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Sheffer Stroke Hilbert Algebras in Connection with Crossing Cubic Structures

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Abstract. The idea of Sheffer stroke Hilbert algebra is studied from the perspective of the structure of the cubic structure. The filter and deductive system of the Sheffer stroke Hilbert algebra are defined and examined through the crossing cubic structure, which is an extension of the fuzziness of these substructures, verifying their many characteristics. Moreover, conditions suitable for the crossing cubic structure are established to be a crossing cubic filter and several characterization theorems are reached. Accordingly, the relationship between crossing cubic filters and filters of Sheffer stroke Hilbert algebras is explained such that the crossing cubic deductive system can handle all of the results for the crossing cubic filter covered above in the same way.

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1. Introduction

In mathematical logic, there are certain logical operations such that conjunction (and), disjunction (or) and negation (not). These logical operations are represented by the symbols "$\wedge$", "$\vee$" and "$\sim$", respectively. In a conjunction operator, if anyone of the statement is false, then the output is false. In a disjunction operator, if anyone of the statements is true, then the output is true. A negation operator gives the opposite result, i.e., if the input is true, then the output is false. Table 1 describes the truth table for the conjunction "$\wedge$" and disjunction "$\vee$" operations.

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Table 1: The truth table for the conjunction "$\wedge$" and disjunction "$\vee$" operations.

Input(A)	Input(B)	Output($A \wedge B$)	Output($A \vee B$)
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

A Sheffer stroke, represented by the symbol "$|$", is a two-input logical operation that produces an incorrect output only when both inputs are true i.e., the Sheffer stroke operation $A \mid B$ is equivalent to $\sim (A \wedge B)$. Table 2 describes the truth table for the Sheffer stroke operation "$|$".

Table 2: The truth table for the Sheffer stroke "$|$" .

Input(A)	Input(B)	Output($A \mid B$)
T	T	F
T	F	T
F	T	T
F	F	T

Here, we will make an example on the truth values of Sheffer stroke operation for the given two statements A and B. Consider the following two statements:

$$A : p \text{ is divisible by } 2,$$
$$B : p \text{ is divisible by } 3.$$

Then, the truth table of the Sheffer stroke "$|$" between the two statements A and B is presented by Table 3.

Table 3: The truth table for the Sheffer stroke "$|$" .

Values of p	A	B	$A \mid B$
$p = 12$	T	T	F
$p = 4$	T	F	T
$p = 9$	F	T	T
$p = 7$	F	F	T

Sheffer [1] used the Sheffer stroke "$|$" (the so-called Sheffer operation) in 1913 as a fundamental axiomatization of Boolean algebra and demonstrated that all Boolean functions may be generated from a single binary operation, as term operations. After that, McCune et al. [2] demonstrated that this operation can represent any Boolean operation. Many researchers were drawn to this operation since it is the only way to define any Boolean operation or function [2]. As a result, many algebraic structures, axioms or formulas, are reduced. Because simple and fewer axiom systems are obtained, it is simple to verify some attributes and notations for a new algebraic structure. Thus, many scientists seek to apply such a reduction to explore algebraic structures using the Sheffer operation such as BG-algebras [3], BM-algebra [4], ortholattices [5], and INK-algebra [6], etc.

Henkin [7] proposed Hilbert algebras in the 1950's as an algebraic equivalent of Hilbert's positive implicative propositional calculus [8], for use in intuitionistic and other non-classical logic studies. These algebraic structures, which include the implication and the distinct element 1, can be understood as parts of propositional logic. In particular, Diego [9] analyzed Hilbert algebras, including their deductive systems and different features, and showed that a variety of Hilbert algebras exist. The idea of a deductive system, also known as implicative filters, is crucial to the overall development of Hilbert algebras because they can be employed to symbolize several logical theories. Furthermore, Busneag [10, 11] and Jun [12] analyzed Hilbert algebras, associated concepts, and deductive systems. The Sheffer stroke was used in Hilbert algebras by Oner et al. in 2021, and they presented its several properties, deductive systems, and ideals [13].

Fuzzy mathematics is a new field in which ordinary notions are being transferred to uncertainty cases. Zadeh [14] adopted the concept of fuzzy structures through his popular paper as a development of standard structure, which only allows elements to be fully in or fully out of a set. This concept is a significant mathematical framework for dealing with uncertainty and ambiguity in data. Zadeh [15] made an extension of the theory of fuzziness concept by an interval-valued fuzziness concept. As another extension of uncertainty sets, Zhang [16] presented the notion bipolar fuzzy sets as a very useful tool for considering positive and negative data at the same time. By ignore the positive part of a bipolar uncertainty set, Jun et al. [17] proposed a new concept, namely N-structures. After that, there were many extensions of fuzziness structures, such as the intuitionistic fuzzy, neutrosophic set and Lukasiewicz fuzzy set, see [18–20].

By incorporating the concept of interval-valued fuzziness sets with N-structures, which creates an expansion of a bipolar-valued fuzziness set to produce the thought of crossing cubic structures [21], that floated for the first time in 2021 by Jun and Song [22], where they applied it to BCK/BCI-algebra. Additionally, Ozturk et al. [23] developed the concept of crossing cubic on semigroup structures, and commutative ideal in BCK-algebras. Al-Masarwah et al. [24] established the idea of crossing cubic Lie subalgebra in the context of Lie algebras. They utilized the notions of Lie ideals, homomorphisms and some fundamental properties of Lie algebras.

In the fuzzification of the Sheffer stroke Hilbert algebras, the concept of fuzzy (uncertainty) sets was connected with a filter and a deductive system of Sheffer stroke Hilbert algebra by Oner et al. [25]. Vasuki et al. [26] presented the concept of anti-Q-fuzzy deductive systems of Hilbert algebras and proved some results. In the connection between intuitionistic fuzzy sets and Sheffer stroke Hilbert algebras, Saeid et al. [27] implemented the concept of intuitionistic fuzzy deductive system and intuitionistic fuzzy filter in Sheffer stroke Hilbert algebras, and investigated some of its features. Oner et al. [28] studied the idea of Sheffer stroke Hilbert algebras via neutrosophic N-structures. They discussed neutrosophic N-subalgebras and neutrosophic N-ideals in Sheffer stroke Hilbert algebras.

Concentrating on exploring the filter and deductive system of the Sheffer stroke Hilbert algebra using the crossing cubic structure, we propose the idea of crossing cubic filter and crossing cubic deductive system, give examples, and then discuss certain features. We create a crossing cubic filter by applying appropriate conditions to a specific crossing cubic structure. We study characterizations of crossing cubic filters. We construct crossing cubic filters that are associated with filters. Eventually, we prove that crossing cubic deductive system and crossing cubic filter are a complementary notion.

2. Preliminaries

This section covers some fundamental definitions and results of Sheffer stroke Hilbert algebras, filters, deductive systems, crossing cubic structures, and some results that will be utilized throughout the study.

