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On Exploring the r-Stirling Fibonacci Numbers and Polynomials

Romeo A. Coronel[2], Roberto B. Corcino[1,2,*]

[1] *Research Institute for Computational Mathematics and Physics, Cebu Normal University, 6000 Cebu City, Philippines*
[2] *Mathematics Department, Cebu Normal University, 6000 Cebu City, Philippines*

Abstract. This paper introduces and investigates a novel class of combinatorial constructs termed the r-Stirling Fibonacci numbers and polynomials of the first and second kind. By integrating the exponential generating functions of classical Fibonacci numbers with those of the signed r-Stirling numbers, we establish an enriched algebraic framework that advances the theory of special numbers and polynomials. The study yields new identities - including horizontal generating functions, explicit formulas, and convolution relations - that extend classical combinatorial results. In addition, we define the r-Stirling Chebyshev polynomials of both kinds by employing hyperbolic functions and exponential techniques, thereby forging a functional link between Fibonacci-type and Stirling-type sequences. These results are rigorously validated through series expansion and the Cauchy product method. The theoretical contributions of this work highlight the interplay between combinatorics, algebra, and analysis, with potential applications in number theory, orthogonal polynomials, and symbolic computation.

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1. Introduction

James Stirling, a Scottish mathematician in his book "Methodus Differentialis" (Stirling, [1]) introduced Stirling numbers within a purely algebraic framework. These numbers are considered the fundamental concept in mathematics particularly in the field of combinatorics, analysis and algebra. Stirling numbers were categorized into two types; (1) first kind Stirling numbers which enumerates the number of permutations of n elements with exactly k disjoint cycles; (2) the second kind Stirling numbers which counts the number of ways to partition n elements into k non-empty subsets. Additionally, The Stirling numbers are the coefficients of the relation

*Corresponding author.
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Email addresses: coronelromeo78@gmail.com R.A. Coronel), rcorcino@yahoo.com ((R. B. Corcino)

$$x^{\underline{n}} = \sum_{n=0}^{\infty} s(n,k)x^k$$

$$x^n = \sum_{n=0}^{\infty} S(n,k)x^{\underline{k}},$$

where $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \prod_{k=1}^{n}(x-k+1)$ is the falling factorial of x of degree n. The coefficient $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first kind and Stirling numbers of the second kind respectively. Over the years, numerous mathematicians have studied these numbers and explored properties, generalizations and its application. Common results include generating functions, recurrence relations, and explicit formulas all of which helps to understand their behavior. (see [2–7])

One of the most notable contributions to the exploration of Stirling numbers was made by A.Z. Broder [8], who introduced the r- Stirling numbers by adding a parameter r which leads to the r- Stirling numbers of the first and second kind. Broders' study employs a combinatorial approach in deriving properties and identities such as generating functions (exponential, ordinary, rational, horizontal, vertical etc.), recurrence relations (cross recurrence, triangular, etc.), explicit formulas, and orthogonality and inverse relation and their generalizations.

Recently, Corcino et al. (2023,[9]) modified Broder's results by presenting the r-Stirling numbers in an algebraic method and obtained results parallel to that of Broder. On the other hand, Fibonacci is named after Leonardo Pisa, an Italian mathematician who introduced the sequence in his book, Liber Abaci (1202, [10]). Before Fibonacci, this sequence appeared in Indian Mathematics. The exponential generating function of the Fibonacci number is due to the study of C.A Church et. al.[11] which is given by

$$\frac{t}{1-t-t^2} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \tag{1.1}$$

This can further be expressed as follows

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}$$

where $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n-1}$ and $\alpha = \frac{(1+\sqrt{5})}{2}$, $\beta = \frac{(1-\sqrt{5})}{2}$.

In mathematics, special functions and number sequences can be generalized through various methods. One common approach is to integrate them with concepts from other well-known functions. Another method involves introducing parameters by modifying the defining generating function - either through addition or multiplication with certain expressions. For example, the exponential generating function of the Fibonacci numbers can be extended or combined with other numerical sequences. Similar techniques are employed in works such as [12–14], where the generating functions of degenerate Hermite, Bernoulli, Euler, and Genocchi polynomials

$$(1+\alpha t)^{\frac{x}{\alpha}}(1+\alpha t^2)^{\frac{y}{\alpha}} = \sum_{n=0}^{\infty} \mathcal{H}_n(x,y,\alpha)\frac{t^n}{n!}$$
$$\frac{t}{(1+\alpha t)^{\frac{1}{\alpha}}-1}(1+\alpha t^2)^{\frac{x}{\alpha}} = \sum_{n=0}^{\infty} \mathcal{B}_n(x,\alpha)\frac{t^n}{n!}$$
$$\frac{2}{(1+\alpha t)^{\frac{1}{\alpha}}+1}(1+\alpha t^2)^{\frac{x}{\alpha}} = \sum_{n=0}^{\infty} \mathcal{E}_n(x,\alpha)\frac{t^n}{n!}$$
$$\frac{2t}{(1+\alpha t)^{\frac{1}{\alpha}}+1}(1+\alpha t^2)^{\frac{x}{\alpha}} = \sum_{n=0}^{\infty} \mathcal{G}_n(x,\alpha)\frac{t^n}{n!}$$

are modified by introducing new parameter λ as follows:

$$\left(\frac{2^{\mu}t^{\nu}}{\lambda(1+at)^{\frac{1}{a}}+1}\right)^{\alpha}(1+at)^{\frac{x}{a}}(1+at^2)^{\frac{y}{a}} = \sum_{n=0}^{\infty} {}_H\mathcal{P}_n^{(\alpha)}(x,y;a;\lambda;\mu;\nu)\frac{t^n}{n!}$$
$$\left(\frac{t^m}{\lambda(1+at)^{\frac{1}{a}}-\sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha}(1+at^2)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x,a,b;\lambda)\frac{t^n}{n!}$$
$$\left(\frac{2^m}{\lambda(1+at)^{\frac{1}{a}}+\sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha}(1+at^2)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{E}_n(x,a,b;\lambda)\frac{t^n}{n!}$$
$$\left(\frac{(2t)^m}{\lambda(1+at)^{\frac{1}{a}}+\sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha}(1+at^2)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{G}_n(x,a,b;\lambda)\frac{t^n}{n!}.$$

In this paper we will introduce another novel variant of r-Stirling numbers which combines the exponential generating function of the Fibonacci number to the exponential generating function of the signed r-Stirling numbers of the first kind and r-Stirling number of the second kind. These numbers are called the r-Stirling Fibonacci numbers. Various formulas were derived including Horizontal Generating Function, Explicit Formulas and Convolution.

