)] \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-G_m,\delta) d\delta, \quad (55)$$

$$v = \frac{H(t)}{\ln(\Pi_2/\Pi_1)} \Big[V_1 g_3(t) \ln(\Pi_2/\rho) + V_2 g_4(t) \ln(\rho/\Pi_1) \Big] - \pi H(t) \sum_{n=1}^{\infty} \frac{J_0(\Pi_1\rho_m) D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m) - J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty} (-\nu\rho_m^2)^i \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} \left(\frac{\alpha^\beta a_0}{\nu}\right)^j \sum_{k=0}^{\infty} \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^{\infty} (-a_1)^l \sum_{h=0}^{l} \frac{l!}{h!(l-h)!} (G_m)^h$$

$$\times \int_0^t [V_2 J_0(\Pi_1\rho_m) g_4(t-\delta) - V_2 J_0(\Pi_2\rho_m) g_3(t-\delta)] \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-G_m,\delta) d\delta, \quad (56)$$

$$S_{\rho\theta}(\rho,t) = \frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2} [U_2 g_2(t)\Pi_1 - U_1 g_1(t)\Pi_2][\mu+\alpha^\beta a_o] - \frac{H(t)\alpha^\beta a_0}{(\Pi_2^2-\Pi_1^2)\rho^2} \int_0^t [U_2\Pi_1 g_2(t-\delta)$$

$$-U_1\Pi_2 g_1(t-\delta)] \mathbf{M}_0^{\beta,1}(-a_1,\delta) d\delta + \pi H(t) \sum_{n=1}^{\infty} \frac{J_1(\Pi_1\rho_n) \widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n) - J_1^2(\Pi_2\rho_n)} \sum_{i=0}^{\infty} (-\nu\rho_n^2)^i$$

$$\times \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} \left(\frac{\alpha^\beta a_0 s_\beta}{\nu}\right)^j \sum_{k=0}^{\infty} \frac{(i-j)!}{k!(i-j-k)!} a_1^k \sum_{l=0}^{\infty} (-a_1)^l \sum_{h=0}^{l} \frac{l!}{h!(l-h)!} (G_m)^h$$

$$\times \int_0^t [U_2 J_1(\Pi_1\rho_n) g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n) g_1(t-\delta)] \Big[\big(\mu + \alpha^\beta a_0\big) \mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-G_m,\delta)$$

$$+\mu a_1 \mathbf{M}^{1,l+i}_{-\beta(1+k+l)+l-h}(-G_m,\delta) \Big] d\delta, \quad (57)$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)](\mu+\alpha^\beta a_o)-\frac{H(t)\alpha^\beta a_0}{\rho\ln(\Pi_2/\Pi_1)}$$

$$\times\int_0^t[V_2g_4(t-\delta)-V_1g_3(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta)d\delta+\pi H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i$$

$$\times\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0 s_\beta}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}(a_1G_m)^h$$

$$\times\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_1J_0(\Pi_2\rho_m)g_3(t-\delta)]\Big[\big(\mu+\alpha^\beta a_0\big)\mathbf{M}^{1,l+i}_{-\beta(k+l)+l-h}(-G_m,\delta)$$

$$+\mu a_1\mathbf{M}^{1,l+i}_{-\beta(1+k+l)+l-h}(-G_m,\delta)\Big]d\delta. \tag{58}$$

5.5. Fractionalized Second grade Fluid ($G_m\to 0$ and $G_p\to 0$)

By assuming $G_m\to 0$ and $G_p\to 0$ to Eqs. (35), (36), (41) and (42), so we achieved the solutions for second grade fluid.

$$w=\frac{H(t)}{(\Pi_2^2-\Pi_1^2)\rho}\Big[U_1g_1(t)\Pi_1(\Pi_2^2-\rho^2)+U_2g_2(t)\Pi_2(\rho^2-\Pi_1^2)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j\sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k\sum_{l=0}^{\infty}(-a_1)^l$$

$$\times\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}^{1,i}_{-\beta(k+l)}(0,\delta)d\delta, \tag{59}$$

$$v=\frac{H(t)}{\ln(\Pi_2/\Pi_1)}\Big[V_1g_3(t)\ln(\Pi_2/\rho)+V_2g_4(t)\ln(\rho/\Pi_1)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_m^2)^i \sum_{j=0}^{i} \frac{i!}{j!(i-j)!}\nu^{i-j}\left(\frac{\alpha^\beta a_0}{\nu}\right)^j \sum_{k=0}^{\infty}\frac{(i-j)!}{k!(i-j-k)!}a_1^k \sum_{l=0}^{\infty}(-a_1)^l$$

$$\times \int_0^t [V_2 J_0(\Pi_1\rho_m)g_4(t-\delta) - V_2 J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}^{1,i}_{-\beta(k+l)}(0,\delta)d\delta, \tag{60}$$

$$S_{\rho\theta}(\rho,t) = \frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}[U_2 g_2(t)\Pi_1 - U_1 g_1(t)\Pi_2][\mu+\alpha^\beta a_o] - \frac{H(t)\alpha^\beta a_0}{(\Pi_2^2-\Pi_1^2)\rho^2}$$

$$\times \int_0^t [U_2\Pi_1 g_2(t-\delta) - U_1\Pi_2 g_1(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta)d\delta + \pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0 s_\beta}{\nu}\right)^j \sum_{k=0}^{\infty}\frac{(i-j-1)!}{k!(i-j-1-k)!}a_1^k \sum_{l=0}^{\infty}(-a_1)^l$$

$$\times \int_0^t [U_2 J_1(\Pi_1\rho_n)g_2(t-\delta) - U_1 J_1(\Pi_2\rho_n)g_1(t-\delta)]\Big[\big(\mu+\alpha^\beta a_0\big)\mathbf{M}^{1,i}_{-\beta(k+l)}(0,\delta)$$

$$+\mu a_1 \mathbf{M}^{1,i}_{-\beta(1+k+l)}(0,\delta)\Big]d\delta, \tag{61}$$

$$S_{\rho z}(\rho,t) = \frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2 g_4(t) - V_1 g_3(t)](\mu+\alpha^\beta a_o) - \frac{H(t)\alpha^\beta a_0}{\rho\ln(\Pi_2/\Pi_1)}$$

$$\int_0^t [V_2 g_4(t-\delta) - V_1 g_3(t-\delta)]\mathbf{M}_0^{\beta,1}(-a_1,\delta) + \pi H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times \sum_{i=0}^{\infty}(-\nu\rho_n^2)^i \sum_{j=0}^{i}\frac{i!}{j!(i-j)!}\left(\frac{\alpha^\beta a_0 s_\beta}{\nu}\right)^j \sum_{k=0}^{\infty}\frac{(i-j-1)!}{k!(i-j-1-k)!}a_1^k \sum_{l=0}^{\infty}(-a_1)^l$$

$$\times\int_0^t [V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_1J_0(\Pi_2\rho_m)g_3(t-\delta)]\Big[(\mu+\alpha^\beta a_0)\mathbf{M}^{1,i}_{-\beta(k+l)}(0,\delta)$$

$$+\mu a_1\mathbf{M}^{1,i}_{-\beta(1+k+l)}(0,\delta)\Big]d\delta. \tag{62}$$

5.6. Newtonian with Porous effect $(\alpha, G_m \to 0)$

By assuming $\alpha \to 0$ and $G_m \to 0$ in Eqs. (35), (36), (41) and (42), so we acquire the solutions for Newtonian with porous effect .

