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Analytic and Numerical Approaches for Solving Nonlinear Painlevé Equations I and II

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Abstract. Nonlinear Painlevé equations play a pivotal role in various branches of mathematical physics, integrable systems, and applied mathematics. These equations, characterized by their complex and highly nonlinear nature, present significant challenges for analytical and numerical investigation. In this paper, we develop and present both analytical and numerical solutions for specific classes of nonlinear Painlevé equations. The analytical approach employs transformation techniques, perturbative expansions, and exact solution methods where applicable, while the numerical solutions are obtained using robust algorithms such as finite difference schemes, spectral methods, and iterative solvers. We validate the numerical results by comparing them with known analytical solutions and explore their accuracy, stability, and convergence properties. Furthermore, we discuss the implications of these solutions in physical models and highlight the intricate structures exhibited by the solutions, such as pole dynamics and asymptotic behaviors. This work contributes to the broader understanding of Painlevé equations and provides a framework for tackling similar nonlinear differential equations in applied contexts.

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Key Words and Phrases: Painlevé equations, DTM, MADM, LDM, NDM, Finite difference method, Runge-Kutta method

1. Introduction

The Painlevé equations were first introduced between 1895 and 1910 through the work of two French mathematicians, Paul Painlevé and Bertrand Gambier. They were identified as second-order differential equations that play a crucial role in various fields of mathematics and physics. They have found applications in areas such as nonlinear waves, plasma physics, statistical mechanics, fiber optics, and more [1–4]. Despite their importance, Painlevé equations are highly nonlinear and do not admit general closed-form solutions, and traditional methods such as separation of variables or perturbation techniques often

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fail due to the equations' complexity. As a result, researchers have focused on developing analytical and numerical approaches to approximate and analyze these equations effectively. Numerous analytical and numerical methods have been successfully developed to solve the Painlevé equations. For example, Differential Transform Method (DTM) [5, 6], Modified Adomian decomposition method (MADM) [7, 8], Laplace decomposition Method (LDM) [9, 10] and Natural decomposition Method [11, 12], Finite difference method [13, 14], Runge Kutta method [15, 16].

This paper aims to develop and compare analytical and numerical techniques for solving Painlevé I and II equations. It contributes to the broader understanding of providing reliable tools and bridging the gap between analytical and numerical techniques for applied scientists and engineers to study nonlinear differential equations in real-world applications.

This paper is organized as follows: Section 2 introduces the Analytical methods. Subsection 2.1 Differential Transform Method. In subsection 2.2. Modified Adomian decomposition Method. Subsection 2.3. Laplace decomposition Method. In subsection 2.4. Natural Decomposition Method. In Section 3 Numerical Methods. Subsection 3.1. Finite difference method. In subsection 3.2. Runge Kutta method. In section 4 Comparative Analysis of Analytical and Numerical Methods. Finally, Section 5 offers concluding remarks for the paper.

Painlevé equations of the first and second types will be defined by the following formulas:

$$y''(x) = 6y^2(x) + x, \quad 0 < x < 1 \tag{1}$$

With the given initial conditions: $y(0) = 0,\ y'(0) = 1$.

$$y''(x) = 2y^3(x) + xy(x) + \mu, \quad 0 < x < 1 \tag{2}$$

With the initial conditions: $y(0) = 1,\ y'(0) = 0$, where μ is a known parameter.

Error analysis

Since the exact solutions for Painlevé I and II are unknown, we utilized the maximal error remainder MER_n to evaluate the accuracy of the approximate solutions. The MER_n for DTM, MADM, LDM, and NDM was computed using the software *Mathematica* code over the domain $[0.01, 0.1]$.

$$ER_n = L(y) - N(y) - f(x). \tag{3}$$

The maximal error remainder is

$$MER_n = \max_{0.01 \leq x \leq 0.1} |ER_n(x)|. \tag{4}$$

The selection of the interval $[0.01, 0.1]$ for the maximal error remainder (MER) analysis is based on consideration of analytic stability and accuracy. Limiting the analysis to this range allows for a more precise assessment of the approximation methods. Furthermore, this interval is specifically chosen to focus on the region where analytical approximations are most effective for accuracy evaluation, as extending the domain further could lead to additional computational complexities without significantly impacting the overall conclusions.