2. Fibonacci Polynomials

Fibonacci polynomials $F_n^{(1)}(x)$ are a family of polynomials that extend the *Fibonacci sequence* into the realm of algebra. Instead of producing just numbers, they produce polynomials in x, while still obeying a Fibonacci-style recurrence. That is,

$$F_0^{(1)}(x) = 0, \quad F_1^{(1)}(x) = 1,$$
$$F_n^{(1)}(x) = xF_{n-1}^{(1)}(x) + F_{n-2}^{(1)}(x), \quad \text{for } n \tag{2.1}$$

This recurrence is linear and non-homogeneous, and it reflects a combination of growth (via multiplication by x) and memory (via the $F^{(1)}_{n-2}(x)$ term).

In the same way the Fibonacci numbers appear in natural growth, recursion, and combinatorics, Fibonacci polynomials provide a parameterized version of Fibonacci numbers. When we evaluate $F^{(1)}_n(x)$ at specific values of x, we recover numerical sequences with meaningful interpretations. For instance:

- $F^{(1)}_n(1)$ gives the standard Fibonacci numbers.
- $F^{(1)}_n(2)$ gives the Pell numbers.
- They also show up in algebraic identities and generating function theory.

First few terms are given as follows

$$\begin{aligned}
F^{(1)}_0(x) &= 0 \\
F^{(1)}_1(x) &= 1 \\
F^{(1)}_2(x) &= x \\
F^{(1)}_3(x) &= x^2 + 1 \\
F^{(1)}_4(x) &= x^3 + 2x \\
F^{(1)}_5(x) &= x^4 + 3x^2 + 1 \\
F^{(1)}_6(x) &= x^5 + 4x^3 + 3x
\end{aligned}$$

These are monic polynomials (leading coefficient is 1) and have alternating degrees, increasing by 1 with each n.

The closed form uses the roots of the characteristic equation associated with the recurrence:

Characteristic Equation:

$$r^2 - xr - 1 = 0 \quad \Rightarrow \quad r = \frac{x \pm \sqrt{x^2+4}}{2}$$

Let:

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2}, \quad \beta(x) = \frac{x - \sqrt{x^2+4}}{2}$$

Then:

$$F^{(1)}_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$$

This generalizes the Binet formula for Fibonacci numbers and offers a fast way to compute Fibonacci polynomials for large n.

Generating Function

The generating function encodes all Fibonacci polynomials into a single expression:

$$G(x,t) = \sum_{n=0}^{\infty} F_n^{(1)}(x) t^n = \frac{t}{1 - xt - t^2} \tag{2.2}$$

This rational function captures all of the recurrence information and is extremely useful in solving recurrence relations, analyzing growth behavior and deriving identities.

Properties and Identities

Some elegant identities include:

(i) Derivative Identity:

$$\frac{d}{dx} F_n^{(1)}(x) = \sum_{k=1}^{n-1} F_k^{(1)}(x) F_{n-k}^{(1)}(x)$$

(ii) Addition Formula:

$$F_{m+n}^{(1)}(x) = F_m^{(1)}(x) F_{n+1}^{(1)}(x) + F_{m-1}^{(1)}(x) F_n^{(1)}(x)$$

These resemble classical Fibonacci identities and are useful in proving combinatorial or algebraic results.

In this section, we define another form of Fibonacci polynomials and call this as the second form of Fibonacci polynomials. Throughout this paper, we call the Fibonacci polynomials defined in (2.2) as the first form of Fibonacci polynomials.

Definition 2.1. The second form of Fibonacci polynomials, denoted by $F_n^{(2)}(x)$, is defined by

$$\frac{t}{1 - t - t^2} e^{xt} = \sum_{n=0}^{\infty} F_n^{(2)}(x) \frac{t^n}{n!}. \tag{2.3}$$

Note that, when $x = 0$, the generating function in (2.3) reduces to the generating function of Fibonacci numbers in (1.1). This implies that

$$F_n^{(2)}(0) = F_n$$

Now, using Cauchy's rule for the product of two power series, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(2)}(x) \frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} F_{n-j} x^j \right) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Theorem 2.2. *The second form of Fibonacci polynomials is given by*

$$F_n^{(2)}(x) = \sum_{j=0}^{n} \binom{n}{j} F_{n-j} x^j. \tag{2.4}$$

The following theorem gives the addition formula for the second form of Fibonacci polynomials.

Theorem 2.3. *The second form of Fibonacci polynomials is given by*

$$F_n^{(2)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} F_{n-j}^{(2)}(x) y^j. \tag{2.5}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(2)}(x+y)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} F_n^{(2)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} y^n \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} F_{n-j}^{(2)}(x) y^j\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Applying derivative to both sides of (2.3) with respect to x yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{d}{dx} F_n^{(2)}(x)\frac{t^n}{n!} &= t\sum_{n=0}^{\infty} F_n^{(2)}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} F_n^{(2)}(x)\frac{t^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty} nF_{n-1}^{(2)}(x)\frac{t^n}{n!}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Theorem 2.4. *The second form of Fibonacci polynomials satisfies the following differential equation*

$$\frac{d}{dx} F_n^{(2)}(x) = nF_{n-1}^{(2)}(x). \tag{2.6}$$

This means that the second form of Fibonacci polynomials can be classified as Appell polynomials.

Definition 2.5. The third form of Fibonacci polynomials, denoted by $F_n^{(3)}(x)$, is defined by

$$\frac{t}{1-t-t^2}(1+t)^x = \sum_{n=0}^{\infty} F_n^{(3)}(x)\frac{t^n}{n!}. \tag{2.7}$$

Note that, when $x = 0$, the generating function in (2.3) reduces to the generating function of Fibonacci numbers in (1.1). This implies that

$$F_n^{(3)}(0) = F_n$$

Now, using Cauchy's rule for the product of two power series, we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(3)}(x)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} F_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(x)_n}{n!}t^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(x)_j F_{n-j}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Theorem 2.6. *The third form of Fibonacci polynomials is given by*

$$F_n^{(3)}(x) = \sum_{j=0}^{n}\binom{n}{j}(x)_j F_{n-j}. \tag{2.8}$$

The following theorem gives the addition formula for the third form of Fibonacci polynomials.

