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General Deformations of Sprays on Finsler Manifolds

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Abstract. In this paper, we investigate the concept of general deformations of a spray S on a manifold M. We then focus on a specific case, which we call a projective-like deformation. This type of deformation extends the notion of projective deformation but, unlike projective deformation, it does not necessarily preserve geodesics. We derive an explicit formula for the Jacobi endomorphism under projective-like deformations and analyze the conditions under which it remains invariant. As applications, we consider (α, β)-metrics and spherically symmetric metrics. We find a necessary and sufficient condition for an (α, β)-metric and the Riemannian metric α to be projectively related. Additionally, we provide and examine several explicit examples.

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Key Words and Phrases: Sprays, projective deformation, projective-like deformation, Jacobi endomorphism

1. Introduction

In Finsler geometry, the concepts of sprays and its projective deformation play a crucial role in understanding how the geodesic structure of a manifold changes under transformations that preserve the projective class of paths. A projective deformation of a spray refers to a modification of the spray such that the new spray generates the same set of unparameterized geodesics as the original one. This concept is closely related to projective geometry, where only the direction of geodesics matters rather than their specific parameterization. For example, see [1–11].

A projective deformation [5, 7, 10, 12] of S results in a new spray $\overline{S}$ given by:

$$\overline{S} = S - 2\mathcal{P}(x, y)\mathcal{C}, \tag{1}$$

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where $\mathcal{P}(x,y)$ is a scalar function known as the projective factor, and $\mathcal{C}$ is the canonical Liouville vector field. This transformation ensures that the integral curves of S and $\overline{S}$ remain the same up to reparameterization, meaning the geodesics of the manifold remain unchanged as point sets.

Projective deformations [1, 7] are particularly useful in Finsler geometry for studying conditions under which a given Finsler metric can be related to another one via projectively equivalent sprays. These transformations are fundamental in various areas, including the study of curvature properties, and projective flatness.

The theory of projective changes in Riemannian geometry has been deeply studied (*locally and intrinsically*) by many authors. As regards to Finsler geometry, a complete local theory of projective changes has been established ([5, 7, 8, 13, 14]). Moreover, an intrinsic theory of projective changes (resp. semi-projective changes) has been investigated in [10, 15]) following the KG-approach.

In this paper, considering two sprays S_1 and S_2 on a manifold M, and using the fact that that the difference $S_1 - S_2$ is always vertical, we introduce the general deformation of a spray S as follows:

$$\widetilde{S} = S - 2\zeta,$$

where ζ is a vertical vector field $\zeta \in \mathfrak{X}^v(TM)$. Then, we focus our attentions to an interesting special case. That is, we introduce what we called projective-like deformation. Precisely, let S be a spray on a manifold M, and consider any vertical vector field $\xi \in \mathfrak{X}^v(TM)$. The projective-like deformation of S is given by

$$\widetilde{S} = S - 2\mathcal{P}(x,y)\xi,$$

where $\mathcal{P}(x,y) \in C^\infty(\mathcal{T}M)$ is a smooth, and positively homogeneous function of degree 1 in y, and called the deformation factor.

We consider two special classes of Finsler metrics, namely, the class of (α,β)-metrics and the class of spherically symmetric metrics. For these classes, we figure out the relation between the background spray and the new one showing the explicit formula of the vertical vector which identifies the difference between the two sprays. Moreover, As by-product, we discuss when the background spray and new one are projective. For example, for an (α,β)-metric $F = \alpha\phi(s)$, then the associated sprays with F and the Riemannian metric α are projectively related if and only if

$$2\alpha\phi' s^i_0 + r_{00}\phi'' b^i - r_{00}\phi'' \frac{\beta}{\alpha^2} y^i = 0.$$

Under the projective-like deformation, we establish the relation between the two associated Jacobi endomorphisms. Moreover, we discuss the invariance property of the jacobi endomorphism. Some explicit examples are studied.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with its tangent bundle represented by (TM, π_M, M), and let $(\mathcal{T}M, \pi, M)$ denote the subbundle comprising nonzero tangent vectors. The local coordinates on M are expressed as (x^i), while the associated coordinates on TM are (x^i, y^i), where y^i are the components of the tangent vectors. The tangent bundle TM is naturally equipped with an almost-tangent structure J, locally given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$. Let's recall some basics and properties of the Klein-Grifone approach to Finsler geometry. The canonical (or Liouville) vector field $\mathcal{C}$ on TM is a vertical vector field defined as:

$$\mathcal{C} = y^i \frac{\partial}{\partial y^i}. \tag{2}$$

More details can be found in [1, 9, 16].

A spray S is a vector field $S \in \mathfrak{X}(\mathcal{T}M)$ that satisfies $JS = \mathcal{C}$ and $[\mathcal{C}, S] = S$. Locally, it has the expression:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{3}$$

where $G^i = G^i(x, y)$, known as the spray coefficients, are functions that exhibit 2-homogeneity in y.

A nonlinear connection is an n-dimensional distribution $H(\mathcal{T}M)$ that serves as a complement to the vertical distribution $V(\mathcal{T}M) := \ker \pi_*$. This implies that, for each $z \in \mathcal{T}M$, the tangent space at z decomposes as:

$$T_z(\mathcal{T}M) = H_z(\mathcal{T}M) \oplus V_z(\mathcal{T}M). \tag{4}$$

Each spray S naturally induces a nonlinear connection $\Gamma = [J, S]$ (see [16]) characterized by horizontal and vertical projectors:

$$h = \frac{1}{2}(\mathrm{Id} + [J, S]), \quad v = \frac{1}{2}(\mathrm{Id} - [J, S]). \tag{5}$$

Locally, these projectors take the forms:

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \tag{6}$$

where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y)\frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N^j_i(x, y)dx^i, \quad N^j_i(x, y) = \frac{\partial G^j}{\partial y^i}. \tag{7}$$