Definition 1 ([1]). *A groupoid is denoted by* $\varphi_\circ := (\varphi, |)$. *Then, the operation "*$|$*" is called a Sheffer stroke (Sheffer operation) if the following identities hold:* $\forall \mathfrak{n}, \mathfrak{v}, \mathfrak{w}, \in \varphi$,

(1) $\mathfrak{n} \mid \mathfrak{v} = \mathfrak{v} \mid \mathfrak{n}$;

(2) $(\mathfrak{n} \mid \mathfrak{n}) \mid (\mathfrak{n} \mid \mathfrak{v}) = \mathfrak{n}$;

(3) $\mathfrak{n} \mid ((\mathfrak{v} \mid \mathfrak{w}) \mid (\mathfrak{v} \mid \mathfrak{w})) = ((\mathfrak{n} \mid \mathfrak{v}) \mid (\mathfrak{n} \mid \mathfrak{v})) \mid \mathfrak{w}$;

(4) $(\mathfrak{n} \mid ((\mathfrak{n} \mid \mathfrak{n}) \mid (\mathfrak{v} \mid \mathfrak{v}))) \mid (\mathfrak{n} \mid ((\mathfrak{n} \mid \mathfrak{n}) \mid (\mathfrak{v} \mid \mathfrak{v}))) = \mathfrak{n}$.

Definition 2 ([13]). *A Sheffer stroke Hilbert algebra with a Sheffer stroke "*$|$*" is a groupoid* $\vartheta_\circ := (\vartheta, |)$ *that satisfies the axioms,* $\forall \mathfrak{n}, \mathfrak{v}, \mathfrak{w} \in \vartheta$:

(1) $(\mathfrak{n} \mid ((X) \mid (X))) \mid (((Y) \mid ((Z) \mid (Z))) \mid ((Y) \mid ((Z) \mid (Z)))) = \mathfrak{n} \mid (\mathfrak{n} \mid \mathfrak{n})$, *where* $X := \mathfrak{v} \mid (\mathfrak{w} \mid \mathfrak{w}), Y := \mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})$ *and* $Z := \mathfrak{n} \mid (\mathfrak{w} \mid \mathfrak{w})$;

(2) $\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) = \mathfrak{v} \mid (\mathfrak{n} \mid \mathfrak{n}) = \mathfrak{n} \mid (\mathfrak{n} \mid \mathfrak{n}) \Rightarrow \mathfrak{n} = \mathfrak{v}$.

For a Sheffer stroke Hilbert algebra $\vartheta_\circ := (\vartheta, |)$, see[13] a relation "$\leq$", which is a partial order on ϑ, is defined as

$$(\mathbf{A})(\mathfrak{n} \leq \mathfrak{v} \Leftrightarrow \mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) = 1, \forall \mathfrak{n}, \mathfrak{v} \in \vartheta).$$

Proposition 1 ([13]). *If* $\vartheta_\circ := (\vartheta, |)$ *is a Sheffer stroke Hilbert algebra, then* ϑ *satisfies:* $\forall \mathfrak{n}, \mathfrak{v}, \mathfrak{w} \in \vartheta$,

(1) $\mathfrak{n} \mid (\mathfrak{n} \mid \mathfrak{n}) = 1$;

(2) $\mathfrak{n} \mid (1 \mid 1) = 1$;

(3) $1 \mid (\mathfrak{n} \mid \mathfrak{n}) = \mathfrak{n}$;

(4) $\mathfrak{n} \leq \mathfrak{v} \mid (\mathfrak{n} \mid \mathfrak{n})$;

(5) $(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})) \mid (\mathfrak{v} \mid \mathfrak{v}) = (\mathfrak{v} \mid (\mathfrak{n} \mid \mathfrak{n})) \mid (\mathfrak{n} \mid \mathfrak{n})$;

(6) $((\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})) \mid (\mathfrak{v} \mid \mathfrak{v})) \mid (\mathfrak{v} \mid \mathfrak{v}) = \mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})$;

(7) $\mathfrak{n} \mid ((\mathfrak{v} \mid (\mathfrak{w} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid (\mathfrak{w} \mid \mathfrak{w}))) = \mathfrak{v} \mid ((\mathfrak{n} \mid (\mathfrak{w} \mid \mathfrak{w})) \mid (\mathfrak{n} \mid (\mathfrak{w} \mid \mathfrak{w})))$.

Definition 3 ([25]). *A subset Ψ of a Sheffer stroke Hilbert algebra $\vartheta_\circ := (\vartheta, |)$ is called a filter of ϑ if the conditions are satisfied: $\forall \mathfrak{n}, \mathfrak{v}, \mathfrak{w} \in \vartheta$*

(1) $1 \in \Psi$;

(2) $\mathfrak{v} \in \Psi \Rightarrow \mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) \in \Psi$;

(3) $\mathfrak{v}, \mathfrak{w} \in \Psi \Rightarrow (\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w}) \in \Psi$.

Definition 4 ([13]). *A subset Ψ of a Sheffer stroke Hilbert algebra $\vartheta_\circ := (\vartheta, |)$ is called a deductive system of ϑ if the conditions are satisfied: $\forall \mathfrak{n}, \mathfrak{v} \in \vartheta$*

(1) $1 \in \Psi$;

(2) $\mathfrak{n} \in \Psi, \mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) \in \Psi \Rightarrow \mathfrak{v} \in \Psi$.

Definition 5 ([15]). *Let $\vartheta \neq \phi$ be a set. An interval valued fuzzy set on ϑ is an object of the following form:*

$$\begin{aligned} \widetilde{\Upsilon}_\Pi &= \{(\mathfrak{n}, \overset{\mp}{\Upsilon}_\Pi(\mathfrak{n})) : \mathfrak{n} \in \vartheta\} \\ &= \{(\mathfrak{n}, [\Upsilon_\Pi^-(\mathfrak{n}), \Upsilon_\Pi^+(\mathfrak{n})]) : \mathfrak{n} \in \vartheta\}, \end{aligned}$$

where $\overset{\mp}{\Upsilon}_\Pi$: $\vartheta \to \mathbb{I}[0,1]$ is a function and $\mathbb{I}[0,1]$ denotes the set containing every closed subinterval of $[0,1]$. The members of $\mathbb{I}[0,1]$ are called interval numbers. If we take an interval number $\widetilde{\upsilon} = [\upsilon^-, \upsilon^+] \in \mathbb{I}[0,1]$, then $0 \leq \upsilon^- \leq \upsilon^+ \leq 1$.

For each two interval numbers $\widetilde{\upsilon} = [\upsilon^-, \upsilon^+]$ and $\widetilde{\sigma} = [\sigma^-, \sigma^+]$ in $\mathbb{I}[0,1]$, we define

(1) $\widetilde{\upsilon} \preceq \widetilde{\sigma}$(or $\widetilde{\sigma} \succeq \widetilde{\upsilon}$) $\Leftrightarrow \upsilon^- \leq \sigma^-, \upsilon^+ \leq \sigma^+$;

(2) $\widetilde{\upsilon} = \widetilde{\sigma} \Leftrightarrow \widetilde{\upsilon} \preceq \widetilde{\sigma}$ and $\widetilde{\sigma} \preceq \widetilde{\upsilon}$;

(3) $\widetilde{min}\{\widetilde{\upsilon}, \widetilde{\sigma}\} = [min\{\upsilon^-, \sigma^-\}, min\{\upsilon^+, \sigma^+\}]$.