$$w=\frac{H(t)}{(\Pi_2^2-\Pi_1^2)\rho}\Big[U_1g_1(t)\Pi_1(\Pi_2^2-\rho^2)+U_2g_2(t)\Pi_2(\rho^2-\Pi_1^2)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\int_0^t [U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}^{1,i}_0(-\nu G_p,\delta)d\delta, \tag{63}$$

$$v=\frac{H(t)}{\ln(\Pi_2/\Pi_1)}\Big[V_1g_3(t)\ln(\Pi_2/\rho)+V_2g_4(t)\ln(\rho/\Pi_1)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_m^2)^i\int_0^t [V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_2J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}^{1,i}_0(-\nu G_p,\delta)d\delta, \tag{64}$$

$$S_{\rho\theta}(\rho,t)=\frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}\Big[U_2g_2(t)\Pi_1-U_1g_1(t)\Pi_2\Big]\mu+\pi\mu H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\int_0^t [U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}^{1,i}_0(-\nu G_p,\delta)d\delta, \tag{65}$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)]\mu+\pi\mu H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_m^2)^i\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_1J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}_0^{1,i}(-\nu G_p,\delta)d\delta. \quad (66)$$

5.7. Newtonian with Magnetic effect $(\alpha, G_p\to 0)$

By assuming $\alpha\to 0$ and $G_p\to 0$ into Eqs. (35), (36), (41) and (42), so we acquire the solutions for Newtonian with magnetic effect .

$$w=\frac{H(t)}{(\Pi_2^2-\Pi_1^2)\rho}\Big[U_1g_1(t)\Pi_1(\Pi_2^2-\rho^2)+U_2g_2(t)\Pi_2(\rho^2-\Pi_1^2)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-G_m,\delta)d\delta, \quad (67)$$

$$v=\frac{H(t)}{\ln(\Pi_2/\Pi_1)}\Big[V_1g_3(t)\ln(\Pi_2/\rho)+V_2g_4(t)\ln(\rho/\Pi_1)\Big]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_m^2)^i\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_2J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}_0^{1,i}(-G_m,\delta)d\delta, \quad (68)$$

$$S_{\rho\theta}(\rho,t)=\frac{2\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}H(t)\Big[U_2g_2(t)\Pi_1-U_1g_1(t)\Pi_2\Big]\mu+\pi\mu H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_n^2)^i(G_m)^h\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)]\mathbf{M}_0^{1,i}(-G_m,\delta)d\delta, \quad (69)$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)]\mu+\pi\mu H(t)\sum_{m=1}^{\infty}\frac{\rho_m J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}H(t)$$

$$\times\sum_{i=0}^{\infty}(-\nu\rho_m^2)^i\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_1J_0(\Pi_2\rho_m)g_3(t-\delta)]\mathbf{M}_0^{1,i}(-G_m,\delta)d\delta. \quad (70)$$

5.8. Newtonian ($\alpha, G_p, G_m \to 0$)

By assuming $\alpha \to 0$, $G_p \to 0$ and $G_m \to 0$ into Eqs. (35), (36), (41) and (42), so we acquire the solutions for Newtonian.

$$w=\frac{H(t)}{(\Pi_2^2-\Pi_1^2)\rho}\left[U_1g_1(t)\Pi_1(\Pi_2^2-\rho^2)+U_2g_2(t)\Pi_2(\rho^2-\Pi_1^2)\right]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)D_0(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)][1-e^{-\nu\rho_n^2\delta}]d\delta, \quad (71)$$

$$v=\frac{H(t)}{\ln(\Pi_2/\Pi_1)}\left[V_1g_3(t)\ln(\Pi_2/\rho)+V_2g_4(t)\ln(\rho/\Pi_1)\right]-\pi H(t)\sum_{n=1}^{\infty}\frac{J_0(\Pi_1\rho_m)D_1(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$

$$\times\int_0^t[V_2J_0(\Pi_1\rho_m)g_4(t-\delta)-V_2J_0(\Pi_2\rho_m)g_3(t-\delta)][1-e^{-\nu\rho_m^2\delta}]d\delta, \quad (72)$$

$$S_{\rho\theta}(\rho,t)=\frac{2H(t)\Pi_1\Pi_2}{(\Pi_2^2-\Pi_1^2)\rho^2}\left[U_2g_2(t)\Pi_1-U_1g_1(t)\Pi_2\right]\mu+\pi\mu H(t)\sum_{n=1}^{\infty}\frac{J_1(\Pi_1\rho_n)\widetilde{D}_1(\rho,\rho_n)}{J_1^2(\Pi_1\rho_n)-J_1^2(\Pi_2\rho_n)}$$

$$\times\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)][1-e^{-\nu\rho_n^2\delta}]d\delta, \quad (73)$$

$$S_{\rho z}(\rho,t)=\frac{H(t)}{\rho\ln(\Pi_2/\Pi_1)}[V_2g_4(t)-V_1g_3(t)\mu+\pi\mu H(t)\sum_{m=1}^{\infty}\frac{J_0(\Pi_1\rho_m)\widetilde{D}_0(\rho,\rho_m)}{J_0^2(\Pi_1\rho_m)-J_0^2(\Pi_2\rho_m)}$$
$$\times\int_0^t[U_2J_1(\Pi_1\rho_n)g_2(t-\delta)-U_1J_1(\Pi_2\rho_n)g_1(t-\delta)][1-e^{-\nu\rho_m^2\delta}]d\delta. \quad (74)$$

If we take $w(\Pi_1,t)=A_1t,\ w(\Pi_2,t)=A_2t$, then the solutions given by Eqs.(71) and (73) are agree to those obtained by [43] in Eqs. (20) and (28). Furthermore, if we take $w(\Pi_1,t)=W_1sin(\omega_1t),\ w(\Pi_2,t)=W_2sin(\omega_2t)$, then the solutions given by Eqs.(71) and (73) are agree to those obtained by [44] in Eqs. (24) and (25). Similarly, we take $v(\Pi_1,t)=A_1t,\ w(\Pi_2,t)=A_2t$, then the solutions given by Eqs.(72) and (74) are agree to those obtained by [45] in Eqs. (4.3) and (4.4).