Theorem 2.7. *The third form of Fibonacci polynomials is given by*

$$F_n^{(2)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}(y)_j F_{n-j}^{(2)}(x). \tag{2.9}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} F_n^{(3)}(x+y)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} F_n^{(3)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} (y)_n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(y)_j F_{n-j}^{(2)}(x)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Note that

$$\frac{d}{dx}(1+t)^x = (1+t)^x ln(1+t).$$

Hence, applying derivative to both sides of (2.7) with respect to x yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d}{dx}F_n^{(3)}(x)\frac{t^n}{n!} &= \ln(1+t)\sum_{n=0}^{\infty}F_n^{(3)}(x)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\cdot\frac{F_{n-k}^{(3)}(x)}{(n-k)!}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the following theorem.

Theorem 2.8. *The second form of Fibonacci polynomials satisfies the following differential equation*

$$\frac{d}{dx}F_n^{(3)}(x) = n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\cdot\frac{F_{n-k}^{(3)}(x)}{(n-k)!}. \tag{2.10}$$

3. Formulas for r-Stirling Fibonacci Number

In this section, we introduce and explore new classes of combinatorial numbers which we call the r-Stirling Fibonacci numbers of the first and second kinds. These numbers are defined through exponential generating functions involving the classical Fibonacci numbers and generalized r-Stirling numbers. The definitions extend the framework of both Fibonacci and r-Stirling numbers by integrating their combinatorial structures. We derive convolution formulas that relate these numbers to the classical Fibonacci sequence and investigate their generating functions. In particular, we establish horizontal generating functions and a Schlömilch-type formula that highlights their intricate combinatorial nature.

Definition 3.1. The r-Stirling Fibonacci number of the first kind is defined by means of the following exponential generating functions:

$$\sum_{n=0}^{\infty}SF_n^1\left(k;r\right)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r\frac{(e^{\alpha t}-e^{\beta t})(\ln^k\left(1+t\right))}{k!\left(\alpha-\beta\right)} \tag{3.1}$$

where $\alpha = \frac{(1+\sqrt{5})}{2}$, $\beta = \frac{(1-\sqrt{5})}{2}$ and $\alpha-\beta=\sqrt{5}$.

Definition 3.2. The r-Stirling Fibonacci number of the second kind is defined by means of the following exponential generating functions:

$$\sum_{n=0}^{\infty}SF_n^2\left(k;r\right)\frac{t^n}{n!} = \frac{(e^{\alpha t}-e^{\beta t})(e^{rt}(e^t-1)^k)}{k!\left(\alpha-\beta\right)} \tag{3.2}$$

where $\alpha = \frac{(1+\sqrt{5})}{2}$, $\beta = \frac{(1-\sqrt{5})}{2}$ and $\alpha-\beta=\sqrt{5}$.

Theorem 3.3. The r-Stirling Fibonacci number of the first and second kind satisfy the convolution formula respectively;

$$SF_n^1(k;r) = \sum_{m=k}^{n} \widehat{\begin{bmatrix} m+r \\ k+r \end{bmatrix}}_r \binom{n}{m} F_{n-m} \tag{3.3}$$

$$SF_n^2(k;r) = \sum_{m=k}^{n} \begin{Bmatrix} m+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m} \tag{3.4}$$

where $n \geq k$, otherwise $SF_n^1(k;r) = SF_n^2(k;r) = 0$.

Proof The Exponential Generating Function (3.1) can be written as:

$$\sum_{n=0}^{\infty} SF_n^1(k;r) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \right)$$

By Cauchy Product we have,

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \widehat{\begin{bmatrix} m+r \\ k+r \end{bmatrix}}_r \binom{n}{m} F_{n-m} \right\} \frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ we have,

$$\begin{aligned} SF_n^1(k;r) &= \sum_{m=0}^{n} \widehat{\begin{bmatrix} m+r \\ k+r \end{bmatrix}}_r \binom{n}{m} F_{n-m} \\ &= \sum_{m=k}^{n} \widehat{\begin{bmatrix} m+r \\ k+r \end{bmatrix}}_r \binom{n}{m} F_{n-m} \end{aligned}$$

Similary, (3.2) can be written as:

$$\sum_{n=0}^{\infty} SF_n^2(k;r) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \right)$$

By Cauchy Product we have,

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m} \right\} \frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ we have,

$$\begin{aligned} SF_n^2(k;r) &= \sum_{m=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m} \\ &= \sum_{m=k}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m} \quad \square \end{aligned}$$

Theorem 3.4. The Horizontal Generating function of the r-Stirling Fibonacci number of the first and second kind is given by respectively;

$$\sum_{k=0}^{n} SF_n^1(k;r)\, z^k = F_n^{(3)}(z-r) \tag{3.5}$$

$$\sum_{k=0}^{n} SF_n^2(k;r)\, z^{\underline{k}} = F_n^{(2)}(z+r) \tag{3.6}$$

Proof The exponential generating function in (3.1) can be written as

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SF_n^1(k;r)\frac{t^n}{n!}\right\} z^k &= \left\{\left(\frac{1}{1+t}\right)^r \left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\sum_{k\geq 0}\frac{\ln^k(1+t)}{k!}\right\} z^k \\ &= \left(\sum_{n\geq 0}(z-r)_n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}\right)\end{aligned}$$

By Cauchy Product the RHS becomes,

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} SF_n^1(k;r)\, z^k\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}(z-r)_m\binom{n}{m}F_{n-m}\right\}\frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ we have,

$$\begin{aligned}\sum_{k=0}^{n} SF_n^1(k;r)\, z^k &= \sum_{m=0}^{n}(z-r)_m\binom{n}{m}F_{n-m} \\ &= F_n^{(3)}(z-r)\end{aligned}$$

Similarly, The exponential generating function in (3.2) can be written as

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} SF_n^2(k;r)\frac{t^n}{n!}\right\} z^{\underline{k}} &= \left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\left(\sum_{k=0}^{\infty}\frac{e^{rt}(e^t-1)^k}{k!}z^{\underline{k}}\right) \\ &= \left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)\left(e^{rt}\sum_{k=0}^{\infty}\frac{z^{\underline{k}}}{k!}(e^t-1)^k\right) \\ &= \left(\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}\right)e^{rt}(1-e^t-1)^z = \frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}e^{rt+zt} \\ &= \left(\sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}\right)\end{aligned}$$