Definition 6 ([17]). *An N-structure ϖ_Π of ϑ is an object of the following form:*

$$\varpi_\Pi = \{(\mathfrak{n}, \varpi^-(\mathfrak{n})) : \mathfrak{n} \in \vartheta\},$$

where $\varpi^- : \vartheta \to [-1, 0]$.

Definition 7 ([21]). *Let $\vartheta \neq \phi$ be a set. A crossing cubic structure of ϑ is an object having the following form:*

$$\Pi = \{(\mathfrak{n}, \widetilde{\Upsilon}_\Pi(\mathfrak{n}), \varpi_\Pi(\mathfrak{n})) \mid \mathfrak{n} \in \vartheta\},$$

where $\widetilde{\Upsilon}_{\Pi} = [\Upsilon^-_{\Pi}, \Upsilon^+_{\Pi}] : \vartheta \to \mathbb{I}[0,1]$ *is an interval valued fuzzy set on* ϑ *and* $\varpi_{\Pi} : \vartheta \to [-1,0]$ *is an N-structure on* ϑ.

For the purpose of simplicity, we will employ the symbol $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *for the crossing cubic structure* $\Pi = \{(\mathfrak{n}, \widetilde{\Upsilon}_{\Pi}(\mathfrak{n}), \varpi_{\Pi}(\mathfrak{n})) \mid \mathfrak{n} \in \vartheta\}$.

Definition 8 ([21])**.** *Let* $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *be a crossing cubic structure over* ϑ. *Then,* $(\widetilde{\tau}, \kappa)$*-level of* $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *is the crisp set in* ϑ *denoted by* $\Gamma\Big(\widetilde{\Upsilon}_{\Pi}; (\widetilde{\tau}, \kappa)\Big)$ *and is defined as*

$$\Gamma\Big(\widetilde{\Upsilon}_{\Pi}; (\widetilde{\tau}, \kappa)\Big) = \Big\{\mathfrak{n} \in \vartheta \mid \widetilde{\Upsilon}_{\Pi}(\mathfrak{n}) \succeq \widetilde{\tau}, \varpi_{\Pi}(\mathfrak{n}) \leq \kappa\Big\},$$

where $\widetilde{\tau} \in \mathbb{I}[0,1]$ *and* $\kappa \in [-1,0]$. *For* $\widetilde{\tau} \in \mathbb{I}[0,1]$ *and* $\kappa \in [-1,0]$. *The* $\widetilde{\tau}$*-level* $\Gamma\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$ *and* κ*-level* $\Gamma\Big(\varpi_{\Omega}; \kappa\Big)$ *subsets of* $\prod$ *can be defined as:*

$$\Gamma\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big) = \Big\{\mathfrak{n} \in \vartheta \mid \widetilde{\Upsilon}_{\Pi}(\mathfrak{n}) \succeq \widetilde{\tau}\Big\} \text{ and } \Gamma\Big(\varpi_{\Pi}; \kappa\Big) = \Big\{\mathfrak{n} \in \vartheta \mid \varpi_{\Pi}(\mathfrak{n}) \leq \kappa\Big\}.$$

3. Crossing Cubic Filters

Here, we present the idea of crossing cubic filters. Then, we study certain characterizations of it. Also, we discuss a relation between a crossing cubic filter and a crisp filter.

Definition 9. *A crossing cubic structure* $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *over* ϑ *is called a crossing cubic filter of* ϑ *if it satisfies the conditions:* $\forall n, v, w \in \vartheta$,

(i)

$$(\forall n \in \vartheta) \begin{pmatrix} \widetilde{\Upsilon}_{\Pi}(1) \succeq \widetilde{\Upsilon}_{\Pi}(n), \\ \varpi_{\Pi}(1) \leq \varpi_{\Pi}(n) \end{pmatrix};$$

(ii)

$$(\forall n, v \in \vartheta) \begin{pmatrix} \widetilde{\Upsilon}_{\Pi}(n \mid (v \mid v)) \succeq \widetilde{\Upsilon}_{\Pi}(v), \\ \varpi_{\Pi}(n \mid (v \mid v)) \leq \varpi_{\Pi}(v) \end{pmatrix};$$

(iii)

$$(\forall n, v, w \in \vartheta) \begin{pmatrix} \widetilde{\Upsilon}_{\Pi}((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi}(v), \widetilde{\Upsilon}_{\Pi}(w)\}, \\ \varpi_{\Pi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi}(v), \varpi_{\Pi}(w)\} \end{pmatrix}.$$

Example 1. *Let* $\vartheta = \{0, 1, 2_{\vartheta}, 3_{\vartheta}, 4_{\vartheta}, 5_{\vartheta}, 6_{\vartheta}, 7_{\vartheta}\}$ *be a set. The Hasse diagram and Sheffer stroke "*$|$*" on* ϑ*, which are shown in Figure 1 and Table 4, respectively.*

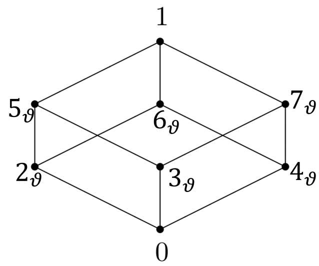

Figure 1: Hasse Diagram

Table 4: Cayley table for the Sheffer stroke "|"

$\mid$	0	2_ϑ	3_ϑ	4_ϑ	5_ϑ	6_ϑ	7_ϑ	1
0	1	1	1	1	1	1	1	1
2_ϑ	1	7_ϑ	1	1	7_ϑ	7_ϑ	1	7_ϑ
3_ϑ	1	1	6_ϑ	1	6_ϑ	1	6_ϑ	6_ϑ
4_ϑ	1	1	1	5_ϑ	1	5_ϑ	5_ϑ	5_ϑ
5_ϑ	1	7_ϑ	6_ϑ	1	4_ϑ	7_ϑ	6_ϑ	4_ϑ
6_ϑ	1	7_ϑ	1	5_ϑ	7_ϑ	3_ϑ	5_ϑ	3_ϑ
7_ϑ	1	1	6_ϑ	5_ϑ	6_ϑ	5_ϑ	2_ϑ	2_ϑ
1	1	7_ϑ	6_ϑ	5_ϑ	4_ϑ	3_ϑ	2_ϑ	0

Then, $\vartheta_\circ := (\vartheta, |)$ *is a Sheffer stroke Hilbert algebra (see [13]). Let* $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ *be a crossing cubic structure in* ϑ*, which is shown in Table 5.*

Table 5: $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is represented tabularly