6. Numerical Results and Discussion

The present work estimates the helical flow of fraction MHD second grade fluid for very generalized boundary conditions under the porous nature between uniaxial annular cylinders. We are living in the era of fractional calculus and concept of fractional calculus has become essential in all sciences and engineering. Therefore, we fractionalized governing equations of the helical flow of second grade fluid with the modern Atangana Baleanu Caputo (ABC) fractional operator. The exact analytical outcomes determine the rotational and longitudinal velocities and shear stresses between two infinite circular cylinders for generalized boundary conditions on the surfaced inner and outer cylinders. The outcomes are obtained with the help of infinite Laplace and finite Hankel transforms while using ABC time derivatives. The outcomes are handover in integral and series forms with generalized $\mathbf{M}_c^{a,b}(\cdot,t)$ function. Furthermore, these outcomes need the requirement of all natural restrictions, we can impose on them. For example, these outcomes can satisfy the governing equations and initial and boundary conditions. We mentioned here some important worth of present research. We first time introduced the ABC fractional operator in the helical flow between coaxial cylinders. Another is that we used the generalized $\mathbf{M}_c^{a,b}(\cdot,t)$ function which shortened our exact analytical solutions as compared to some important previous work [30, 36–38, 42–45]. Another aspect of the present outcomes is that we used a very generalized function on the surface of the inner and outer cylinders. Additionally, the outcomes of second grade fluid and Newtonian fluid with and without MHD and porous effect can also be obtained very easily from the present generalized solution.

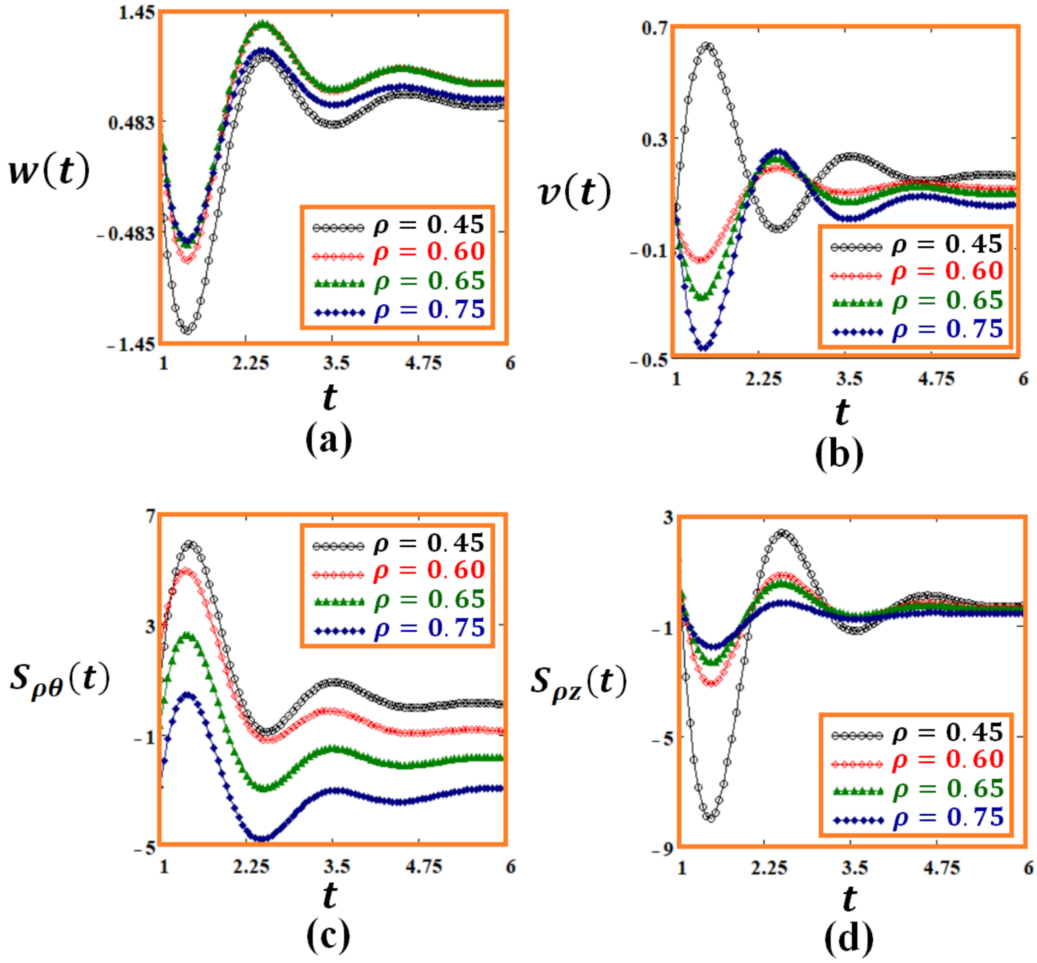

Figure 2: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha = 2.5,\ G_m = 0.5,\ G_p = 0.4$ dissimilar rates of ρ.

In the limiting cases of our outcomes, by assigning the fractional parameter $\beta \to 1$ and material parameter $\alpha \to 0$, we earn ordinary second grade fluid in section 5.1 and Newtonian fluid with both porous and magnetic effects in section 5.2 respectively. By considering magnetic constant $G_m \to 0$ and porous constant $G_p \to 0$, we acquire the outcome as the fractionalized second grade with porous effect in section 5.3 and fractionalized second grade with magnetic effect in section 5.4 respectively while in this case, both porous and magnetic effects eliminate together then it becomes fractionalized second grade in section 5.5. The fractionalized second grade with porous and fractionalized second grade with magnetic effect are recognized as Newtonian with porous section 5.6 and Newtonian with magnetic effect section 5.7 respectively, when we vanish the material parameter. At last section 5.8, we attain the Newtonian fluid as the material parameter abolished in the fractionalized second grade fluid.