By Cauchy Product the RHS becomes,

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} (z+r)^m \binom{n}{m} F_{n-m} \right\} \frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ we have,

$$\begin{aligned} \sum_{k=0}^{n} SF_n^2(k;r)\, z^{\underline{k}} &= \sum_{m=0}^{n} (z+r)^m \binom{n}{m} F_{n-m} \\ &= F_n^{(2)}(z+r) \quad \square \end{aligned}$$

Theorem 3.5. The Schlömilch-type formula of the r-Stirling Fibonacci number is given by

$$\begin{aligned} SF_n^1(k;r) = \sum_{m=k}^{n} \sum_{j=0}^{m} \sum_{a=0}^{m-k} \sum_{b=0}^{a} (-1)^{n-j+b+a} \binom{a}{b} \binom{n}{j} \binom{j}{m} F_{j-m} \binom{m-1+a}{m-k+a} \\ \times \binom{2m-k}{m-k-a} \frac{(a-b)^{m-k+a}}{a!} r^{\overline{n-j}} \end{aligned} \tag{3.7}$$

Proof To derive the Schlömilch-type formula we have to decompose the exponential generating function in (3.1) into product of three functions as follows: The first function can be expressed as:

$$\begin{aligned} \left(\frac{1}{1+t}\right)^r &= (1+t)^{-r} \\ &= \sum_{n\geq 0} \binom{-r}{n} t^n \\ &= \binom{-r}{0} (-t)^0 + \sum_{n\geq 0} \binom{-r}{n} t^n \end{aligned}$$

where $\binom{-r}{n}$ is the Newton's generalized binomial coefficients. By Newton's Binomial Theorem this can be further computed as

$$\begin{aligned} \left(\frac{1}{1+t}\right)^r &= 1 + \sum_{n>0} \frac{(-r)(-r-1)\cdots(-r-n+1)}{n!} t^n \\ &= 1 + \sum_{n>0} (-1)^n \frac{(r)(r+1)\cdots(r+n-1)}{n!} t^n \\ &= 1 + \sum_{n>0} (-1)^n \left((r)(r+1)\cdots(r+n-1)\right) \frac{t^n}{n!} \end{aligned}$$

by the definition of the rising factorial $r^{\overline{n}} = r(r+1)(r+1)\cdots(r+n-1)$ we have,

$$\left(\frac{1}{1+t}\right)^r = 1 + \sum_{n>0} (-1)^n r^{\overline{n}} \frac{t^n}{n!}$$

$$= \sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}.$$

The second function is the Fibonacci polynomial which can be expressed as its exponential generating function given by;

$$\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}=\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}.$$

Lastly, the third function can be written as

$$\frac{[\ln(1+t)]^k}{k!} = \sum_{n\geq k}s(n,k)\frac{t^n}{n!}.$$

Hence, using Cauchy's Rule for the product we have,

$$\begin{aligned}
\sum_{k=0}^{\infty}SF_n^1(k;r)\frac{t^n}{n!} &= \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}\right)\left(\sum_{n\geq k}s(n,k)\frac{t^n}{n!}\right)\\
&= \left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left\{\sum_{m=k}^{n}s(m,k)\binom{n}{m}F_{n-m}\right\}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\sum_{j=m}^{n}(-1)^{n-j}r^{\overline{n-j}}\binom{n}{j}s(m,k)\binom{j}{m}F_{j-m}\right\}\frac{t^n}{n!}
\end{aligned}$$

Comparing coefficients of $\frac{t^n}{n!}$ we have,

$$SF_n^1(k;r) = \sum_{m=0}^{n}\sum_{j=m}^{n}(-1)^{n-j}s(m,k)\binom{n}{j}\binom{j}{m}F_{j-m}r^{\overline{n-j}}$$

The Schlömilch-type formula for the signed r-Stirling number of the first kind is given by[15],

$$s(n,k) = \sum_{r=0}^{n-k}\sum_{j=0}^{r}(-1)^{j+r}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k-r}\frac{(r-j)^{n-k+r}}{r!}$$

so the Schlömilch-type formula for $SF_n^1(k;r)$ is

$$\begin{aligned}
SF_n^1(k;r) &= \sum_{m=k}^{n}\sum_{j=0}^{m}\sum_{a=0}^{m-k}\sum_{b=0}^{a}(-1)^{n-j+b+a}\binom{a}{b}\binom{n}{j}\binom{j}{m}F_{j-m}\binom{m-1+a}{m-k+a}\\
&\times \binom{2m-k}{m-k-a}\frac{(a-b)^{m-k+a}}{a!}r^{\overline{n-j}} \quad \square
\end{aligned}$$

Theorem 3.6. *The Explicit formula of the r-Stirling Fibonacci number of the second kind is given by*

$$SF_n^2(k;r) = \sum_{m=0}^{n} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \binom{n}{m} F_{n-m} \tag{3.8}$$

Proof The exponential generating function (3.2)

$$\begin{aligned} k!\sum_{n=0}^{\infty} SF_n^2(k;r,x)\frac{t^n}{n!} &= \frac{(e^{\alpha t}-e^{\beta t})(e^{rt}(e^t-1)^k)}{(\alpha-\beta)} \\ &= \left(\sum_{n=0}^{\infty}\left\{(-1)^i\binom{k}{i}((k-i)+r)^n\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}\right) \\ &= \left(\sum_{n=0}^{\infty}k!\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}F_n\frac{t^n}{n!}\right) \end{aligned}$$

By Cauchy Product we have,

$$SF_n^2(k;r,x) = \sum_{m=0}^{n} \left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r \binom{n}{k} F_{n-m} \quad \square$$

4. Formulas r-Stirling Fibonacci Polynomials

In this section, we introduce and derive various identities and properties associated with the r-Stirling Fibonacci polynomials of the first and second kinds. These polynomials are generalizations that intertwine the concepts of r-Stirling numbers and classical Fibonacci polynomials. Their structure is made explicit through exponential generating functions and convolution-type formulas that extend known results in combinatorics and special functions. We also establish their horizontal generating functions, a Schlömilch-type expansion, and an explicit representation, all of which showcase the richness and applicability of these polynomials in combinatorial analysis.