ϑ	$\widetilde{\Upsilon}_\Pi(n)$	$\varpi_\Pi(n)$
0	[0.28, 0.65]	-0.41
2_ϑ	[0.28, 0.65]	-0.55
3_ϑ	[0.28, 0.65]	-0.41
4_ϑ	[0.32, 0.73]	-0.41
5_ϑ	[0.28, 0.65]	-0.63
6_ϑ	[0.38, 0.76]	-0.55
7_ϑ	[0.32, 0.73]	-0.41
1	[0.42, 0.91]	-0.71

It is customary to check that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ *is a crossing cubic filter of* ϑ*.*

Proposition 2. *Every crossing cubic filter* $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ *of* ϑ *satisfies:*

(i)

$$(\forall n, v \in \vartheta) \left(\begin{array}{c} \widetilde{\Upsilon}_\Pi((n \mid (v \mid v)) \mid (v \mid v)) \succeq \widetilde{\Upsilon}_\Pi(n), \\ \varpi_\Pi((n \mid (v \mid v)) \mid (v \mid v)) \leq \varpi_\Pi(n) \end{array} \right);$$

(ii)
$$(\forall n, v \in \vartheta)\left(n \leq v \Rightarrow \begin{cases} \widetilde{\Upsilon}_{\Pi}(v) \succeq \widetilde{\Upsilon}_{\Pi}(n), \\ \varpi_{\Pi}(v) \leq \varpi_{\Pi}(n) \end{cases}\right).$$

Proof. Let $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ be a crossing cubic filter of ϑ. Then, by using (5) of Proposition 1 and (iii) of Definition 9, $\forall n, v \in \vartheta$, we have

$$\begin{aligned} \widetilde{\Upsilon}_{\Pi}((n \mid (v \mid v)) \mid (v \mid v)) &= \widetilde{\Upsilon}_{\Pi}((v \mid (n \mid n)) \mid (n \mid n)) \\ &\succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi}(n), \widetilde{\Upsilon}_{\Pi}(n)\} \\ &= \widetilde{\Upsilon}_{\Pi}(n) \end{aligned}$$

and

$$\begin{aligned} \varpi_{\Pi}((n \mid (v \mid v)) \mid (v \mid v)) &= \varpi_{\Pi}((v \mid (n \mid n)) \mid (n \mid n)) \\ &\leq max\{\varpi_{\Pi}(n), \varpi_{\Pi}(n)\} \\ &= \varpi_{\Pi}(n). \end{aligned}$$

Therefore, the proof of (i) is completed. Now, let $n, v \in \vartheta$ be such that $n \leq v$. Then $n \mid (v \mid v) = 1$ by condition (**A**). Using (3) of Proposition 1 and (i) of Proposition 2, we have
$$\widetilde{\Upsilon}_{\Pi}(v) = \widetilde{\Upsilon}_{\Pi}(1 \mid (v \mid v)) = \widetilde{\Upsilon}_{\Pi}((n \mid (v \mid v)) \mid (v \mid v)) \succeq \widetilde{\Upsilon}_{\Pi}(n)$$
and
$$\varpi_{\Pi}(v) = \varpi_{\Pi}(1 \mid (v \mid v)) = \varpi_{\Pi}((n \mid (v \mid v)) \mid (v \mid v)) \leq \varpi_{\Pi}(n).$$
This completes the proof of (ii).

Next example illustrates that if a crossing cubic structure $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ satisfies condition (ii) of Proposition 2, then it is not enough for a crossing cubic filter of ϑ.

Example 2. *Let $\vartheta = \{0, 1, 2_{\vartheta}, 3_{\vartheta}\}$ be a set. The Hasse diagram and Sheffer stroke "$|$" on ϑ, which are shown in Figure 2 and Table 6, respectively.*

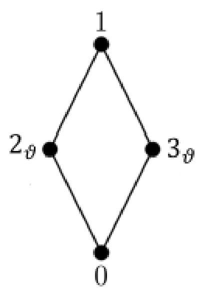

Figure 2: Hasse Diagram

Table 6: Cayley table for the Sheffer stroke "|"

$\mid$	1	2_ϑ	3_ϑ	0
1	0	3_ϑ	2_ϑ	1
2_ϑ	3_ϑ	3_ϑ	1	1
3_ϑ	2_ϑ	1	2_ϑ	1
0	1	1	1	1

Then, $\vartheta_\circ := (\vartheta, |)$ is a Sheffer stroke Hilbert algebra (see [13]). Let $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ be a crossing cubic structure in ϑ which is shown in Table 7.

Table 7: $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is represented tabularly

ϑ	$\widetilde{\Upsilon}_\Pi(n)$	$\varpi_\Pi(n)$
0	[0.29,0.63]	-0.13
2_ϑ	[0.32,0.67]	-0.38
3_ϑ	[0.36,0.75]	-0.55
1	[0.47,0.89]	-0.82

It is customary to check that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ in ϑ satisfies condition (ii) of Proposition 2, but it is not a crossing cubic filter of ϑ since

$$\widetilde{\Upsilon}_\Pi((0 \mid (2_\vartheta \mid 3_\vartheta)) \mid (2_\vartheta \mid 3_\vartheta)) = \widetilde{\Upsilon}_\Pi(0) = [0.29, 0.63] \not\succeq [0.32, 0.67] = \widetilde{min}\{\widetilde{\Upsilon}_\Pi(2_\vartheta), \widetilde{\Upsilon}_\Pi(3_\vartheta)\}$$

or

$$\varpi_\Pi((0 \mid (2_\vartheta \mid 3_\vartheta)) \mid (2_\vartheta \mid 3_\vartheta)) = \varpi_\Pi(0) = -0.13 \not\leq -0.38 = max\{\varpi_\Pi(2_\vartheta), \varpi_\Pi(3_\vartheta)\}.$$

Next, we propose conditions under which a crossing cubic structure $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ.

Theorem 1. *Let $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ be a crossing cubic structure in ϑ. Then, $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ if and only if it satisfies (ii) of Proposition 2 and condition (**B**), where*

$$(\boldsymbol{B})\ (\forall n, v \in \vartheta) \left(\begin{array}{c} \widetilde{\Upsilon}_\Pi((n \mid v) \mid (n \mid v)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(n), \widetilde{\Upsilon}_\Pi(v)\}, \\ \varpi_\Pi((n \mid v) \mid (n \mid v)) \leq max\{\varpi_\Pi(n), \varpi_\Pi(v)\} \end{array} \right).$$

Proof. Let $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ be a crossing cubic filter of ϑ. Proposition 2 establishes the validity of condition (ii). Now, using (1) and (2) of Definition 1, as well as (2) and (3) of Proposition 1, we get

$$\begin{aligned} ((1 \mid 1) \mid (n \mid v)) \mid (n \mid v) &= ((n \mid v) \mid (1 \mid 1)) \mid (n \mid v) \\ &= 1 \mid (n \mid v) \\ &= 1 \mid (((n \mid v) \mid (n \mid v)) \mid ((n \mid v) \mid (n \mid v))) \\ &= (n \mid v) \mid (n \mid v) \end{aligned}$$