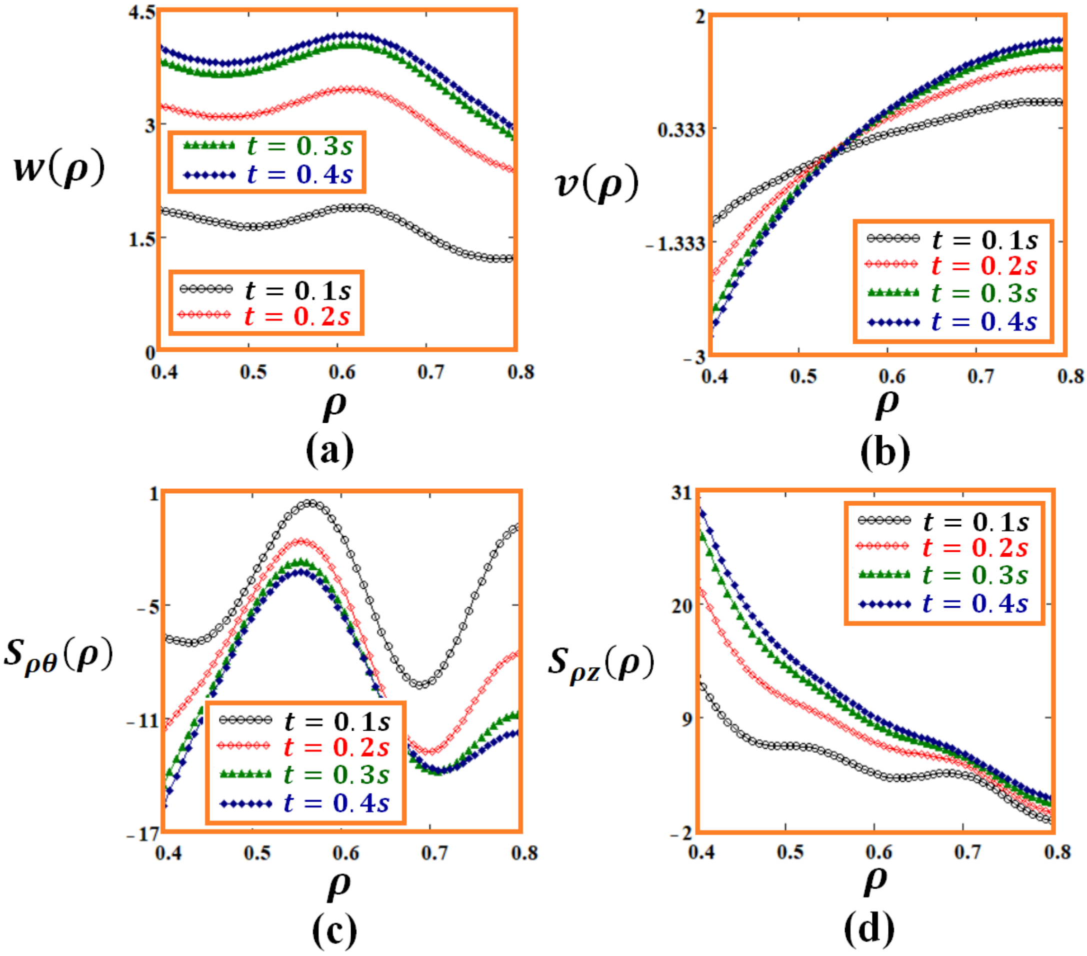

Figure 3: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha = 2.5,\ G_m = 0.5,\ G_p = 0.4$ dissimilar rates of t.

What physical represent the outcomes, and what are the impacts of physical parameters that appear in the outcomes? For these concern, we made graphs for velocity fields $\omega(\rho, t)$, $v(\rho, t)$ and shear stresses $S_{\rho z}(\rho, t)$, $S_{\rho\theta}(\rho, t)$. We have two choices for making these graphs either we use independent variable radial distances ρ or time t. We select here to represent the velocity field component and shear stresses against the time t because we take boundary conditions $g_1(t)$, $g_2(t)$, $g_3(t)$ and $g_4(t)$ as $e^{-t} sin(\omega t)$. Such type of boundary conditions in literature are very few but it seems to be intrinsic because the amplitude of the oscillation tends to zero when time goes to infinity, and other parameters fixed in all graphs are $\varrho = 11.12$, $\eta = 0.042$, $\Pi_1 = 0.4$, $\Pi_2 = 0.8$, $U_1 = U_2 = V_1 = V_2 = 2$,

$\omega = 3$, $\alpha = 2.5$, fractional parameter $\beta = 0.5$, $G_m = 0.5$, $G_p = 0.4$, and the radial distance off course vary from 0.4 to 0.8 and variation in t, we consider 1 to 6, unless otherwise stated in the diagram. For plotting these graphs, we used Mathcad program and for final makeup, we used MS Paint.

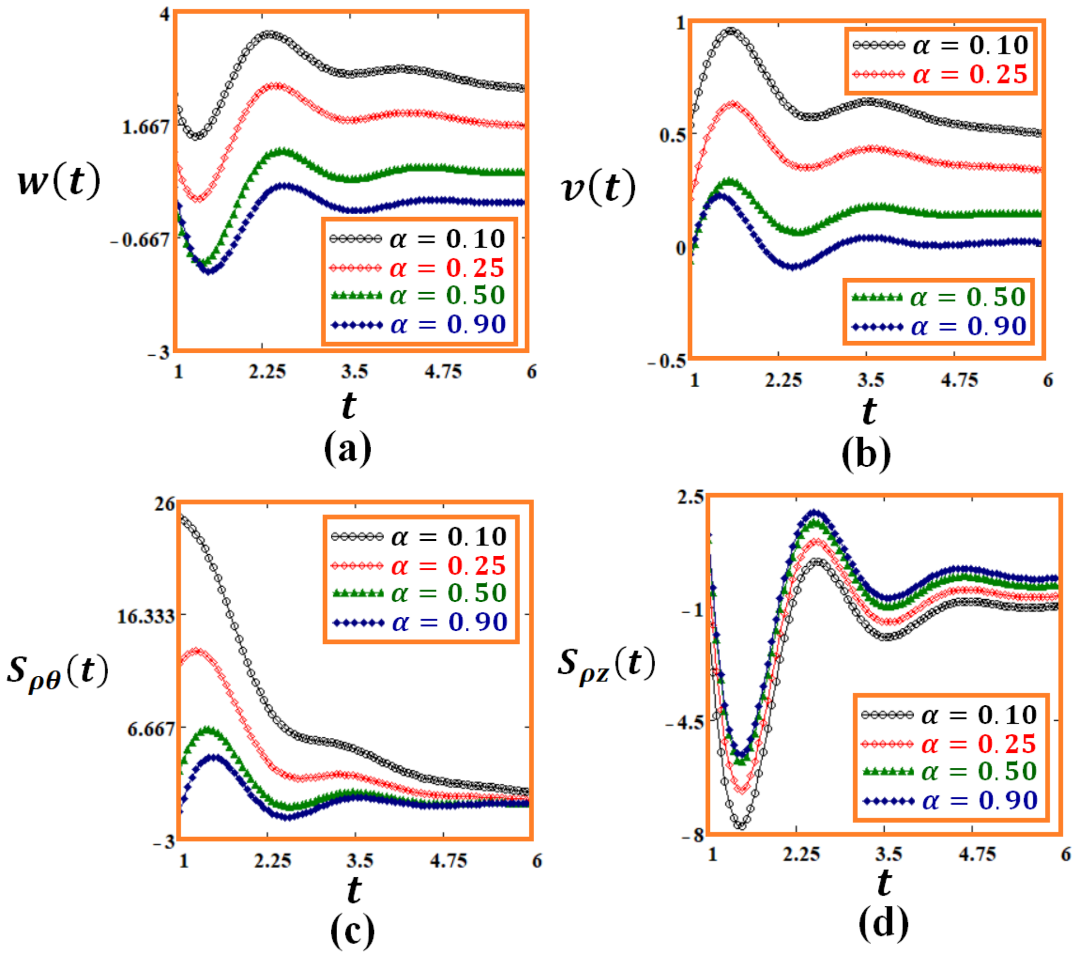

Figure 4: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ presented in Eqs. (35), (36), (41) and (42), for $G_m = 0.5,\ G_p = 0.4$ dissimilar rates of α.

The effect of the radial parameter of ρ for small and large values of time on velocity fields and shear stresses portrait is highlighted in Fig. 2. It is noted that the amplitude of oscillation of these entities decreases with the increase of radial distance. However for $v(\rho, t)$ these above comments are not appropriate as shown in Fig. 2(b). The large time effect is clearly shown in all diagrams. It is observed that the portrait of four entities tends to zero for large time t, due to boundary condition $e^{-t}sin(\omega t)$. Further, , it has been noted that $\omega(\rho,t)$ and $v(\rho,t)$ and similarly $S_{\rho\theta}(\rho,t)$, $S_{\rho z}(\rho,t)$ have quite opposite behavior in their respective portraits. It is also clear from these pictures $v(\rho,t)$ and $S_{\rho z}(\rho,t)$, and $\omega(\rho,t)$ and $S_{\rho\theta}(\rho,t)$ have again quite opposite behavior in their portrait.