Definition 4.1. The r-Stirling Fibonacci polynomials of the first and second kinds are respectively defined by means of the following exponential generating functions:

$$\phi(t,x) = \sum_{n=0}^{\infty} SF_n^1(k;r,x)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{\left(2e^{\frac{xt}{2}}\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)\right)(\ln^k(1+t))}{k!\left(\sqrt{x^2+4}\right)} \tag{4.1}$$

$$\gamma(t,x) = \sum_{n=0}^{\infty} SF_n^2(k;r,x)\frac{t^n}{n!} = \frac{\left(2e^{\frac{xt}{2}}\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)\right)e^{rt}(e^t-1)^k}{k!\left(\sqrt{x^2+4}\right)} \tag{4.2}$$

Theorem 4.2. *The r-Stirling Fibonacci polynomial of the first and second kind satisfy the convolution formula respectively;*

$$SF_n^1(k;r,x) = \sum_{m=k}^{n} \widehat{\begin{bmatrix} m+r \\ k+r \end{bmatrix}}_r \binom{n}{m} F_{n-m}(x) \tag{4.3}$$

$$SF_n^2(k;r,x) = \sum_{m=k}^{n} \begin{Bmatrix} m+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m}(x) \tag{4.4}$$

where $n \geq k$, *otherwise* $SF_n^1(k;r,x) = SF_n^2(k;r,x) = 0$.

Proof The proof follows similarly to **Theorem 2.3**. □

Theorem 4.3. *The Horizontal Generating function of the r-Stirling Fibonacci polynomials of the first and second kind is given by respectively;*

$$\sum_{n=0}^{\infty} SF_n^1\,(k;r,x)\, z^k = F_n^{(3)}(x+z-r) \tag{4.5}$$

$$\sum_{n=0}^{\infty} SF_n^2\,(k;r,x)\, z^{\underline{k}} = F_n^{(2)}(x+z+r) \tag{4.6}$$

Proof. The proof follows similarly to **Theorem 2.4**. □

Theorem 4.4. *The Schlömilch-type formula of the r-Stirling Fibonacci polynomial is given by*

$$SF_n^1(k;r,x) = \sum_{m=k}^{n} \sum_{j=0}^{m} \sum_{a=0}^{m-k} \sum_{b=0}^{a} (-1)^{n-j+b+a} \binom{a}{b} \binom{n}{j} \binom{j}{m} F_{j-m}(x) \tag{4.7}$$
$$\times \binom{m-1+a}{m-k+a} \binom{2m-k}{m-k-a} \frac{(a-b)^{m-k+a}}{a!} r^{\overline{n-j}}$$

Proof The proof follows similarly to **Theorem 2.5**. □

Theorem 4.5. *The Explicit formula of the r-Stirling Fibonacci polynomial of the second kind is given by*

$$SF_n^2\,(k;r,x) = \sum_{m=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \binom{n}{m} F_{n-m}(x) \tag{4.8}$$

Proof The proof follows similarly to **Theorem 2.6**. □

5. r-Stirling Fibonacci Polynomials Identities via Exponential Generating Function

In this section, we derive new identities for the r-Stirling Fibonacci polynomials by employing exponential generating functions. These identities establish explicit connections between modified Stirling-type polynomials and Fibonacci-type structures enriched with Chebyshev polynomial components. We begin by introducing two forms of r-Stirling Chebyshev polynomials of the first and second kind, which serve as essential generating tools in the subsequent derivations. These definitions form the foundation for constructing closed-form expressions and recurrence-type relations that generalize classical results through the lens of hyperbolic and exponential function techniques. The identities are validated using series expansions and Cauchy product methods, culminating in elegant representations that link special polynomial sequences through combinatorial and analytic perspectives.

Definition 5.1. The first form of r-Stirling Chebyshev polynomial of the first and second kind is defined by;

$$\begin{aligned} \tau_1(t,x) &= \sum_{n=0}^{\infty} ST_n^1\left(k;r,x\right)\frac{t^n}{n!} \qquad (5.1)\\ &= \left(\frac{1}{1+t}\right)^r \frac{\left(2e^{xt}\cosh\left(\sqrt{x^2-1}t\right)\right)\left(\ln^k\left(1+t\right)\right)}{k!} \end{aligned}$$

$$\begin{aligned} \tau_2(t,x) &= \sum_{n=0}^{\infty} ST_n^2\left(k;r,x\right)\frac{t^n}{n!} \qquad (5.2)\\ &= \frac{\left(2e^{xt}\cosh\left(\sqrt{x^2-1}t\right)\right)\left(e^{rt}(e^t-1)\right)^k}{k!} \end{aligned}$$

Definition 5.2. The second form of r-Stirling Chebyshev polynomial of the first and second kind is defined by;

$$\begin{aligned} \omega_1(t,x) &= \sum_{n=0}^{\infty} SU_n^1\left(k;r,x\right)\frac{t^n}{n!} \qquad (5.3)\\ &= \left(\frac{1}{1+t}\right)^r \frac{e^{xt}\left(x\sinh\left(\sqrt{x^2-1}t\right)+\sqrt{x^2-1}\cosh(\sqrt{x^2-1}t)\right)\left(\ln^k\left(1+t\right)\right)}{k!} \end{aligned}$$

$$\begin{aligned} \omega_2(t,x) &= \sum_{n=0}^{\infty} SU_n^2\left(k;r,x\right)\frac{t^n}{n!} \qquad (5.4)\\ &= \frac{e^{xt}\left(x\sinh\left(\sqrt{x^2-1}t\right)+\sqrt{x^2-1}\cosh(\sqrt{x^2-1}t)\right)\left(e^{rt}(e^t-1)^k\right)}{k!} \end{aligned}$$

Theorem 5.3. *For $n \geq 0$ the formula holds,*

$$\sum_{k=0}^{n-1} ST_k^1(m;r,x)\binom{n}{k}\left(\left(\sqrt{x^2+4}\right)^{n-k-1}\left(1-(-1)^{n-k}\right)\right) \tag{5.5}$$

$$= \sum_{k=1}^{n}\left(\frac{2+2t}{2+t}\right)^r \ln^m\left(\frac{-t}{2}\right) SF_k^1(m;r,x)\binom{n}{k}2^{k-1}$$

$$\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(1+(-1)^{n-k}\right)\right) \tag{5.6}$$

where $t < 0$ and $t \neq -2$.