$\forall n, v \in \vartheta$, it follows from (iii) of Definition 9 that

$$\widetilde{\Upsilon}_\Pi((n \mid v) \mid (n \mid v)) = \widetilde{\Upsilon}_\Pi(((1 \mid 1) \mid (n \mid v)) \mid (n \mid v)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(n), \widetilde{\Upsilon}_\Pi(v)\}$$

and

$$\varpi_\Pi((n \mid v) \mid (n \mid v)) = \varpi_\Pi(((1 \mid 1) \mid (n \mid v)) \mid (n \mid v)) \leq max\{\varpi_\Pi(n), \varpi_\Pi(v)\}.$$

In contrast, let us assume that a crossing cubic structure $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ satisfies (ii) of Proposition 2 and the condition (**B**). Since $n \leq 1$ and $v \leq n \mid (v \mid v) \forall n, v \in \vartheta$, using (ii) of Proposition 2, we have

$$(\forall n, v \in \vartheta) \begin{pmatrix} \widetilde{\Upsilon}_\Pi(1) \succeq \widetilde{\Upsilon}_\Pi(n), \\ \varpi_\Pi(1) \leq \varpi_\Pi(n) \end{pmatrix} \text{and} \begin{pmatrix} \widetilde{\Upsilon}_\Pi(n \mid (v \mid v)) \succeq \widetilde{\Upsilon}_\Pi(v), \\ \varpi_\Pi(n \mid (v \mid v)) \leq \varpi_\Pi(v) \end{pmatrix},$$

respectively. In (4) of Proposition 1, if $\mathfrak{n} = (v \mid w) \mid (v \mid w)$ and $\mathfrak{v} = n \mid (v \mid w)$, and use (2) of Definition 1, then $\forall n, v, w \in \vartheta$,

$$\begin{aligned} (v \mid w) \mid (v \mid w) &\leq (n \mid (v \mid w)) \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w))) \\ &= (n \mid (v \mid w)) \mid (v \mid w). \end{aligned}$$

Using (ii) of Proposition 2 and the condition (**B**), we have

$$\widetilde{\Upsilon}_\Pi((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{\Upsilon}_\Pi((v \mid w) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}_\Pi(w)\}$$

and

$$\varpi_\Pi((n \mid (v \mid w)) \mid (v \mid w)) \leq \varpi_\Pi((v \mid w) \mid (v \mid w)) \leq max\{\varpi_\Pi(v), \varpi_\Pi(w)\}.$$

Therefore, $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ.

Theorem 2. *If $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ, then the nonempty sets $\xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big)$ and $\xi\Big(\varpi_\Pi; \kappa\Big)$ are filters of ϑ, for all $\widetilde{\tau} = [\tau^-, \tau^+] \in \mathbb{I}[0, 1]$ and $\kappa \in [-1, 0]$.*

Proof. Suppose that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ. Let $\widetilde{\tau} = [\tau^-, \tau^+] \in \mathbb{I}[0, 1]$ and $\kappa \in$ [-1, 0] be such that $\xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big)$ and $\xi\Big(\varpi_\Pi; \kappa\Big)$ are nonempty. Clearly, $1 \in \xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big) \cap \xi\Big(\varpi_\Pi; \kappa\Big)$ by (i) of Definition 9. Let $n \in \vartheta$ and $v \in \xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big) \cap \xi\Big(\varpi_\Pi; \kappa\Big)$. Then,

$$\widetilde{\Upsilon}_\Pi(v) \succeq \widetilde{\tau} \text{ and } \varpi_\Phi(v) \leq \kappa.$$

Using (ii) of Definition 9, we have $\widetilde{\Upsilon}_\Pi(n \mid (v \mid v)) \succeq \widetilde{\Upsilon}_\Pi(v) \succeq \widetilde{\tau}$ and $\varpi_\Pi(n \mid (v \mid v)) \leq \varpi_\Pi(v) \leq \kappa$. Hence, $n \mid (v \mid v) \in \xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big) \cap \xi\Big(\varpi_\Pi; \kappa\Big)$. Let $n \in \vartheta$ and $v, w \in \xi\Big(\widetilde{\Upsilon}_\Pi; \widetilde{\tau}\Big) \cap \xi\Big(\varpi_\Pi; \kappa\Big)$. Then $\widetilde{\Upsilon}_\Pi(v) \succeq \widetilde{\tau}, \varpi_\Pi(v) \leq \kappa, \widetilde{\Upsilon}_\Pi(w) \succeq \widetilde{\tau}$ and $\varpi_\Pi(w) \leq \kappa$. It follows from (iii) of Definition 9 that

$$\widetilde{\Upsilon}_\Pi((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}(w)\} \succeq \widetilde{\tau}$$

and

$$\varpi_{\Pi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi}(v), \varpi_{\Pi}(w)\} \leq \kappa.$$

Thus, $(n \mid (v \mid w)) \mid (v \mid w) \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big) \cap \xi\Big(\varpi_{\Pi}; \kappa\Big)$. Therefore, $\xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$ and $\xi\Big(\varpi_{\Pi}; \kappa\Big)$ are filters of ϑ.

Theorem 3. *Given a crossing cubic structure* $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *in* ϑ. *If the nonempty sets* $\xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$ *and* $\xi\Big(\varpi_{\Pi}; \kappa\Big)$ *are filters of* ϑ *for all* $\widetilde{\tau} = [\tau^{-}, \tau^{+}] \in \mathbb{I}[0,1]$ *and* $\kappa \in [-1, 0]$, *then* $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ *is a crossing cubic filter of* ϑ.