For a vector k-form K on M, the graded derivations i_K and d_K on the Grassmann algebra of M are given by [17]:

$$i_K f = 0, \quad i_K df = df \circ K, \tag{8}$$

where $f \in C^\infty(M)$ and df represents the exterior derivative of f. Furthermore, the derivation d_K is defined as:

$$d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d i_K. \tag{9}$$

The Jacobi endomorphism (or Riemann curvature, as seen in [9]) Φ is formulated as:

$$\Phi = v \circ [S, h] = R^i{}_j \frac{\partial}{\partial y^i} \otimes dx^j, \tag{10}$$

where:

$$R^i{}_j = 2\frac{\partial G^i}{\partial x^j} - S(N^i_j) - N^i_k N^k_j. \tag{11}$$

The curvature tensor R of a spray S is given by:

$$R = -\frac{1}{2}[h, h] = R^\ell_{ij} \frac{\partial}{\partial y^\ell} \otimes dx^i \otimes dx^j,$$

with:

$$R^\ell_{ij} = \frac{\delta G^\ell_i}{\delta x^j} - \frac{\delta G^\ell_j}{\delta x^i}. \tag{12}$$

The two curvature tensors are related as follows (see, for example, [3]):

$$3R = [J, \Phi], \quad \Phi = i_S R. \tag{13}$$

Definition 1. *A Finsler manifold (M, F) consists of an n-dimensional manifold M along with a function $F : TM \to \mathbb{R}$ that meets the following conditions:*

a) *F is smooth and strictly positive on $\mathcal{T}M$.*

b) *F is positively homogeneous of degree 1 in y: $\mathcal{L}_\mathcal{C} F = F$.*

c) *The Hessian matrix $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ has full rank n on $\mathcal{T}M$, where $E = \frac{1}{2}F^2$.*

The geodesic spray S associated with F is determined uniquely by the Euler-Lagrange equation:

$$i_S dd_J E = -dE.$$

For simplicity, we use the notations

$$\delta_i := \frac{\delta}{\delta x^i}, \quad \partial_i := \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}.$$

For a Finsler manifold (M, F), the coefficients G^i of the geodesic spray of F are given by

$$G^i = \frac{1}{4} g^{ih} (y^r \partial_r \dot{\partial}_h F^2 - \partial_h F^2), \tag{14}$$

where g^{ij} are the components of the inverse metric tensor.

2.1. Projective deformations of sprays

The concepts of sprays and its projective deformation play a crucial role in understanding how the geodesic structure of a manifold changes under transformations that preserve the projective class of paths. A complete local theory of projective changes has been established ([5, 7, 8, 13, 14]).

A regular curve is a curve $\gamma\colon I \to M$ with the tangent lift $\gamma'\colon I \to TM$. A regular curve γ on M is a geodesic of a spray S if it satisfies the condition

$$S \circ \gamma' = \gamma''.$$

Locally, if $\gamma(t) = (x^i(t))$, then the geodesic equation is given by:

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{15}$$

In other words, the curve $\gamma(t)$ is a geodesic of S if its velocity $y^i = \frac{dx^i}{dt}$ satisfies the above equation.

Two sprays, S and $\overline{S}$, are said to be projectively related if their geodesics remain unchanged, up to an orientation-preserving reparameterization. In this case, one spray can be regarded as a projective deformation of the other.

Locally, (1) modifies the spray coefficients as follows:

$$\overline{G}^i = G^i + \mathcal{P}y^i.$$

To see why the geodesics remain unchanged under this deformation, consider the new geodesic equation:

$$\frac{d^2x^i}{dt^2} + 2\overline{G}^i\left(x, \frac{dx}{dt}\right) = 0.$$

Expanding $\overline{G}^i$, we obtain:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) + 2\mathcal{P}\left(x, \frac{dx}{dt}\right)\frac{dx^i}{dt} = 0. \tag{16}$$

Now, consider a reparameterization $\tilde{t} = \tilde{t}(t)$ with $\frac{d\tilde{t}}{dt} > 0$, then (15) takes the form

$$\frac{d^2x^i}{d\tilde{t}^2} + 2G^i\left(x, \frac{dx}{d\tilde{t}}\right) + \frac{\frac{d^2\tilde{t}}{dt^2}}{\left(\frac{d\tilde{t}}{dt}\right)^2}\frac{dx^i}{d\tilde{t}} = 0. \tag{17}$$

This equation suggests that the new geodesic parameter $\tilde{t}$ is related to the original parameter t by a reparameterization. Specifically, the function $\mathcal{P}\left(x, \frac{dx}{dt}\right)$ induces a reparameterization of the geodesic equation, but the actual path traced by the geodesics remains unchanged. To see this explicitly, comparing (16) and (17), then the reparameterization $\tilde{t} = \tilde{t}(t)$ satisfies

$$\frac{d^2\tilde{t}}{dt^2} = 2\mathcal{P}\left(x, \frac{dx}{dt}\right)\frac{d\tilde{t}}{dt}.$$

The key observation is that a projective deformation modifies the acceleration along the geodesics but does not alter their direction. The new spray $\overline{S}$ still determines the same set of geodesics in terms of their geometric trajectories, though their parameterization may differ.

3. General deformations of sprays

For any two sprays S_1 and S_2 on a manifold M, one can see that the difference $S_1 - S_2$ is always vertical. Indeed,

$$J(S_1 - S_2) = JS_1 - JS_2 = \mathcal{C} - \mathcal{C} = 0.$$

Using the fact that a vector field X on TM is vertical if and only if $JX = 0$, we conclude that $S_1 - S_2$ is vertical.

Now, using the fact that the difference of two sprays is vertical, we can define the general deformation of a spray S as follows.