Proof To prove (5.5) we use the exponential generating functions in (4.1) and (5.1), which gives a functional equation,

$$2\tau_1\left(\frac{t}{2},x\right)\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)$$

$$= \frac{2^r(1+t)^r \ln^k\left(\frac{2+t}{2}\right)\sqrt{x^2+4}\cosh\left(\frac{\sqrt{x^2-1}}{2}t\right)\phi(t,x)}{(2+t)^r \ln^k(1+t)}. \tag{5.7}$$

After solving the equation by expressing the hyperbolic functions in terms of exponential functions and Cauchy Product yields LHS and RHS expressions,
for LHS,

$$LHS = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1} ST_k^1(m;r,x)\binom{n}{k}\frac{1}{2^n}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)\right\}\frac{t^n}{n!}$$

Similarly, the RHS,

$$RHS = \left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)\frac{\sqrt{x^2+4}}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}$$
$$\times\left(\left(\frac{\sqrt{x^2-1}}{2}\right)^{n-k}\left(1+(-1)^{n-k}\right)\right)\frac{t^n}{n!}$$

Now, by (5.7) we have

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1} ST_k^1(m;r,x)\binom{n}{k}\frac{1}{2^n}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)\right\}\frac{t^n}{n!}$$
$$= \left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)\frac{\sqrt{x^2+4}}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}$$

$$\times\left(\left(\frac{\sqrt{x^2-1}}{2}\right)^{n-k}\left(1+(-1)^{n-k}\right)\right)\frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ and expressing 2^{n-k+1} as $2^n 2^{k-1}$ yields,

$$\sum_{k=0}^{n-1} ST_k^1(m;r,x)\binom{n}{k}\frac{1}{2^n}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)$$
$$=\left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)\sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}\frac{\sqrt{x^2+4}}{2^n}2^{k-1}$$
$$\times\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(1+(-1)^{n-k}\right)\right)$$

Multiplying $\frac{2^n}{\sqrt{x^2+4}}$ both sides yields (5.5) □

Remark 5.4. The term

$$\left(\frac{2(1+t)}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)$$

is well defined when $t<0$ and $t\neq-2$.

Theorem 5.5. *For $n\geq 0$ the formula holds,*

$$\sum_{k=0}^{n-1}\binom{n}{k}\left(\sqrt{x^2+4}\right)^{n-k-1}\left(1-(-1)^{n-k}\right)ST_k^2(m;r,x)$$
$$=\sum_{k=1}^{n} SF_k^2(m;r,x)\,e^{\frac{-rt}{2}}\left(\frac{e^{\frac{t}{2}}-1}{e^t-1}\right)^m\binom{n}{k}2^{k-1}$$
$$\times\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(1+(-1)^{n-k}\right)\right) \tag{5.8}$$

where $t\geq 0$.

Proof To prove (5.8) we use the exponential generating functions in (4.2) and (5.2),which gives a functional equation,

$$2\tau_2\left(\frac{t}{2},x\right)\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)$$
$$=e^{\frac{-rt}{2}}\left(\frac{e^{\frac{t}{2}}-1}{e^t-1}\right)^k\gamma(t,x)\sqrt{x^2+4}\cosh\left(\frac{\sqrt{x^2-1}}{2}t\right) \tag{5.9}$$

After solving the equation by expressing the hyperbolic functions in terms of exponential functions and Cauchy Product yields LHS and RHS expressions,
for LHS,

$$LHS = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n-1} ST_k^2\,(m;r,x) \binom{n}{k} \frac{1}{2^n} \left(\left(\sqrt{x^2+4}\right)^{n-k} \left(1-(-1)^{n-k}\right)\right) \right\} \frac{t^n}{n!}$$

Similarly the RHS,

$$RHS = e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}}-1}{e^t-1} \right)^m \frac{\sqrt{x^2+4}}{2} \sum_{n=0}^{\infty} \sum_{k=1}^{n} SF_k^2\,(m;r,x) \binom{n}{k} \times \left(\frac{\sqrt{x^2-1}}{2} \right)^{n-k} (1+(-1)^{n-k}) \frac{t^n}{n!}$$

Now, by (5.9) we have

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n-1} ST_k^2\,(m;r,x) \binom{n}{k} \frac{1}{2^n} \left(\left(\sqrt{x^2+4}\right)^{n-k} \left(1-(-1)^{n-k}\right)\right) \right\} \frac{t^n}{n!}$$
$$= e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}}-1}{e^t-1} \right)^m \frac{\sqrt{x^2+4}}{2} \sum_{n=0}^{\infty} \left\{ \sum_{k=1}^{n} SF_k^2\,(m;r,x) \binom{n}{k} \left(\left(\frac{\sqrt{x^2-1}}{2} \right)^{n-k} (1+(-1)^{n-k}) \right) \right\} \frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ and expressing 2^{n-k+1} as $2^n 2^{k-1}$ yields,

$$\sum_{k=0}^{n-1} ST_k^2\,(m;r,x) \binom{n}{k} \frac{1}{2^n} \left(\left(\sqrt{x^2+4}\right)^{n-k} \left(1-(-1)^{n-k}\right)\right)$$
$$= e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}}-1}{e^t-1} \right)^m \frac{\sqrt{x^2+4}}{2} \sum_{k=1}^{n} SF_k^2\,(m;r,x) \binom{n}{k} \times \left(\left(\frac{\sqrt{x^2-1}}{2} \right)^{n-k} (1+(-1)^{n-k}) \right)$$

Multiplying $\frac{2^n}{\sqrt{x^2+4}}$ both sides yields (5.8) □

Theorem 5.6. For $n \geq 0$ the formula holds,

$$\sum_{k=0}^{n-1} SU_k^1(m;r,x) \binom{n}{k} \left(\left(\sqrt{x^2+4}\right)^{n-k-1} \left(1-(-1)^{n-k}\right)\right)$$

$$= \sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}\left(\frac{2+2t}{2+t}\right)^r \ln^m\left(\frac{-t}{2}\right)2^{k-1}$$
$$\times\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(\alpha(x)-\beta(x)(-1)^{n-k}\right)\right) \tag{5.10}$$

where $\alpha(x)=x+\sqrt{x^2-1}$, $\beta(x)=x-\sqrt{x^2-1}$,$t<0$ and $t\neq -2$.