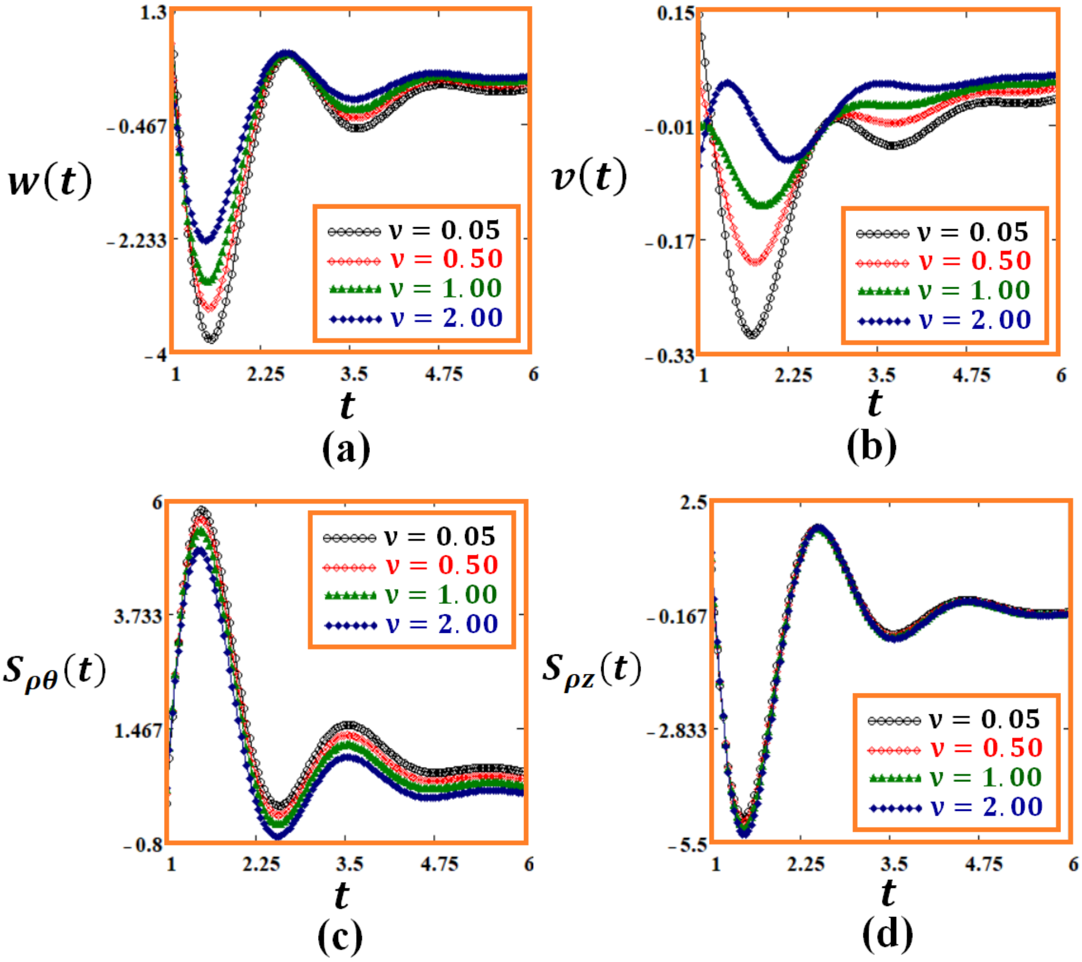

Figure 5: Portraits of the velocity fields $w(\rho,t)$ and $v(\rho,t)$ and shear stresses $S_{\rho\theta}(\rho,t)$ and $S_{\rho z}(\rho,t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha=2,\ G_m=5,\ G_p=4$ dissimilar rates of ν.

The effect of time on ABC fractionalized helical motion of second grade fluid are shown in Fig. 3. The velocity field components $\omega(\rho,t)$ on the entire domain and 60% of $v(\rho,t)$ in the given domain are increasing function of time t. However, shear stresses $S_{\rho\theta}(\rho,t)$ and $S_{\rho z}(\rho,t)$ have opposite portraits to each other with respect to time. It is also clear from these figures that rotational components of velocity fields and shear stresses have oscillating effects in their portrait as compared to translation entities.

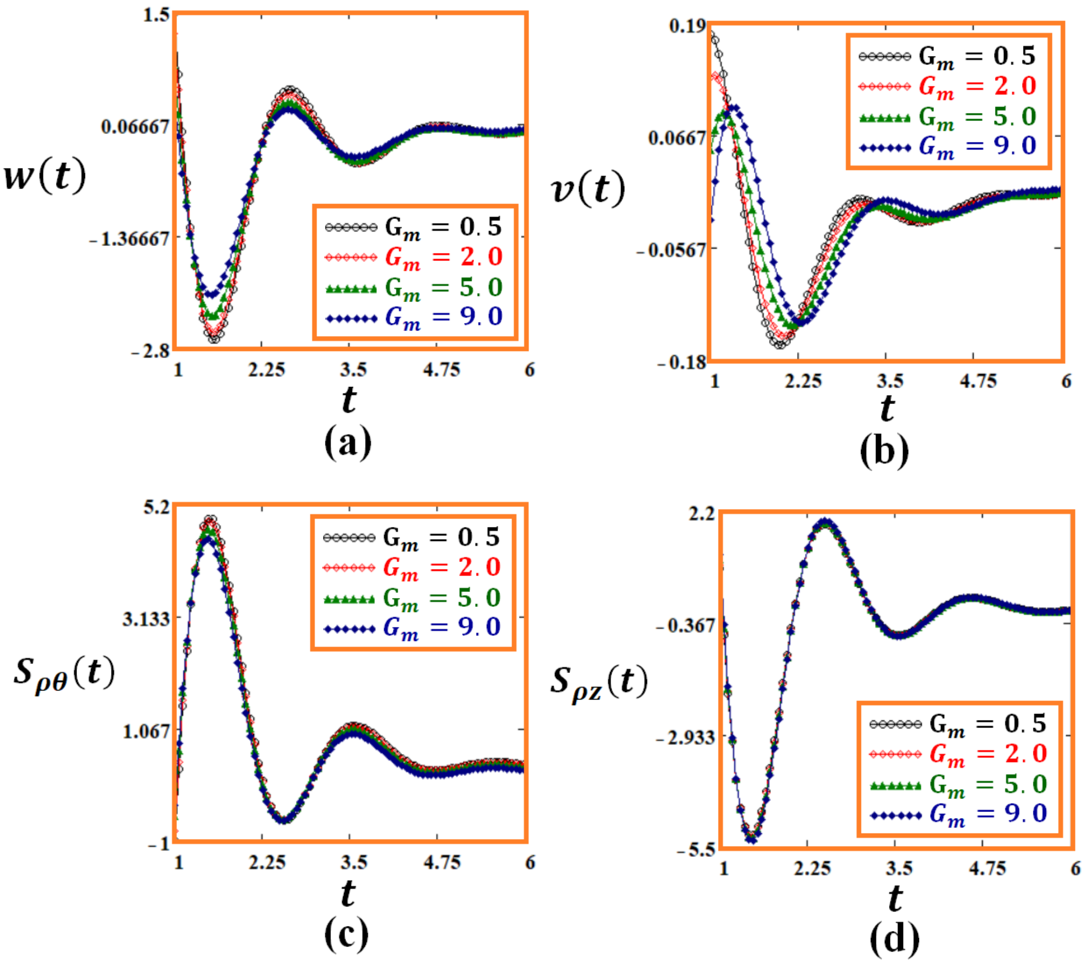

Figure 6: Portraits of the velocity fields $w(\rho,t)$ and $v(\rho,t)$ and shear stresses $S_{\rho\theta}(\rho,t)$ and $S_{\rho z}(\rho,t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha = 2,\ G_p = 2$ dissimilar rates of G_m.