Definition 2. *Let S be a spray on a manifold M, and ζ be a vertical vector field $\zeta \in \mathfrak{X}^v(TM)$. The general deformation of S is given by*

$$\widetilde{S} = S - 2\zeta. \tag{18}$$

Moreover, the coefficients of the two sprays are related by

$$\widetilde{G}^i = G^i + \zeta^i. \tag{19}$$

By making use of (16) and (19), we have the following property: The general deformation (18) of S is projective, i.e., preserves geodesics, if and only if the vertical vector ζ is proportional to the Liouville vector field $\mathcal{C}$ (or equivalently, ζ^i is proportional to y^i).

For a given Finsler manifold (M, F), any deforamtion of F implies a deformation of the geodesic spray. As an example, let's consider the deformation of a a geodesic spray of a Finsler manifold (M, F) due to a conformal change of the Finsler metric F.

Example 1. *Consider the conformal transformation of a Finsler manifold (M, F) with the geodesic spray S*

$$\widetilde{F} = e^{\sigma(x)} F.$$

So the geodesic spray S is deformed as follows:

$$\widetilde{S} = S - 2\zeta.$$

By [18], the spray coefficients are related by

$$\widetilde{G}^i = G^i + \zeta^i = G^i + \frac{1}{2} F^2 \sigma^i - \sigma_0 y^i,$$

where $\sigma_r := \partial_r \sigma$, and $\sigma_0 := \sigma_r y^r$.

Now, using the facts that $\mathcal{C} = y^i\dot{\partial}_i$ and $\ell^i = \frac{y^i}{F}$, we have

$$\zeta = \zeta^i\dot{\partial}_i = \frac{F^2}{2}\left(g^{ir} - 2\ell^i\ell^r\right)\sigma_r(x)\dot{\partial}_i = \frac{F^2}{2}g^{ir}\sigma_r(x)\dot{\partial}_i - \sigma_0(x)\mathcal{C} = \frac{1}{2}F^2\sigma^i\dot{\partial}_i - \sigma_0\mathcal{C},$$

where

$$\zeta^i = \frac{F^2}{2}\left(g^{ir} - 2\ell^i\ell^r\right)\sigma_r(x) = \frac{1}{2}F^2\sigma^i - \sigma_0 y^i.$$

Moreover, the conformal change is projective, i.e., preserves the geodesics of S, if and only if $\zeta = \mathcal{C}$. This occurs if and only if $\sigma^i(x) = 0$. That is, σ is constant.

3.1. Projective-like deformation

One the special and important deformation of a spray is the projective deformation, that is, $\overline{S} = S - 2\mathcal{P}(x,y)\mathcal{C}$. Under the projective deformation, the geodesics of a spray are preserved. So, in this subsection we consider another point of view of a more general deformation than the projective deformation but still a special case of the general deformation of a spray S. In this section, we introduce the porjective-like deformation as follows:

Definition 3. *Let S be a spray on a manifold M, and consider any vertical vector field $\xi \in \mathfrak{X}^v(TM)$. The projective-like deformation of S is given by*

$$\widetilde{S} = S - 2\mathcal{P}(x,y)\xi. \tag{20}$$

where $\mathcal{P}(x,y) \in C^\infty(\mathcal{T}M)$ is a smooth, and positively homogeneous function of degree 1 in y, and called the deformation factor. Moreover, the coefficients of the two sprays are related by

$$\widetilde{G}^i = G^i + \mathcal{P}\xi^i. \tag{21}$$

In a similar manner to what we discuss in Property 3, if $\xi := \mathcal{C}$, then the porjective-like spray deformation (20) reduces to the projective deformation, and $\mathcal{P}(x,y)$ is called projective factor. It is clear that, as $\widetilde{S}$ is a spray, then ξ is homogenous of degree one, that is

$$[\mathcal{C}, \xi] = 0. \tag{22}$$

Taking the following vector one form $\mu := [J, \xi]$, we have:

Lemma 1. *The endomorphism μ has the following properties*

$$\mu J = 0, \quad J\mu = 0, \quad \mu^2 = 0, \quad \Gamma\mu + \mu\Gamma = 0.$$

Proof. Since $J\widetilde{\Gamma} = J\Gamma = J$, then

$$\widetilde{\Gamma} = J(\Gamma + \mu) = J + J\mu = J \Longrightarrow J\mu = 0.$$

Similarly, using the fact that $\widetilde{\Gamma}J = \Gamma J = -J$, we get $\mu J = 0$. Therefore, we conclude that $\mu^2 = 0$. Since $\widetilde{\Gamma}^2 = I$ and $\Gamma^2 = I$, then we obtain that $\Gamma\mu + \mu\Gamma = 0$.

Proposition 1. *Under the projective-like deformation (20) of a spray S with deformation factor $\mathcal{P}(x,y)$, the Barthel connection $\widetilde{\Gamma}$ has the form*

$$\widetilde{\Gamma} = \Gamma - 2(\mathcal{P}\mu + d_J\mathcal{P} \otimes \xi). \tag{23}$$

Consequently, the horizontal and vertical projections $\widetilde{h}$ and $\widetilde{v}$ becomes

$$\widetilde{h} = h - \mathbb{L}, \qquad \widetilde{v} = v + \mathbb{L},$$

where, $\mathbb{L}$ is the vector 1-form defined by

$$\mathbb{L} := \mathcal{P}\mu + d_J\mathcal{P} \otimes \xi. \tag{24}$$

Proof. Using the formula [17]

$$[J, fX] = f[J,X] + d_Jf \otimes X - df \otimes JX,$$

and the fact that ξ is a vertical vector field, we obtain

$$\widetilde{\Gamma} = [J, \widetilde{S}] = [J, S - 2\mathcal{P}\xi] = [J,S] - 2[J,\mathcal{P}\xi] = \Gamma - 2(\mathcal{P}\mu + d_J\mathcal{P} \otimes \xi).$$

Hence, using the facts $\widetilde{h} = \frac{1}{2}(I + \widetilde{\Gamma})$, and $\widetilde{v} = \frac{1}{2}(I - \widetilde{\Gamma})$, the proof completes.