Proof To prove (5.10) we use the exponential generating functions in (4.1) and (5.3),which gives a functional equation,

$$2\sqrt{x^2-1}\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)\omega_1\left(\frac{t}{2},x\right)$$
$$=\frac{2^r(1+t)^r\ln^k\left(\frac{2+t}{2}\right)}{(2+t)^r\ln^k(1+t)}\sqrt{x^2+4}\left(x\sinh\left(\frac{\sqrt{x^2-1}}{2}t\right)\right.$$
$$\left.+\sqrt{x^2-1}\cosh\left(\frac{\sqrt{x^2-1}}{2}t\right)\right)\phi(t,x) \tag{5.11}$$

After solving the equation by expressing the hyperbolic functions in terms of exponential functions and Cauchy Product yields LHS and RHS expressions,
for LHS,

$$LHS=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1}SU_k^1(m;r,x)\frac{1}{2^n}\binom{n}{k}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)\right\}\frac{t^n}{n!}$$

Similarly the RHS,

$$RHS=\sum_{n=0}^{\infty}\sum_{k=0}^{n}SF_k^1(m;r,x)\frac{n!}{k!(n-k)!}\left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)$$
$$\times\frac{\sqrt{x^2+4}}{2}\left(\alpha(x)\left(\frac{\sqrt{x^2-1}}{2}\right)^{n-k}-\beta(x)\left(\frac{\sqrt{x^2-1}}{2}\right)^{n-k}\right)\frac{t^n}{n!}.$$

Now, by (5.11) we have

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1}SU_k^1(m;r,x)\frac{1}{2^n}\binom{n}{k}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)\right\}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=1}^{n}SF_k^1(m;r,x)\binom{n}{k}\left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)$$
$$\times\frac{\sqrt{x^2+4}}{2^{n-k+1}}\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(\alpha(x)-\beta(x)(-1)^{n-k}\right)\right)\frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ and expressing 2^{n-k+1} as $2^n 2^{k-1}$ yields,

$$\sum_{k=0}^{n-1} SU_k^1(m;r,x)\frac{1}{2^n}\binom{n}{k}\left(\left(\sqrt{x^2+4}\right)^{n-k}\left(1-(-1)^{n-k}\right)\right)$$
$$=\sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}\left(\frac{2+2t}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)$$
$$\times\frac{\sqrt{x^2+4}}{2^n}2^{k-1}\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(\alpha(x)-\beta(x)(-1)^{n-k}\right)\right)$$

Multiplying $\frac{2^n}{\sqrt{x^2+4}}$ both sides yields (5.10) □

Remark 5.7. The term

$$\left(\frac{2(1+t)}{2+t}\right)^r\left(\ln^m\left(\frac{-t}{2}\right)\right)$$

is well defined when $t<0$ and $t\neq -2$.

Theorem 5.8. *For $n\geq 0$ the formula holds,*

$$\sum_{k=0}^{n-1} SU_k^1(m;r,x)\binom{n}{k}\left(\left(\sqrt{x^2+4}\right)^{n-k-1}\left(1-(-1)^{n-k}\right)\right)$$
$$=\sum_{k=1}^{n} SF_k^1(m;r,x)\binom{n}{k}\left(e^{\frac{-rt}{2}}\left(\frac{e^{\frac{t}{2}}-1}{e^t-1}\right)^m\right)2^{k-1}$$
$$\left(\left(\sqrt{x^2-1}\right)^{n-k}\left(\alpha(x)-\beta(x)(-1)^{n-k}\right)\right) \tag{5.12}$$

where $\alpha(x)=x+\sqrt{x^2-1}$, $\beta(x)=x-\sqrt{x^2-1}$.

Proof To prove (5.12)we use the exponential generating functions in eq. (4.2) and (5.4),which gives a functional equation,

$$2\sqrt{x^2-1}\sinh\left(\frac{\sqrt{x^2+4}}{2}t\right)\omega_2\left(\frac{t}{2},x\right)$$
$$=e^{\frac{-rt}{2}}\left(\frac{e^{\frac{t}{2}}-1}{e^t-1}\right)^k\sqrt{x^2+4}\left(x\sinh\left(\frac{\sqrt{x^2-1}}{2}t\right)\right.$$
$$\left.+\sqrt{x^2-1}\cosh\left(\frac{\sqrt{x^2-1}}{2}t\right)\right)\gamma(t,x) \tag{5.13}$$

After solving the equation by expressing the hyperbolic functions in terms of exponential functions and Cauchy Product yields LHS and RHS expressions, for LHS,

$$LHS = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n-1} SU_k^2(m;r,x) \frac{1}{2^n} \binom{n}{k} \left(\left(\sqrt{x^2+4} \right)^{n-k} \left(1 - (-1)^{n-k} \right) \right) \right\} \frac{t^n}{n!}$$

Similarly the RHS,

$$RHS = \sum_{n=0}^{\infty} \sum_{k=0}^{n} SF_k^2(m;r,x) \binom{n}{k} \left(e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}} - 1}{e^t - 1} \right)^m \right)$$
$$\times \frac{\sqrt{x^2+4}}{2} \left(\alpha(x) \left(\frac{\sqrt{x^2-1}}{2} \right)^{n-k} - \beta(x) \left(\frac{\sqrt{x^2-1}}{2} \right)^{n-k} \right) \frac{t^n}{n!}$$

Now, by (5.13) we have

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n-1} SU_k^2(m;r,x) \frac{1}{2^n} \binom{n}{k} \left(\left(\sqrt{x^2+4} \right)^{n-k} \left(1 - (-1)^{n-k} \right) \right) \right\} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=1}^{n} SF_k^2(m;r,x) \binom{n}{k} \left(e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}} - 1}{e^t - 1} \right)^m \right)$$
$$\times \frac{\sqrt{x^2+4}}{2^{n-k+1}} \left(\left(\sqrt{x^2-1} \right)^{n-k} \left(\alpha(x) - \beta(x)(-1)^{n-k} \right) \right) \frac{t^n}{n!}$$