Proof. Assume that $\xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$ and $\xi\Big(\varpi_{\Pi}; \kappa\Big)$ are nonempty filters of ϑ for all $\widetilde{\tau} = [\tau^{-}, \tau^{+}] \in \mathbb{I}[0,1]$ and $\kappa \in [-1, 0]$. Let $\mathfrak{n} \in \vartheta$ be such that $\widetilde{\Upsilon}_{\Pi}(\mathfrak{n}) = \widetilde{\tau}$. Then, $\mathfrak{n} \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$ i.e., $\xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big) \neq \phi$, and so $1 \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$. Hence, $\widetilde{\Upsilon}_{\Pi}(1) \succeq \widetilde{\tau} = \widetilde{\Upsilon}_{\Pi}(\mathfrak{n})$. If there is $\mathfrak{n} \in \vartheta$ such that $\varpi_{\Pi}(1) \nleq \varpi_{\Pi}(\mathfrak{n})$, then $\varpi_{\Pi}(1) > \varpi_{\Pi}(\mathfrak{n})$. Hence, $\mathfrak{n} \in \xi\Big(\varpi_{\Pi}; \varpi_{\Pi}(\mathfrak{n})\Big)$ and $1 \notin \xi\Big(\varpi_{\Pi}; \varpi_{\Pi}(\mathfrak{n})\Big)$, a contradiction. Let $\mathfrak{v} \in \vartheta$ be such that $\widetilde{\Upsilon}_{\Pi}(\mathfrak{v}) = \widetilde{\tau}$. Then, $\mathfrak{v} \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$, which implies that $\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big) \forall \mathfrak{n} \in \vartheta$. Hence, $\widetilde{\Upsilon}_{\Pi}(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})) \succeq \widetilde{\tau} = \widetilde{\Upsilon}_{\Pi}(\mathfrak{v})$. If there are $\mathfrak{n}, \mathfrak{v} \in \vartheta$ such that $\varpi_{\Pi}(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})) \nleq \varpi_{\Pi}(\mathfrak{v})$, then $\varpi_{\Pi}(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v})) > \varpi_{\Pi}(\mathfrak{v})$. It follows that $\mathfrak{v} \in \xi\Big(\varpi_{\Pi}; \varpi_{\Pi}(\mathfrak{v})\Big)$ and $\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{v}) \notin \xi\Big(\varpi_{\Pi}; \varpi_{\Pi}(\mathfrak{v})\Big)$, a contradiction. Let $\mathfrak{n}, \mathfrak{v}, \mathfrak{w} \in \vartheta$ be such that $\widetilde{min}\{\widetilde{\Upsilon}_{\Pi}(\mathfrak{v}), \widetilde{\Upsilon}_{\Pi}(\mathfrak{w})\} = \widetilde{\tau}$. Then $\mathfrak{v}, \mathfrak{w} \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$. It follows that $(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w}) \in \xi\Big(\widetilde{\Upsilon}_{\Pi}; \widetilde{\tau}\Big)$. Therefore $\widetilde{\Upsilon}_{\Pi}((\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w})) \succeq \widetilde{\tau} = \widetilde{min}\{\widetilde{\Upsilon}_{\Pi}(\mathfrak{v}), \widetilde{\Upsilon}_{\Pi}(\mathfrak{w})\}$. If there are $\mathfrak{n}, \mathfrak{v}, \mathfrak{w} \in \vartheta$ such that $\varpi_{\Pi}((\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w})) \nleq max\{\varpi_{\Pi}(\mathfrak{v}), \varpi_{\Pi}(\mathfrak{w})\}$, then $\varpi_{\Pi}((\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w})) > max\{\varpi_{\Pi}(\mathfrak{v}), \varpi_{\Pi}(\mathfrak{w})\}$. If we take $\kappa := max\{\varpi_{\Pi}(\mathfrak{v}), \varpi_{\Pi}(\mathfrak{w})\}$, then $\mathfrak{v}, \mathfrak{w} \in \xi\Big(\varpi_{\Pi}; \kappa\Big)$ and $(\mathfrak{n} \mid (\mathfrak{v} \mid \mathfrak{w})) \mid (\mathfrak{v} \mid \mathfrak{w}) \notin \xi\Big(\varpi_{\Pi}; \kappa\Big)$. This is a contradiction. Consequently, $\Pi = (\widetilde{\Upsilon}_{\Pi}, \varpi_{\Pi})$ is a crossing cubic filter of ϑ.

Theorem 4. *Given a nonempty subset* Ψ *of* ϑ, *let* $\Pi_{\Psi} = (\widetilde{\Upsilon}_{\Pi_{\Psi}}, \varpi_{\Pi_{\Psi}})$ *be a crossing cubic structure in* ϑ *defined by*

$$\Pi_{\Psi} = (\widetilde{\Upsilon}_{\Pi_{\Psi}}, \varpi_{\Pi_{\Psi}}) \ : \ \vartheta \to \mathbb{I}[0,1] \times [-1, 0], \quad n \mapsto \begin{cases} (\widetilde{\tau}, \kappa) \ \text{if } n \in \Psi, \\ (\widetilde{\alpha}, \beta) \ \text{otherwise}, \end{cases}$$

where $(\widetilde{\tau}, \kappa), (\widetilde{\alpha}, \beta) \in \mathbb{I}[0,1] \times [-1, 0]$ *such that* $\widetilde{\tau} \succ \widetilde{\alpha}$ *and* $\kappa < \beta$. *Then,* $\Pi_{\Psi} = (\widetilde{\Upsilon}_{\Pi_{\Psi}}, \varpi_{\Pi_{\Psi}})$ *is a crossing cubic filter of* ϑ *if and only if* Ψ *is a filter of* ϑ. *Also,* $\Psi = \vartheta_{\Pi_{\Psi}}$, *where*

$$\vartheta_{\Pi_{\Psi}} = \{n \in \vartheta \mid \widetilde{\Upsilon}_{\Pi_{\Psi}}(n) = \widetilde{\Upsilon}_{\Pi_{\Psi}}(1), \ \text{and } \varpi_{\Pi_{\Psi}}(n) = \varpi_{\Pi_{\Psi}}(1)\}.$$

Proof. Assume that $\Pi_\Psi = (\widetilde{\Upsilon}_{\Pi_\Psi}, \varpi_{\Pi_\Psi})$ is a crossing cubic filter of ϑ. Then, $\widetilde{\Upsilon}_{\Pi_\Psi}(1) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}(1) = \kappa$ by (i) of Definition 9, and so $1 \in \Psi$. Let $n, v \in \vartheta$ be such that $v \in \Psi$. Then, $\widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}(v) = \kappa$. It follows from (ii) of Definition 9 that $\widetilde{\Upsilon}_{\Pi_\Psi}(n \mid (v \mid v)) \succeq \widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}(n \mid (v \mid v)) \leq \varpi_{\Pi_\Psi}(v) = \kappa$. Hence, $\widetilde{\Upsilon}_{\Pi_\Psi}(n \mid (v \mid v)) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}(n \mid (v \mid v)) = \kappa$. This illustrates that $n \mid (v \mid v) \in \Psi$. Let $v, w \in \Psi$. Then, $\widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\tau} = \widetilde{\Upsilon}_{\Pi_\Psi}(w)$ and $\varpi_{\Pi_\Psi}(v) = \kappa = \varpi_{\Pi_\Psi}(w)$. Using (iii) of Definition 9 and for $n \in \vartheta$, we have

$$\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi_\Psi}(v), \widetilde{\Upsilon}_{\Pi_\Psi}(w)\} = \widetilde{\tau}$$

and

$$\varpi_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi_\Psi}(v), \varpi_{\Pi_\Psi}(w)\} = \kappa.$$

It follows that $\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) = \kappa$. Hence, $(n \mid (v \mid w)) \mid (v \mid w) \in \Psi$. Therefore, Ψ is a filter of ϑ.