Proposition 2. *Under the projective-like deformation (20) of a spray S with deformation factor $\mathcal{P}(x,y)$. The Jacobi endomorphism $\widetilde{\Phi}$ is given by*

$$\begin{aligned}
\widetilde{\Phi} \;=\; & \Phi - (\mathcal{P}^2\mu[S,\mu] + S(\mathcal{P})\mu) + (2d_h\mathcal{P} - \mathcal{P}d_\mu\mathcal{P} - \nabla d_J\mathcal{P}) \otimes \xi \\
& -\mathcal{P}\, v[S,\mu] - d_J\mathcal{P} \otimes v[S,\xi] - \mathcal{P}d_\mu\mathcal{P} \otimes \xi - 2d_\xi\mathcal{P}\, d_J\mathcal{P} \otimes \xi \\
& -2\mathcal{P}(v[\xi,h] - v[\xi,\mathbb{L}]) + \mathcal{P}\{\mu[S,h] - d_J\mathcal{P} \otimes \mu[S,\xi] - 2\mathcal{P}\mu[\xi,h]\} \\
& -\{\mathcal{P}d_{J[S,\mu]}\mathcal{P} + d_J\mathcal{P}d_{J[S,\xi]}\mathcal{P} + 2\mathcal{P}d_{J[\xi,h]}\mathcal{P}\} \otimes \xi\,.
\end{aligned}$$

Proof. Under the spray deformation (20), with deformation factor $\mathcal{P}(x,y)$, taking into account Proposition 1, one can show that

$$\begin{aligned}
[\widetilde{S}, \widetilde{h}] &= [S - 2\mathcal{P}\xi, h - \mathbb{L}] \\
&= [S,h] - [S,\mathbb{L}] - 2[\mathcal{P}\xi, h] + 2[\mathcal{P}\xi, \mathbb{L}].
\end{aligned} \tag{25}$$

On the other hand, taking into account the expression of $\mathbb{L}$ given by (24), and using the following relations:

$$\xi \in \mathfrak{X}^v(\mathcal{T}M), \;\; h\xi = \mathbb{L}\xi = 0,$$

together with the facts that, for $f, g \in C^\infty(\mathcal{T}M), K \in \Psi^1(\mathcal{T}M), \omega \in \Lambda^1(\mathcal{T}M)$:

$$[X, fK] = (d_Xf)K + f[X,K],$$

$$[X, \omega \otimes Y] = [X,\omega] \otimes Y + \omega \otimes [X,Y],$$

$$d_{fK+\omega\otimes Y}g = f d_K g + \omega\otimes d_Y g.$$

It follows that

$$\begin{aligned}
[S,\mathbb{L}] &= [S,\mathcal{P}\mu + d_J\mathcal{P}\otimes\xi]\\
&= [S,\mathcal{P}\mu] + [S, d_J\mathcal{P}\otimes\xi]\\
&= S(\mathcal{P})\mu + \mathcal{P}[S,\mu] + [S,d_J\mathcal{P}]\otimes\xi + d_J\mathcal{P}\otimes[S,\xi]\\
&= S(\mathcal{P})\mu + \mathcal{P}[S,\mu] + \mathcal{L}_S d_J\mathcal{P}\otimes\xi + d_J\mathcal{P}\otimes[S,\xi].\\
[\mathcal{P}\xi,h] &= d\mathcal{P}\otimes h\xi - d_h\mathcal{P}\otimes\xi + \mathcal{P}[\xi,h]\\
&= \mathcal{P}[\xi,h] - d_h\mathcal{P}\otimes\xi.\\
[\mathcal{P}\xi,\mathbb{L}] &= d\mathcal{P}\otimes\mathbb{L}\xi - d_{\mathbb{L}}\mathcal{P}\otimes\xi + \mathcal{P}[\xi,\mathbb{L}]\\
&= \mathcal{P}[\xi,\mathbb{L}] - d_{\mathbb{L}}\mathcal{P}\otimes\xi\\
&= \mathcal{P}[\xi,\mathbb{L}] - \mathcal{P}d_\mu\mathcal{P}\otimes\xi - d_\xi\mathcal{P}\, d_J\mathcal{P}\otimes\xi.
\end{aligned}$$

In view of the above relations, Equation (25) becomes

$$\begin{aligned}
[\widetilde{S},\widetilde{h}] = [S,h] &- S(\mathcal{P})\mu - \mathcal{P}[S,\mu] - \mathcal{L}_S d_J\mathcal{P}\otimes\xi - d_J\mathcal{P}\otimes[S,\xi]\\
&+ 2d_h\mathcal{P}\otimes\xi - 2\mathcal{P}d_\mu\mathcal{P}\otimes\xi - 2d_\xi\mathcal{P}\, d_J\mathcal{P}\otimes\xi - 2\mathcal{P}([\xi,h]-[\xi,\mathbb{L}]).
\end{aligned}$$

Now, using the definition of the Jacobi endomorphism $\widetilde{\Phi} := \widetilde{v}\circ[\widetilde{S},\widetilde{h}]$ and we compose to the left both terms in the above formula by $\widetilde{v}$, taking into account Proposition 1, one can show that:

$$\widetilde{\Phi} \;:=\; \widetilde{v}\circ[\widetilde{S},\widetilde{h}] = (v+\mathbb{L})[\widetilde{S},\widetilde{h}] = v[\widetilde{S},\widetilde{h}] + \mathbb{L}[\widetilde{S},\widetilde{h}]. \tag{26}$$