The very internal parameter of second grade fluid is material parameter α. The increasing values of material parameter α increase the non-Newtonian behavior of second grade fluid. The graphs of such an important and interesting study of the present research are depicted in Fig. 4. All graphs agree that large values of material parameter α lower the amplitude of the motion or decay the motion. Furthermore, the rate of decay in shear stress portraits is faster than in velocity field components. It is also clear from these graphs that the shear stress component $S_{\rho\theta}$ decays in minimum and the velocity field decays $w(t)$ in maximum time.

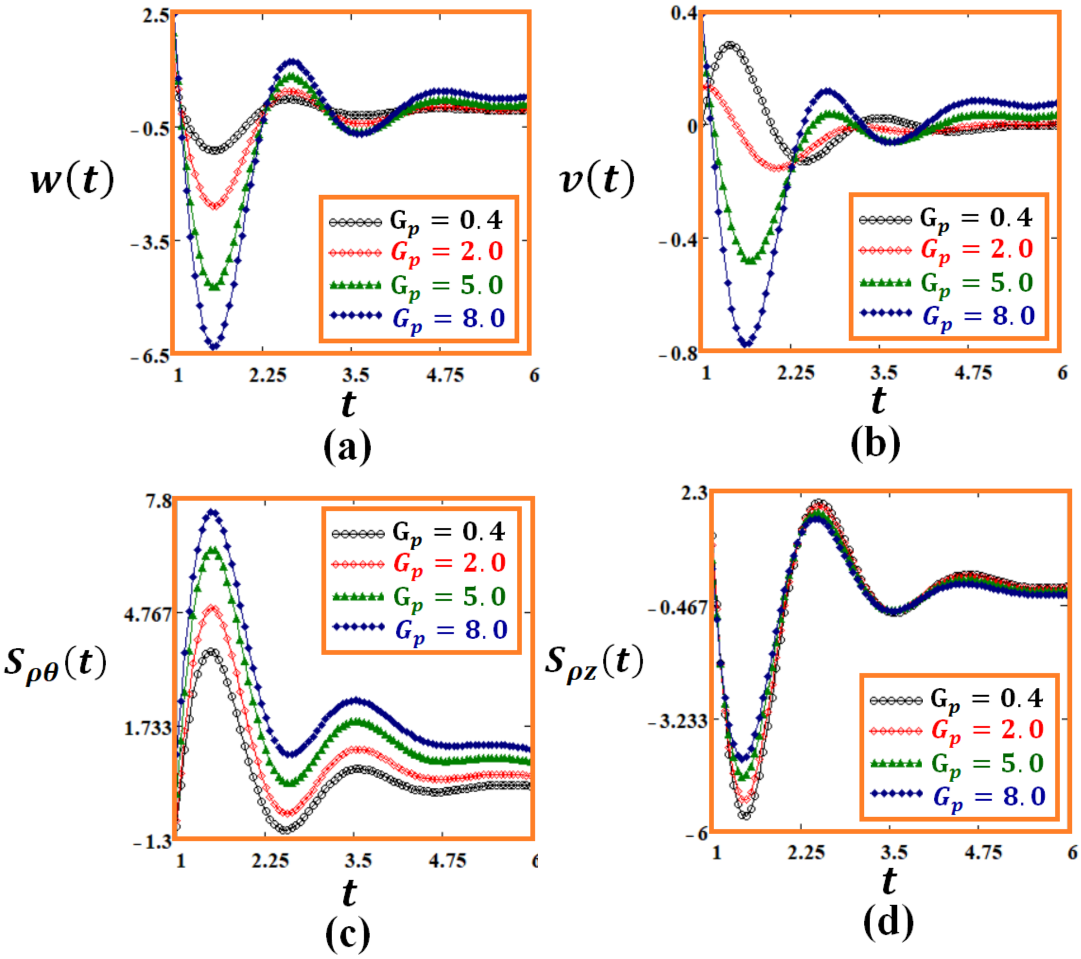

Figure 7: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha = 2,\ G_m = 2$ dissimilar rates of G_p.

The other significant internal parameter that resists the flow is the kinematic viscosity ν. As it is a common phenomenon in Newtonian fluid the greater the viscosity, the thicker it is that is flow is slow. Fig. 5 shows the impact of kinematic viscosity on the present dynamical system. Although the fluid is second grade fluid, it is clear from Figs. 5 that the larger the viscosity the slower the flow and the amplitude of the motion of fluid reduced. However, Fig. 5(d) seems a little bit opposite here but we can ignore it as difference in the four portraits is negligible. The present study is the electronically conducting fluid and quantitative measurement of such fluid is done by the magnetic parameter G_m. For this purpose, we made the Figs. 6, these graphs explicitly show the great impact of magnetic parameters on the present system. It is obvious that increasing the magnetic parameter G_m decreases the amplitude of the motion and the four essences in Figs. 6 tend to zero faster than Figs. 4 and 5.