Conversely, let Ψ be a filter of ϑ. Since $1 \in \Psi$, we have $\widetilde{\Upsilon}_{\Pi_\Psi}(1) = \widetilde{\tau} \succeq \widetilde{\Upsilon}_{\Pi_\Psi}(n)$ and $\varpi_{\Pi_\Psi}(1) = \kappa \leq \varpi_{\Pi_\Psi}(n)$ for all $n \in \vartheta$. Let $n, v \in \vartheta$. Then, we have two cases:
Case(1). If $v \in \Psi$, then $n \mid (v \mid v) \in \Psi$ and thus

$$\widetilde{\Upsilon}_{\Pi_\Psi}(n \mid (v \mid v)) = \widetilde{\tau} = \widetilde{\Upsilon}_{\Pi_\Psi}(v) \text{ and } \varpi_{\Pi_\Psi}(n \mid (v \mid v)) = \kappa = \varpi_{\Pi_\Psi}(v).$$

Case(2). If $v \notin \Psi$, then $\widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\alpha}$ and $\varpi_{\Pi_\Psi}(v) = \beta$. Hence,

$$\widetilde{\Upsilon}_{\Pi_\Psi}(n \mid (v \mid v)) \succeq \widetilde{\Upsilon}_{\Pi_\Psi}(v) \text{ and } \varpi_{\Pi_\Psi}(n \mid (v \mid v)) \leq \varpi_{\Pi_\Psi}(v).$$

Moreover, let $n, v, w \in \vartheta$. Then, we have the following cases:
Case(1). If $v, w \in \Psi$. Then, $(n \mid (v \mid w)) \mid (v \mid w) \in \Psi$. Thus,

$$\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) = \widetilde{\tau} = \widetilde{min}\{\widetilde{\Upsilon}_{\Pi_\Psi}(v), \widetilde{\Upsilon}_{\Pi_\Psi}(w)\}$$

and

$$\varpi_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) = \kappa = max\{\varpi_{\Pi_\Psi}(v), \varpi_{\Pi_\Psi}(w)\}.$$

Case(2). If $v, w \notin \Psi$. Then,

$$\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi_\Psi}(v), \widetilde{\Upsilon}_{\Pi_\Psi}(w)\} = \widetilde{\alpha}$$

and

$$\varpi_{\Pi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi}(v), \varpi_{\Pi}(w)\} = \beta.$$

Case(3). If $v \in \Psi$ and $w \notin \Psi$, then $\widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\tau}, \varpi_{\Pi_\Psi}(v) = \kappa, \widetilde{\Upsilon}_{\Pi_\Psi}(w) = \widetilde{\alpha}$ and $\varpi_{\Pi_\Psi}(w) = \beta$. So,

$$\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi_\Psi}(v), \widetilde{\Upsilon}_{\Pi_\Psi}(w)\} = \widetilde{\alpha}$$

and

$$\varpi_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi_\Psi}(v), \varpi_{\Pi_\Psi}(w)\} = \beta.$$

Case(4). If $v \notin \Psi$ and $w \in \Psi$, then $\widetilde{\Upsilon}_{\Pi_\Psi}(v) = \widetilde{\alpha}, \varpi_{\Pi_\Psi}(v) = \beta, \widetilde{\Upsilon}_{\Pi_\Psi}(w) = \widetilde{\tau}$ and $\varpi_{\Pi_\Psi}(w) = \kappa$. So,

$$\widetilde{\Upsilon}_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \succeq \widetilde{min}\{\widetilde{\Upsilon}_{\Pi_\Psi}(v), \widetilde{\Upsilon}_{\Pi_\Psi}(w)\} = \widetilde{\alpha}$$

and

$$\varpi_{\Pi_\Psi}((n \mid (v \mid w)) \mid (v \mid w)) \leq max\{\varpi_{\Pi_\Psi}(v), \varpi_{\Pi_\Psi}(w)\} = \beta.$$

Hence, $\Pi_\Psi = (\widetilde{\Upsilon}_{\Pi_\Psi}, \varpi_{\Pi_\Psi})$ is a crossing cubic filter of ϑ. Since Ψ is a filter of ϑ, we get

$$\begin{aligned} \vartheta_{\Pi_\Psi} &= \{n \in \vartheta \mid \widetilde{\Upsilon}_{\Pi_\Psi}(n) = \widetilde{\Upsilon}_{\Pi_\Psi}(1), \text{ and } \varpi_{\Pi_\Psi}(n) = \varpi_{\Pi_\Psi}(1)\} \\ &= \{n \in \vartheta \mid \widetilde{\Upsilon}_{\Pi_\Psi}(n) = \widetilde{\tau}, \text{ and } \varpi_{\Pi_\Psi}(n) = \kappa\} \\ &= \{n \in \vartheta | n \in \Psi\} \\ &= \Psi. \end{aligned}$$

4. Crossing Cubic Deductive Systems

Here, we present the idea of crossing cubic deductive system and give an example on it. Then, we prove that a crossing cubic deductive system and a crossing cubic filter are a complementary notion.

Definition 10. *A crossing cubic structure $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is called a crossing cubic deductive system of ϑ if it satisfies (i) of Definition 9 and the condition (**C**), where*

$$(\boldsymbol{C}) \quad (\forall n, v \in \vartheta) \left(\begin{array}{c} \widetilde{\Upsilon}_\Pi(v) \succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(n), \widetilde{\Upsilon}_\Pi(n \mid (v \mid v))\}, \\ \varpi_\Pi(v) \leq max\{\varpi_\Pi(n), \varpi_\Pi(n \mid (v \mid v))\} \end{array} \right).$$

Example 3. *Consider the Sheffer stroke Hilbert algebra $\vartheta_\circ := (\vartheta, |)$ in Example 1 and let $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ be a crossing cubic structure in ϑ, which is displayed in Table 8.*

Table 8: $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is represented tabularly

ϑ	$\widetilde{\Upsilon}_\Pi(n)$	$\varpi_\Pi(n)$
0	[0.19, 0.54]	-0.37
2_ϑ	[0.24, 0.62]	-0.57
3_ϑ	[0.19, 0.54]	-0.37
4_ϑ	[0.19, 0.54]	-0.37
5_ϑ	[0.26, 0.63]	-0.61
6_ϑ	[0.24, 0.62]	-0.57
7_ϑ	[0.19, 0.54]	-0.37
1	[0.41, 0.87]	-0.69

It is routine to check that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic deductive system of ϑ.

Next theorem proves that a crossing cubic deductive system and a crossing cubic filter are a complementary notion.