On the other hand, using the facts that $v\circ\mu = v\mu = \mu,\ v\xi = \xi,\ \mu\circ\mu = J\mu = 0,\ \mu\xi = 0,\ \mathbb{L}\xi = J\xi = 0,\ v = J[S,h]$, we get

$$\begin{aligned}
v[\widetilde{S},\widetilde{h}] &= \Phi - S(\mathcal{P})\mu - \mathcal{P}\,v[S,\mu] - \mathcal{L}_S d_J\mathcal{P}\otimes\xi - d_J\mathcal{P}\otimes v[S,\xi]\\
&\quad +2d_h\mathcal{P}\otimes\xi - 2\mathcal{P}d_\mu\mathcal{P}\otimes\xi - 2d_\xi\mathcal{P}\, d_J\mathcal{P}\otimes\xi - 2\mathcal{P}(v[\xi,h]-v[\xi,\mathbb{L}]).\\
\mathbb{L}[\widetilde{S},\widetilde{h}] &= \mathcal{P}\mu[\widetilde{S},\widetilde{h}] + d_J\mathcal{P}([\widetilde{S},\widetilde{h}])\otimes\xi\\
&= \mathcal{P}\{\mu[S,h] - \mathcal{P}\mu[S,\mu] + d_J\mathcal{P}\otimes\mu[S,\xi] + 2\mathcal{P}\mu[\xi,h]\}\\
&\quad +\{d_{(J[S,h]-\mathcal{P}J[S,\mu]-d_J\mathcal{P}\otimes J[S,\xi]-2\mathcal{P}J[\xi,h])}\mathcal{P}\}\otimes\xi\\
&= \mathcal{P}\{\mu[S,h] + \mathcal{P}\mu[S,\mu] + d_J\mathcal{P}\otimes\mu[S,\xi] + 2\mathcal{P}\mu[\xi,h]\}\\
&\quad +\{d_v\mathcal{P} - \mathcal{P}d_{J[S,\mu]}\mathcal{P} - d_J\mathcal{P}d_{J[S,\xi]}\mathcal{P} - 2\mathcal{P}d_{J[\xi,h]}\mathcal{P}\}\otimes\xi
\end{aligned}$$

In view of, the action of the dynamical covariant derivative ∇ on the semi-basic 1-form $d_J\mathcal{P}$ is given by

$$\nabla d_J\mathcal{P} = \mathcal{L}_S d_J\mathcal{P} - d_v\mathcal{P}.$$

Now, by substituting (26), the required formula of the Jacobi endomorphism $\widetilde{\Phi}$ is obtained, and this completes the proof.

Theorem 1. *The Barthel curvature tensor* $\widetilde{R}$ *associated with the spray deformation (20), with deformation factor* $\mathcal{P}(x,y)$*, is determined by*

$$\widetilde{R} = R + [h, \mathbb{L}] - N_{\mathbb{L}},$$

where $\mathbb{L}$ *defined by (24), and* $N_{\mathbb{L}} := \frac{1}{2}[\mathbb{L}, \mathbb{L}]$ *is the Nijenhuis torsion of* $\mathbb{L}$.

Proof. The proof follows from Proposition 23, together with the fact that

$$\widetilde{R} = -\frac{1}{2}[\widetilde{h}, \widetilde{h}],$$

and taking into account the properties of the Frölicher-Nijenhuis bracket [17].

Corollary 1. *The Berwald connection* $\widetilde{D}^{\circ}$ *associated with the spray deformation (20), with deformation factor* $\mathcal{P}(x,y)$*, is determined by*

$$\begin{aligned} \widetilde{D}^{\circ}_{JX}JY &= D^{\circ}_{JX}JY \\ \widetilde{D}^{\circ}_{\widetilde{h}X}JY &= D^{\circ}_{hX}JY - [\mathbb{L}X, JY] + \mathbb{L}[hX, JY]. \end{aligned}$$

Proof. The proof follows from Proposition 23, together with the facts that (for example, see [19])

$$D^{\circ}_{JX}JY = J[JX, Y], \quad D^{\circ}_{hX}JY = v[hX, JY].$$

This completes the proof.

Now, we retrieve some important results of projective deformation. In the deformation (20), if $\xi = \mathcal{C}$, then the deformation reduces to the projective deformation

$$\overline{S} = S - 2\mathcal{P}(x,y)\,\mathcal{C} \tag{27}$$

Corollary 2. *[3] Under the projective deformation* $\overline{S} = S - 2\mathcal{P}(x,y)\,\mathcal{C}$*, we have*

$$\begin{aligned} \overline{\Gamma} &= \Gamma - 2(\mathcal{P}J + d_J\mathcal{P} \otimes \mathcal{C}), \\ \overline{h} &= h - \mathcal{P}J - d_J\mathcal{P} \otimes \mathcal{C}, \\ \overline{v} &= v + \mathcal{P}J + d_J\mathcal{P} \otimes \mathcal{C}, \\ \overline{\Phi} &= \Phi + (\mathcal{P}^2 - \mathcal{L}_S\mathcal{P})J + (2d_h\mathcal{P} - \mathcal{P}d_J\mathcal{P} - \nabla d_J\mathcal{P}) \otimes \mathcal{C}, \\ \overline{R} &= R + d_J d_h\mathcal{P} \otimes \mathcal{C} + (\mathcal{P}d_J\mathcal{P} - d_h\mathcal{P}) \wedge J, \end{aligned}$$

Proof. The proof follows from the above results, taking into account the fact that, under projective deformation, the vertical vector field $\xi = \mathcal{C}$, the vector form $\mu = [J, \mathcal{C}] = J$, and hence the vector 1-form $\mathbb{L} = \mathcal{P}\,J + d_J\mathcal{P} \otimes \mathcal{C}$, together with the following relations:

$$\mathcal{C}(\mathcal{P}) = \mathcal{P}, \quad [\mathcal{C}, S] = S, \quad [\mathcal{C}, h] = 0, \quad [\mathcal{C}, J] = -J,$$

$$[\mathcal{C}, \mathbb{L}] = 0, \quad h\mathcal{C} = \mathbb{L}\mathcal{C} = 0, \quad [S, J] = -\Gamma,\ v\mathcal{C} = \mathcal{C},\ vS = 0,\ v\Gamma = -v.$$

4. Some special cases

Many of the known special Finsler metrics yield a deformation for the background geodesic spray. As examples, in this section, we consider two rich classes of Finsler metrics. Namely, we consider the class of (α, β)-metrics and the class of spherically symmetric Finsler metrics.