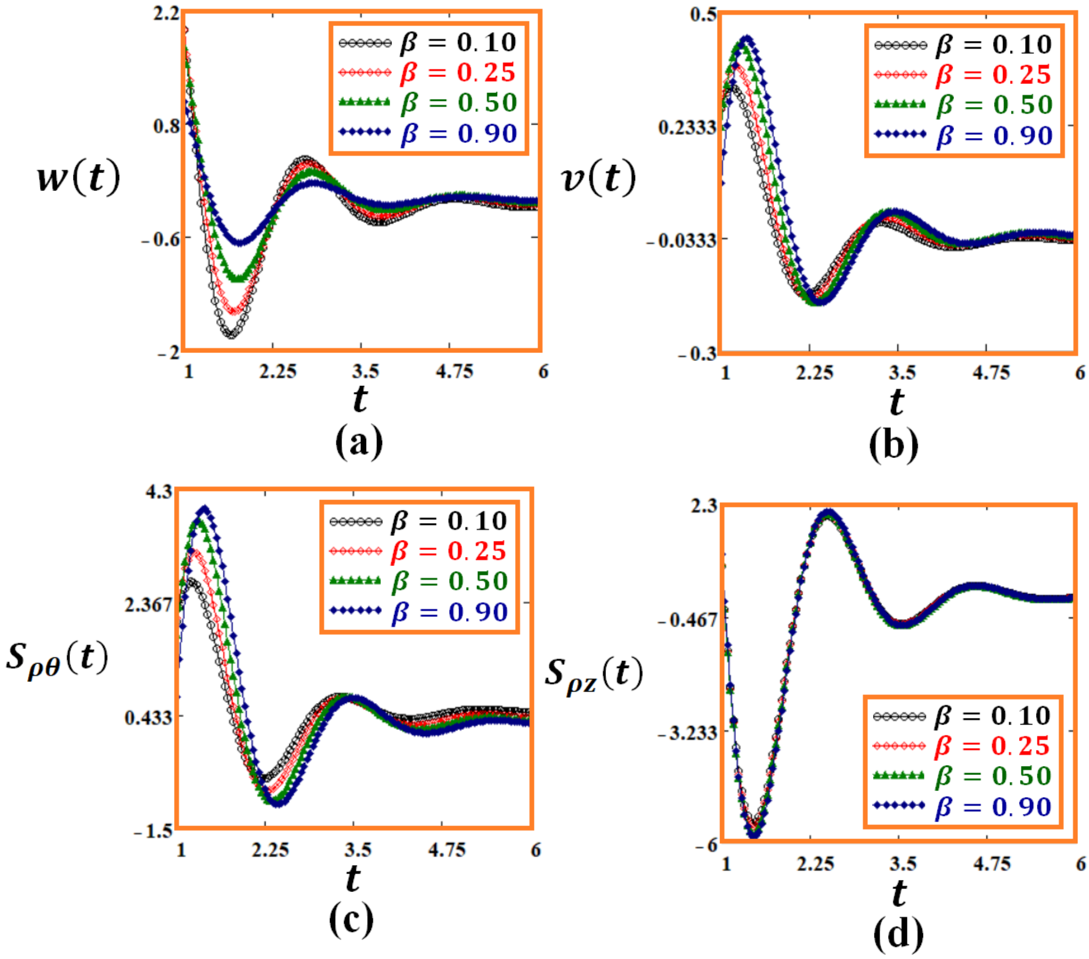

Figure 8: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ presented in Eqs. (35), (36), (41) and (42), for $\alpha = 1.5,\ G_m = 3,\ G_p = 2$ dissimilar rates of β.

The porosity G_p of the material cannot be neglected. It is the effect that decreases the strength of the flow. Without porosity is the ideal condition, which practically does not exist. This resistive measure quantitatively is explained in Figs. 7. It is noted that the present flow conditions put the effect of porosity measure in such that increasing values of porosity will reduce the oscillating motion of the fluid. However, there are different portraits in Fig 7(b), which is due to the oscillating motion of the fluid. Further, the impact of the porosity parameter G_p on the reduction of motion of the fluid is lower than the magnetic parameter G_m. Today's world is a fractionalized world. All sciences, engineering, and social-economic information are nowadays transformed into a fractionalized system. The reason is clear that integer order explanation is not enough to explore the world. Very recent fractional derivative that has already discussed is the Atangana Baleanu Caputo (ABC) operator β that is used in this study. The effect of this operator is explored in the Figs. 8. It brings to the knowledge that three entities $v(t), S_{\rho\theta}$ and $S_{\rho z}$ are increasing functions of the ABC fractional operator β, however, the $w(t)$ is quite opposite behavior in comparison to three.

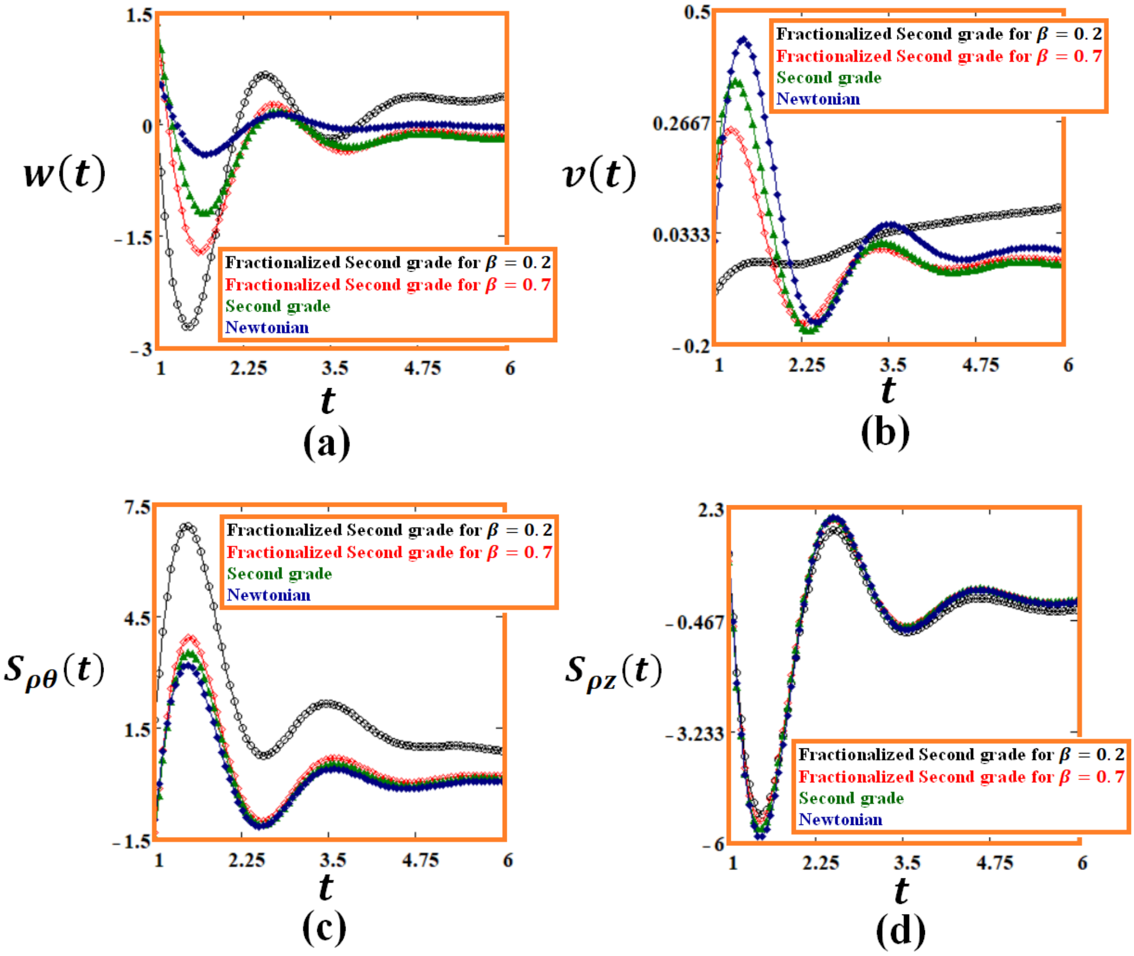

Figure 9: Portraits of the velocity fields $w(\rho, t)$ and $v(\rho, t)$ and shear stresses $S_{\rho\theta}(\rho, t)$ and $S_{\rho z}(\rho, t)$ for $\alpha = 2.5,\ G_m = 2.5,\ G_p = 1.5$ for dissimilar types of fluids.