Theorem 5. *A crossing cubic structure* $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ *in* ϑ *is a crossing cubic deductive system of* ϑ *if and only if it is a crossing cubic filter of* ϑ*.*

Proof. Assume that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic deductive system of ϑ and let $n, v, w \in \vartheta$. Using the condition (**A**) and (4) of Proposition 1 induces $v \mid ((n \mid (v \mid v)) \mid (n \mid (v \mid v))) = 1$. It follows from (i) of Definition 9 and the condition (**C**) that

$$\begin{aligned}\widetilde{\Upsilon}_\Pi(n \mid (v \mid v)) &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}_\Pi(v \mid ((n \mid (v \mid v)) \mid (n \mid (v \mid v))))\}\\ &= \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}_\Pi(1)\} = \widetilde{\Upsilon}_\Pi(v)\end{aligned}$$

and

$$\begin{aligned}\varpi_\Pi(n \mid (v \mid v)) &\leq max\{\varpi_\Pi(v), \varpi_\Pi(v \mid ((n \mid (v \mid v)) \mid (n \mid (v \mid v))))\}\\ &= max\{\varpi_\Pi(v), \varpi_\Pi(1)\} = \varpi_\Pi(v).\end{aligned}$$

Note that by (2) of Definition 1, (1) and (7) of Proposition 1, we have

$$\begin{aligned}v \mid (((v \mid w) \mid w) \quad &\mid \quad ((v \mid w) \mid w))\\ &= v \mid (((v \mid w) \mid ((w \mid w) \mid (w \mid w))) \mid ((v \mid w) \mid ((w \mid w) \mid (w \mid w))))\\ &= (v \mid w) \mid ((v \mid ((w \mid w) \mid (w \mid w))) \mid (v \mid ((w \mid w) \mid (w \mid w))))\\ &= (v \mid w) \mid ((v \mid w) \mid (v \mid w)) = 1.\end{aligned}$$

Using (i) of Definition 9 and the condition (**C**), we get

$$\begin{aligned}\widetilde{\Upsilon}_\Pi((v \mid w) \mid w) &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}_\Pi(v \mid (((v \mid w) \mid w) \mid ((v \mid w) \mid w)))\}\\ &= \widetilde{min}\{\widetilde{\Upsilon}_\Pi(v), \widetilde{\Upsilon}_\Pi(1)\} = \widetilde{\Upsilon}_\Pi(v)\end{aligned}$$

and

$$\begin{aligned}\varpi_\Pi((v \mid w) \mid w) &\leq max\{\varpi_\Pi(v), \varpi_\Pi(v \mid (((v \mid w) \mid w) \mid ((v \mid w) \mid w)))\}\\ &= max\{\varpi_\Pi(v), \varpi_\Pi(1)\} = \varpi_\Pi(v).\end{aligned}$$

Since by (1) and (2) of Definition 1, we get

$$w \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w))) = w \mid (v \mid w) = (v \mid w) \mid w.$$

Then, we obtain

$$\begin{aligned}\widetilde{\Upsilon}_\Pi((v \mid w) \mid (v \mid w)) &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(w), \widetilde{\Upsilon}_\Pi(w \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w))))\}\\ &= \widetilde{min}\{\widetilde{\Upsilon}_\Pi(w), \widetilde{\Upsilon}_\Pi((v \mid w) \mid w)\}\\ &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(w), \widetilde{\Upsilon}_\Pi(v)\}\end{aligned}$$

and

$$\begin{aligned}\varpi_\Pi((v \mid w) \mid (v \mid w)) &\leq max\{\varpi_\Pi(w), \varpi_\Pi(w \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w))))\} \\ &= max\{\varpi_\Pi(w), \varpi_\Pi((v \mid w) \mid w)\} \\ &\leq max\{\varpi_\Pi(w), \varpi_\Pi(v)\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\widetilde{\Upsilon}_\Pi((n \mid (v \mid w)) \mid (v \mid w)) \\ &= \widetilde{\Upsilon}_\Pi((n \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w)))) \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w)))) \\ &\succeq \widetilde{\Upsilon}_\Pi((v \mid w) \mid (v \mid w)) \\ &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(w), \widetilde{\Upsilon}_\Pi(v)\}\end{aligned}$$

and

$$\begin{aligned}&\varpi_\Pi((n \mid (v \mid w)) \mid (v \mid w)) \\ &= \varpi_\Pi((n \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w)))) \mid (((v \mid w) \mid (v \mid w)) \mid ((v \mid w) \mid (v \mid w)))) \\ &\leq \varpi_\Pi((v \mid w) \mid (v \mid w)) \\ &\leq max\{\varpi_\Pi(w), \varpi_\Pi(v)\}\end{aligned}$$

Consequently, $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ.

Conversely, suppose that $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic filter of ϑ. By (1), (2) and (3) of Definition 1; and (1), (2), (3), (5) and (6) of Proposition 1, and for every $n, v \in \vartheta$, we have

$$\begin{aligned}v &= ((n \mid n) \mid (1 \mid 1)) \mid (v \mid v) \\ &= ((n \mid n) \mid ((v \mid (v \mid v)) \mid (v \mid (v \mid v)))) \mid (v \mid v) \\ &= ((((n \mid n) \mid v) \mid ((n \mid n) \mid v)) \mid (v \mid v)) \mid (v \mid v) \\ &= (v \mid ((n \mid n) \mid v)) \mid ((n \mid n) \mid v) \\ &= ((((n \mid n) \mid v) \mid v) \mid v) \mid (((n \mid n) \mid v) \mid v) \\ &= (v \mid (n \mid (n \mid (v \mid v)))) \mid (n \mid (n \mid (v \mid v))).\end{aligned}$$

It follows from (iii) of Definition 9 that

$$\begin{aligned}\widetilde{\Upsilon}_\Pi(v) &= \widetilde{\Upsilon}_\Pi((v \mid (n \mid (n \mid (v \mid v)))) \mid (n \mid (n \mid (v \mid v)))) \\ &\succeq \widetilde{min}\{\widetilde{\Upsilon}_\Pi(n), \widetilde{\Upsilon}_\Pi(n \mid (v \mid v))\}\end{aligned}$$

and

$$\begin{aligned}\varpi_\Pi(v) &= \varpi_\Pi((v \mid (n \mid (n \mid (v \mid v)))) \mid (n \mid (n \mid (v \mid v)))) \\ &\leq max\{\varpi_\Pi(n), \varpi_\Pi(n \mid (v \mid v))\}.\end{aligned}$$

Therefore $\Pi = (\widetilde{\Upsilon}_\Pi, \varpi_\Pi)$ is a crossing cubic deductive system of ϑ.

Remark 1. *By Theorem 5, the crossing cubic deductive system can handle all of the results for the crossing cubic filter covered above in the same way.*

5. Conclusion

The crossing cubic structure creates new opportunities to introduce new tools, allowing experiments and studies that were previously not possible through broad applications in various fields. Combining the interval-value fuzzy set and N-structure leads to acquiring the crossing cubic structure. We utilized the crossing cubic structure on filters and deductive systems in Sheffer stroke Hilbert algebras. Also, we defined the notions of the crossing cubic filter and crossing cubic deductive system. Then, we verified many characteristics of these notions. We established conditions suitable for the crossing cubic structure to be a crossing cubic filter. The crossing cubic filters also received further characterization theorems. To create a relationship between crossing cubic filters and filters, we generated crossing cubic filters that are related with filters. We demonstrated that the crossing cubic deductive system can handle all of the results for the crossing cubic filter in the same way.

This work is designed to stimulate more research on crossing cubic structures, resulting in fresh and unfamiliar findings. This opens up possibilities for additional research and applications. The concept of crossing cubic structures can be applied in the future. Their application extends to numerous algebraic structures such as PI-algebras, Lie algebras, lattices, BCK/BCI-algebras, Hopf algebras, etc. In addition, we believe that the potential of this work will be used in database theory, probability theory, and other fields.

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