4.1. (α, β)-metrics

The geodesic spray $\widetilde{S}$ of an (α, β)-metric F is given by [4, 20]

$$\widetilde{G}^i = G^i + \alpha Q s^i{}_0 + \Theta \left\{-2\alpha Q s_0 + r_{00}\right\} \left\{ \frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i \right\}, \tag{28}$$

where $\widetilde{G}^i$ (resp. G^i) are the coefficients of the geodesic spray of F (resp. α), and

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$r_{00} := r_{ij} y^i y^j, \quad s_{i0} := s_{ij} y^j, \quad s^i{}_j := s_{hj} a^{ih}, \quad s_j := s_{ij} b^i, \quad b^i := b_j a^{ij}$$

$$Q := \frac{\phi'}{\phi - s\phi'}, \tag{29}$$

$$\Theta := \frac{Q - sQ'}{2(1 + sQ + (b^2 - s^2)Q')}, \tag{30}$$

the symbol $|$ refers to the covariant derivative with respect to the Levi-Civita connection of α, and Q' (resp. ϕ') mean the derivative of Q (resp. ϕ) with respect to s.

The two sprays $\widetilde{S}$ (resp. S) of the metrics F (resp. α) are related by

$$\widetilde{S} = S - 2\mathcal{P}\xi,$$

where the vertical vector ξ is given by

$$\mathcal{P}\xi = \mathcal{P}\xi^i \dot{\partial}_i = \left(\alpha Q s^i{}_0 + \Theta \left\{-2\alpha Q s_0 + r_{00}\right\} \left\{ \frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i \right\} \right) \dot{\partial}_i$$

Let's study when the two sprays are protectively related (that is, $\xi = \mathcal{C}$) as shown in the following theorem.

Theorem 2. *Let $F = \alpha\phi(s)$ be an (α, β)-metric, then the associated sprays with F and the Riemannian metric α are projectively related if and only if*

$$2\alpha\phi' s^i_0 + r_{00}\phi'' b^i - r_{00}\phi'' \frac{\beta}{\alpha^2} y^i = 0. \tag{31}$$

In this case, we have

$$S = S_\alpha - 2\mathcal{P}\mathcal{C}, \quad \mathcal{P} = \frac{1}{2} \frac{r_{00}\phi'}{\alpha\phi}.$$

Proof. In [21], C. Shibata introduced the β-change of a Riemannian metric α, that is, $F = f(\alpha, \beta)$ where f is homogeneous function of degree 1 in α and β. He characterized when this change is projective change, that is, the sprays of F and α are projectively related. Namely, the two sprays are projectively related if and only if

$$2qs_{i0} + q_0 r_{00} \left(b_i - \frac{\beta}{\alpha^2} y_i \right) = 0,$$

where $q := f\partial_\beta f$ and $q_0 := f\partial_\beta^2 f$. Now, since f is homogeneous function of degree 1 in α and β, then we can write

$$F = \alpha f\left(1, \frac{\beta}{\alpha}\right) = \alpha f(1, s) = \alpha\phi(s),$$

where we set $\phi(s) := f(1, s)$.

That is,

$$q = f\partial_\beta f = \alpha\phi\phi', \quad q_0 = f\partial_\beta^2 f = \phi\phi''.$$

Moreover, contracting (31) by b_i, we get

$$2\alpha\phi' s_0 = -r_{00}\phi'' \left(b^2 - \frac{\beta^2}{\alpha^2} \right).$$

By substituting from the above equation and (31) into (28), we get

$$G^i = G^i_\alpha + \frac{1}{2} \frac{r_{00}\phi'}{\alpha\phi} y^i.$$

Corollary 3. *Let $F = \alpha\phi(s)$ be an (α, β)-metric and β be a closed 1-form, then the associated sprays with F and the Riemannian metric α are projectively related if and only if F is of Randers type or β is parallel with respect to the Levi-Civita connection of α.*

Proof. Assume that $F = \alpha\phi(s)$ is an (α, β)-metric which is projectively related to the Riemannian metric α as well as that β is a closed 1-form. Then $s_{ij} = 0$ and hence $s_{i0} = 0$. That is (31) reduces to

$$r_{00}\phi'' \left(b^i - \frac{\beta}{\alpha^2} y^i \right) = 0.$$

Using [22, Lemma 3.2], $b^i - \frac{\beta}{\alpha^2} y^i \neq 0$. Then, we have $r_{00} = 0$ or $\phi'' = 0$. The later implies that $\phi = c_1 s + c_2$ which means that F is of Randers type. The choice $r_{00} = 0$ implies

$$r_{00} = r_{ij} y^i y^j = 0.$$

Taking the derivative twice with respect to y^h and y^k respectively together with using the fact that β is closed, we get

$$r_{hk} = \frac{1}{2}(b_{h|k} + b_{k|h}) = b_{h|k} = 0.$$

That is, β is parallel and this complete the proof.

4.2. Spherically symmetric metrics

A spherically symmetric Finsler metric F on $\mathbb{B}^n(r_0) \subset \mathbb{R}^n$ is given by

$$F(x, y) = u\ \varphi(r, \mathbf{s}),$$

where

$$r = |x|, \quad u = |y|, \quad \mathbf{s} = \frac{\langle x, y\rangle}{|y|},$$

$|\cdot|$ denotes the standard Euclidean norm and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$, for more details, we refer for example to [23–25].