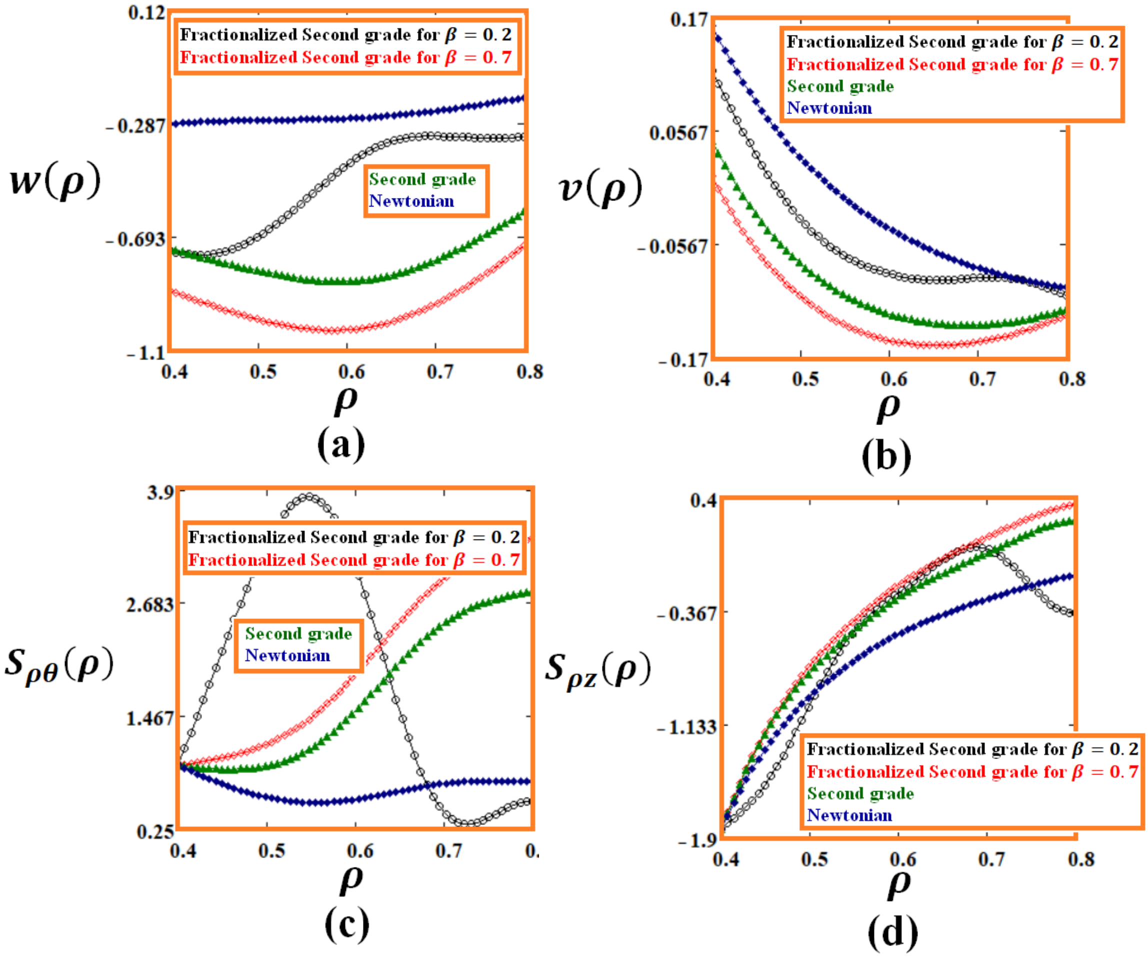

Figure 10: Portraits of the velocity fields $w(\rho,t)$ and $v(\rho,t)$ and shear stresses $S_{\rho\theta}(\rho,t)$ and $S_{\rho z}(\rho,t)$ for $\alpha = 2.5,\ G_m = 2.5,\ G_p = 1.5$ for dissimilar types of fluids.

In order to make this study more interesting and draw some more extract and conclusions, we made Figs 9 and 10. In these graphs we present the four fluids, fractionalized second grade fluid (given by Eqs. (35), (36), (41), and (42)) for $\beta = 0.2$ and $\beta = 0.7$, second grade (obtained from Eqs. (59 - 62) by putting $\beta \to 1$) and Newtonian (given by Eqs. 71- 74) fluids with respect to time t and radial parameter ρ. Quite the opposite phenomenon is observed in Figs. 9 that in rotational quantities $w(t)$, $S_{\rho\theta}$ the fractionalized second grade fluid for $\beta = 0.2$ have the largest values and Newtonian fluid has least values. In spite of that the translational quantities $v(t)$, $S_{\rho z}$ have quite contrary attitude in comparison of rotational quantities. With regard to radial distance as shown in Figs. 10, it is observed that fractionalized second grade fluid for $\beta = 0.2$ have largest amplitude and Newtonian fluid has smallest. On the other hand, the fractionalized second grade fluid for $\beta = 0.7$ and second grade fluid have similar behavior but the second grade fluid has deeper amplitude. All these pictures are depicted in Mathcad software and using MKS units system. At the end, we used Paint software to bring graphs in good condition.

7. Conclusion

This research paper deals with the application and influence of the newly defined fractional derivative Atangana Baleanu Caputo (ABC) on the unsteady helical motion of second grade fluid with MHD effect between porous annular cylinders. In this work after using a helical model of second grade fluid, we introduced the ABC fractional derivative to the governing equations, and analytical outcomes are obtained for rotational and translation quantities in porous annular region. Dual integral transformations including Laplace and Hankel are used to eliminate the partial derivatives and then the resulting algebraic expression will be solved by inverse transformations in the form of series, integral, convolution product, and generalized functions. These outcomes satisfy all restrictions and give many particular solutions of second grade and Newtonian fluid with/without MHD and porous attachment. Some of our special solutions coincide with previous solutions, which prove the correctness of our results. The major novel findings of the present work are:

The ABC fractional derivative is used in the helical motion of second grade and Newtonian fluids. Introducing the generalized **M** functions, which are natural for such types of calculations. The fluid motion is a decreasing function of the material parameter α and kinematic viscosity ν. The magnetic and porous parameters G_m and G_p respectively, have almost contrary impacts on the fluid vibration. The fractional parameter β has opposite influence on velocity field and shear stress portraits. With respect to time t, the rotational quantities have large values of fractionalized second grade fluid for $\beta = 0.2$ and Newtonian has least. With respect to radial distance ρ the fractionalized second grade fluid for $\beta = 0.2$ and Newtonian fluids have respectively maximum and minimum oscillations.

8. Future Recommendation

This research work can be extendable to the other non-Newtonian fluid such as Maxwell fluid, Burgers material, Oldroyd-B model by considering various geometrical configuration like helical flow or flow in channel along with some additional effects through Atangana Baleanu Caputo (ABC) fractional operator. The outcome can be obtained in terms of a series of trigonometric or any other generalized Wright functions.

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