Comparing coefficients of $\frac{t^n}{n!}$ and expressing 2^{n-k+1} as $2^n 2^{k-1}$ yields,

$$\sum_{k=0}^{n-1} SU_k^1(m;r,x) \frac{1}{2^n} \binom{n}{k} \left(\left(\sqrt{x^2+4} \right)^{n-k} \left(1 - (-1)^{n-k} \right) \right)$$
$$= \sum_{k=1}^{n} SF_k^1(m;r,x) \binom{n}{k} \left(e^{\frac{-rt}{2}} \left(\frac{e^{\frac{t}{2}} - 1}{e^t - 1} \right)^m \right)$$
$$\times \frac{\sqrt{x^2+4}}{2^n} 2^{k-1} \left(\left(\sqrt{x^2-1} \right)^{n-k} \left(\alpha(x) - \beta(x)(-1)^{n-k} \right) \right)$$

Multiplying $\frac{2^n}{\sqrt{x^2+4}}$ both sides yields (5.12) □

6. Conclusion and Recommendations

This study introduced and developed a novel class of combinatorial structures known as the r-Stirling Fibonacci numbers and polynomials of the first and second kind. By integrating the exponential generating functions of Fibonacci numbers with those of the signed r-Stirling numbers, we established an enriched framework that deepens the mathematical theory of special numbers and polynomials.

Through this construction, we derived new and elegant identities, including horizontal generating functions, explicit expressions, and convolution formulas that generalize classical results in combinatorics. Furthermore, we introduced the r-Stirling Chebyshev polynomials of the first and second kind, using hyperbolic functions and exponential techniques to create a functional bridge between Fibonacci-type and Stirling-type sequences. These derivations were rigorously validated through series expansions and the Cauchy product rule, confirming their consistency and opening new pathways in discrete mathematics.

The results of this research offer significant insights into the interplay between combinatorics, algebra, and analysis. The structures revealed by the presence of alternating signs, binomial coefficients, and hyperbolic identities not only broaden our understanding of Fibonacci-related sequences but also suggest deeper connections to number theory, orthogonal polynomials, and symbolic computation.

Given the promising findings of this work, we propose the following directions for further research and exploration:

(i) *Generalizations via q-Analogues and* (p, q)*-Extensions.* We recommend investigating q-analogues or (p, q)-extensions of the r-Stirling Fibonacci numbers and polynomials. These generalizations can yield new recurrence relations, identities, and applications, particularly in the context of quantum algebra and combinatorial enumeration.

(ii) *Combinatorial Models and Interpretations.* It is essential to explore combinatorial models - such as labeled graphs, partition structures, or lattice paths - that offer tangible interpretations of the r-Stirling Fibonacci sequences and polynomials. Such models can enrich their applicability and enhance intuition for their properties.

(iii) *Applications to Number Theory and Discrete Structures.* The derived sequences have potential applications in number theory, tiling problems, recurrence relations, and discrete dynamical systems. We recommend investigating their role in coding theory, integer partitions, and modular arithmetic.

(iv) *Extension to Other Polynomial Families.* The exponential generating function framework used here may be applied to other orthogonal polynomial families - such as Legendre, Hermite, Laguerre, and Jacobi polynomials - by introducing r-Stirling analogues. This could uncover new algebraic and combinatorial relationships and extend the utility of the approach.

(v) *Symbolic and Computational Implementations.* Developing efficient symbolic algorithms for computing r-Stirling Fibonacci numbers and polynomials will support further research and educational use. Tools such as Mathematica, Maple, and SageMath may be utilized to visualize, manipulate, and analyze these sequences at higher orders.

(vi) *Multivariate and Matrix Extensions.* Future work may explore multivariate versions or matrix representations of these identities, which could reveal richer symmetries and deeper structural insights-especially relevant in linear algebra and applied fields.

By pursuing these directions, researchers can build upon the foundational results of this study and continue to expand the theoretical and practical landscape of r-Stirling Fibonacci numbers and polynomials within mathematics and related disciplines.

References

[1] J. Stirling. *Methodus Differentialis, Sire Tractatus De Summatione et Interpolatione Serierum Infinitorum*. London, 1730.

[2] R. B. Corcino and C. B. Corcino. The hankel transform of generalized bell numbers and its q-analogue. *Utilitas Mathematica*, 89:297–309, 2012.

[3] R. B. Corcino and C. B. Montero. A q-analogue of rucinski-voigt numbers. *ISRN Discrete Mathematics*, page 818, 2012.

[4] R. B. Corcino and C. Barrientos. Some theorems on the q-analogue of the generalized stirling numbers. *Bulletin of the Malaysian Mathematical Sciences Society*, 34(3):487–501, 2011.

[5] R. B. Corcino, L. C. Hsu, and E. L. Tan. A q-analogue of generalized stirling numbers. *The Fibonacci Quarterly*, 44(2):154–165, 2006.

[6] R. B. Corcino and C. B. Corcino. An asymptotic formula for the r-bell numbers. *Matimyas Matematika*, 24(1):9–18, 2001.

[7] I. Mező and R. B. Corcino. The estimation of the zeros of the bell and r-bell polynomials. *Applied Mathematics and Computation*, 250:727–732, 2015.

[8] A. Z. Broder. The r-stirling numbers. *Discrete Mathematics*, 49(3):241–259, 1984.

[9] R. Corcino, V. Dechosa, R. Coronel, and V. M. Deveraturda. The sm r-stirling numbers: An algebraic approach. *CNU-Journal of Higher Education*, 17:1–14, 2023.

[10] L. Sigler. *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*. Springer, New York, 2002.

[11] C. A. Church and M. Bicknell. Exponential generating functions for fibonacci identities. *The Fibonacci Quarterly*, 11(3):275–281, 1973.

[12] C. Cesarano, W. Ramirez, and S. Khan. A new class of degenerate apostol-type hermite polynomials and applications. *Dolomites Research Notes on Approximation*, 15(1):1–10, 2022.

[13] W. Ramirez and C. Cesarano. Some new classes of degenerated generalized apostol-bernoulli, apostol-euler and apostol-genocchi polynomials. *Carpathian Mathematical Publications*, 14(2):354–363, 2022.

[14] W. Ramirez, C. Cesarano, and S. Diaz. New results for degenerated generalized apostol-bernoulli, apostol-euler and apostol-genocchi polynomials. *WSEAS Transactions on Mathematics*, 21:604–608, 2022.

[15] R. B. Corcino, M. B. Montero, and S. L. Ballenas. Schlömilch-type formula for r-whitney numbers of the first kind. *European Journal of Pure and Applied Mathematics*, 2024.