Moreover, the geodesic spray coefficients G^i of F are given by

$$G^i = uPy^i + u^2Qx^i, \tag{32}$$

where the functions P and Q have the following formulae

$$Q := \frac{1}{2r}\frac{-\varphi_r + \mathbf{s}\varphi_{rs} + r\varphi_{\mathbf{ss}}}{\varphi - \mathbf{s}\varphi_{\mathbf{s}} + (r^2 - \mathbf{s}^2)\varphi_{\mathbf{ss}}}, \quad P := -\frac{Q}{\varphi}(\mathbf{s}\varphi + (r^2 - \mathbf{s}^2)\varphi_{\mathbf{s}}) + \frac{1}{2r\varphi}(\mathbf{s}\varphi_r + r\varphi_{\mathbf{s}}). \tag{33}$$

The spray S of a spherically symmetric metric F is given by

$$S = S_0 - 2\mathcal{P}\xi,$$

where S_0 is the spray of the Euclidean metric and

$$\mathcal{P}\xi = uP\mathcal{C} + u^2Qx^i\dot{\partial}_i.$$

From which we get the following known corollary.

Corollary 4. *The spray of a spherically symmetric metric $F = u\varphi(r, \boldsymbol{s})$ is projectively flat if and only if $Q = 0$.*

5. Invariant Jacobi endomorphism and curvature

In this section, under the projective-like deformation (18), we study some properties of the Jacobi endomorphism, Barthel connection, and the curvature of Barthel connection [26, 27].

Using the property (13)

$$3R = [J, \Phi], \quad \Phi = i_SR,$$

we have the following theorem.

Theorem 3. *Under the general deformation* (18), *the Jacobi endomorphism Φ is invariant if and only if the curvature R is invariant.*

Proof. Let S be a spray on a manifold M with the Jacobi endomorphism Φ and curvature R. Now, consider the general deformation (18) of S. Assume that Φ is invarinat under the deforamtion (18), that is, $\widetilde{\Phi} = \Phi$. Then, we have

$$3\widetilde{R} = [J, \widetilde{\Phi}] = [J, \Phi] = 3R.$$

That is, the curvature R is invariant.

Conversely, let R be invariant, then we get

$$\widetilde{\Phi} = i_{\widetilde{S}}\widetilde{R} = i_{S-2\xi}R = i_S R - 2i_\zeta R = \Phi - 2i_\xi R.$$

But using the fact that R is semi-basic, we have $i_\zeta R = 0$. Consequently, $\widetilde{\Phi} = \Phi$, i.e., Φ is invariant.

Proposition 3. *Under the projective-like deformation (18), with deformation factor $\mathcal{P}(x, y)$, the curvature Barthel connection R is invariant if and only if the vertical one form $\mathbb{L}$ satisfies the following relation*

$$[h, \mathbb{L}] = \frac{1}{2}[\mathbb{L}, \mathbb{L}],$$

where h is the corresponding horizontal projection of the spray S.

Proof. The proof is clear and we omit it.

From which together with Theorem 3, we conclude that

Theorem 4. *Under the projective-like deformation (18), with deformation factor $\mathcal{P}(x, y)$, the following assertions are equivalent*

(i) the vertical one form $\mathbb{L}$ satisfies $[h, \mathbb{L}] = \frac{1}{2}[\mathbb{L}, \mathbb{L}]$.

(ii) the Jacobi endomorphism Φ is invariant.

(iii) the curvature Barthel connection R is invariant.

Proposition 4. *The projective deformation* (27) *preserves the curvature the projective factor $\mathcal{P}$ is a Funk function, that is,*

$$d_h\mathcal{P} = \mathcal{P} d_J \mathcal{P}.$$

Proof. It is easy to see that the curvature of a spray S is preserved under any deformation of S if and only if $\widetilde{R} = R$, that is

$$d_J d_h \mathcal{P} \otimes \mathcal{C} + (\mathcal{P} d_J \mathcal{P} - d_h \mathcal{P}) \wedge J = 0.$$

The above equation is satisfied if and only if

$$\mathcal{P} d_J \mathcal{P} - d_h \mathcal{P} = 0.$$

In the following two examples, for simplicity, we consider

$$\widetilde{S} = S - 2\mathcal{P}\xi = S - 2\zeta,$$

where we set $\zeta = \mathcal{P}\xi$.

For calculations, we use the Finsler package [28], moreover, PDF and Maple files of the calculation are posted on
https://github.com/salahelgendi/Calculations-for-the-paper-Deformation-of-sprays.git

Example 2. *Consider $M = \{(x^1, x^2) \in \mathbb{R}^2 : \ x^1 > 0$ with the Finsler function F*

$$F = \sqrt{x^1((y^1)^2 + (y^2)^2)},$$

where $(x^1, x^2; y^1, y^2) \in T\mathbb{R}^2 = \mathbb{R}^4$. The coefficients G^i of the geodesic spray are given by

$$G^1 = \frac{1}{4}\frac{(y^1)^2 - (y^2)^2}{x^1}, \quad G^2 = \frac{1}{4}\frac{y^1y^2}{x^1}.$$

The components of the Jacobi endomorphism are given by

$$R^1_1 = -\frac{1}{2}\frac{(y^2)^2)^2}{(x^1)^2}, \quad R^1_2 = \frac{1}{2}\frac{y^1y^2)^2}{(x^1)^2}, \quad R^2_1 = \frac{1}{2}\frac{y^1y^2)^2}{(x^1)^2}, R^2_2 = -\frac{1}{2}\frac{(y^1)^2}{(x^1)^2}.$$

Now, consider the conformal transformation

$$\widetilde{F} = e^{-\frac{1}{2}((x^1)^2 - (x^2)^2 + x^1 + x^2))}F.$$

Then the spray coefficients of $\widetilde{F}$ are given by

$$\widetilde{G}^1 = G^1 - \frac{1}{4}\frac{2(x^1)^2(y^1)^2 - 2(x^1)^2(y^2)^2 - 4x^1x^2y^1y^2 + x^1(y^1)^2 + 2x^1y^1y^2 - x^1(y^2)^2}{x^1}.$$

$$\widetilde{G}^2 = G^2 - \frac{1}{4}\frac{4(x^1)^2y^1y^2 + 2x^1x^2(y^1)^2 - 2x^1x^2(y^2)^2 - x^1(y^1)^2 + 2x^1y^1y^2 + x^1(y^2)^2}{x^1}.$$

We can write $\widetilde{S} = S - 2\mathcal{P}\xi = S - 2\zeta$, then the coefficients $\widetilde{G}^i$ as follows:

$$\widetilde{G}^1 = G^1 + \zeta^1, \quad \widetilde{G}^2 = G^2 + \zeta^2.$$

Hence the vector ζ is given by

$$\zeta = \zeta^1\dot{\partial}_1 + \zeta^2\dot{\partial}_2$$

where

$$\zeta^1 = -\frac{1}{4}\frac{2(x^1)^2(y^1)^2 - 2(x^1)^2(y^2)^2 - 4x^1x^2y^1y^2 + x^1(y^1)^2 + 2x^1y^1y^2 - x^1(y^2)^2}{x^1}$$

$$\zeta^2 = -\frac{1}{4}\frac{4(x^1)^2y^1y^2 + 2x^1x^2(y^1)^2 - 2x^1x^2(y^2)^2 - x^1(y^1)^2 + 2x^1y^1y^2 + x^1(y^2)^2}{x^1}$$

The components of R^i_j of the Jacobi endomorphism are equal since the Jacobi endomorphism is preserved under this transformation, moreover the components of $\widetilde{R}^i_j$ are given by

$$\widetilde{R}^1_1 = R^1_1 = \frac{1}{2}\frac{(y^2)^2}{(x^1)^2}, \quad \widetilde{R}^1_2 = R^1_2 = -\frac{1}{2}\frac{y^1y^2}{(x^1)^2},$$
$$\widetilde{R}^1_2 = R^2_1 = -\frac{1}{2}\frac{y^1y^2}{(x^1)^2}, \quad \widetilde{R}^1_2 = R^2_2 = \frac{1}{2}\frac{(y^1)^2}{(x^1)^2}.$$

The components of R^i_{jk} of the curvature is preserved under this transformation, moreover the components of $\widetilde{R}^i_{jk}$ are given by

$$\widetilde{R}^1_{12} = R^1_{12} = -\frac{1}{2}\frac{y^2}{(x^1)^2}, \quad \widetilde{R}^1_{21} = R^1_{21} = \frac{1}{2}\frac{y^2}{(x^1)^2},$$
$$\widetilde{R}^2_{12} = R^2_{12} = \frac{1}{2}\frac{y^1}{(x^1)^2}, \quad \widetilde{R}^2_{21} = R^2_{21} = -\frac{1}{2}\frac{y^1}{(x^1)^2}.$$

Example 3. *Consider $M = \mathbb{R}^3$ with the Finsler function F*

$$F = \sqrt[4]{(y^1)^4 + (y^2)^4 + (y^3)^4},$$

where $(x^1, x^2, x^3; y^1, y^2, y^3) \in T\mathbb{R}^3 = \mathbb{R}^6$. The coefficients G^i of the geodesic spray are given by

$$G^i = 0.$$

The components of the Jacobi endomorphism are given by

$$R^i_j = 0, \quad R^i_{jk} = 0.$$

Now, consider the deformation of F as follows

$$\widetilde{F} = \sqrt[4]{f_1(x^1)(y^1)^4 + f_2(x^2)(y^2)^4 + f_3(x^3)(y^3)^4}.$$

Then the spray coefficients of $\tilde{F}$ are given by

$$\widetilde{G}^1 = \frac{(y^1)^2}{8f_1(x^1)}\frac{df_1(x^1)}{dx^1}, \quad \widetilde{G}^2 = \frac{(y^2)^2}{8f_2(x^2)}\frac{df_2(x^2)}{dx^2}, \quad \widetilde{G}^3 = \frac{(y^3)^2}{8f_3(x^3)}\frac{df_3(x^3)}{dx^3}.$$

Consider $\widetilde{S} = S - 2\mathcal{P}\xi = S - 2\zeta$, then the coefficients $\widetilde{G}^i$ as follows:

$$\widetilde{G}^1 = G^1 + \zeta^1, \quad \widetilde{G}^2 = G^2 + \zeta^2, \quad \widetilde{G}^3 = G^3 + \zeta^3.$$

Hence the vector ζ is given by

$$\zeta = \zeta^1\dot{\partial}_1 + \zeta^2\dot{\partial}_2 + \zeta^3\dot{\partial}_3$$

where

$$\zeta^1 = \frac{(y^1)^2}{8f_1(x^1)}\frac{df_1(x^1)}{dx^1}, \quad \zeta^2 = \frac{(y^2)^2}{8f_2(x^2)}\frac{df_2(x^2)}{dx^2}, \quad \zeta^3 = \frac{(y^3)^2}{8f_3(x^3)}\frac{df_3(x^3)}{dx^3}.$$

The components of R^i_j of the Jacobi endomorphism are equal since the Jacobi endomorphism is preserved under this transformation. That is, we have

$$\widetilde{R}^i_j = R^i_j = 0, \quad \widetilde{R}^i_{jk} = R^i_{jk} = 0.$$

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