2. Analytical methods for solving Painlevé Equations I, II

Analytical methods provide powerful techniques for solving nonlinear ordinary differential equations. In this section, we apply the Differential Transform Method (DTM), Modified Adomian Decomposition Method (MADM), Laplace Decomposition Method (LDM), and Natural Decomposition Method (NDM) to obtain approximate or closed-form solutions. These methods simplify nonlinear equations by transforming them into more manageable forms, offering advantages in accuracy, convergence, and efficiency. The objective is to analyze and compare their effectiveness in solving Painlevé equations, highlighting their role in nonlinear differential equation research [1, 2].

2.1. Differential Transform Method (DTM)

The Differential Transform, initially introduced by Zhou [5, 6, 17–20] is a technique for solving differential equations. This method determines the coefficients of a Taylor series of the function by solving a recursive equation induced by the given differential equation. The Differential Transform Method has been effectively applied to solve initial value problems represented by strongly ordinary differential equations [21–25].

The differential transform of a function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\left(\frac{d^k y(x)}{dx^k}\right)\bigg|_{x=0}. \tag{5}$$

Where $y(x)$ is the original function and $Y(k)$ is the transformed function. Differential inverse transform of $Y(k)$ is defined as:

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_n(x) = \sum_{k=0}^{n} Y(k)x^k. \tag{6}$$

By substituting (5) in (6) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=0}. \tag{7}$$

The differential transform verified the following properties: [26, 27]

- If $u(x) = u_1(x) \pm u_2(x)$, then $U(k) = U_1(k) \pm U_2(k)$.
- If $u(x) = cu_1(x)$ then $U(k) = cU_1(k)$, where c is constant.
- If $u(x) = \frac{d^n u_1(x)}{dx^n}$, then $U(k) = \frac{(k+n)!}{k!}U_1(k+n)$.

- If $u(x) = u_1(x)\, u_2(x)$, then $U(k) = \sum_{r=0}^{k} U_1(r)U_2(k-r)$.
- If $u(x) = u_1(x)u_2(x)\ldots u_n(x)$, then

$$U(k) = \sum_{r_{n-1}=0}^{k} \sum_{r_{n-2}=0}^{r_{n-1}} \cdots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} U_1(r_1)U_2(r_2-r_1)\ldots U_{n-1}(r_{n-1}-r_{n-2})U_n(k-r_{n-1}).$$

- If $u(x) = ax^m$, then $U(k) = a\delta(k-m)$, where

$$\delta(k-m) = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

- If $u(x) = u_1(x)\frac{du_2(x)}{dx}$, then $U(k) = \sum_{r=0}^{k}(k-r+1)U_1(r)U_2(k-r+1)$.

2.1.1. The DTM for solving the Painlevé I differential equation

Rewrite (1)

$$\frac{d^2y}{dx^2} = 6y^2 + x, \quad y(0) = 0, \quad y'(0) = 1.$$

Assume $y(x)$ can be expressed as a power series:

$$y(x) = \sum_{k=0}^{\infty} Y_k x^k.$$

From the initial conditions:

$$Y_0 = y(0) = 0, \quad Y_1 = y'(0) = 1.$$

Apply the DTM to the differential equation.
The second derivative is:

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(k+2)(k+1)Y_{k+2}x^k. \tag{8}$$

$$6y^2 = 6\left(\sum_{k=0}^{\infty} Y_k x^k\right)^2 = 6\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} Y_m Y_{k-m}\right)x^k. \tag{9}$$

$$x = \sum_{k=0}^{\infty}\delta(k-1)x^k. \tag{10}$$

Substitute (8),(9) and (10) into the equation.

$$\sum_{k=0}^{\infty}(k+2)(k+1)Y_{k+2}x^k = 6\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} Y_m Y_{k-m}\right)x^k\right) + \sum_{k=0}^{\infty}\delta(k-1)x^k.$$

For each k we get

$$(k+2)(k+1)Y_{k+2} = 6\sum_{m=0}^{k} Y_m Y_{k-m} + \delta(k-1). \tag{11}$$

$k = 0$

$$(0+2)(0+1)Y_{0+2} = 6\sum_{m=0}^{0} Y_m Y_{0-m} + \delta(0-1).$$

$$2Y_2 = 6Y_0Y_0 + 0 = 0.$$

$$Y_2 = 0.$$

$k = 1$

$$(1+2)(1+1)Y_{1+2} = 6\sum_{m=0}^{1} Y_m Y_{1-m} + \delta(1-1).$$

$$6Y_3 = 6(Y_0Y_1 + Y_1Y_0) + 1 = 6(0(1) + 1(0)) + 1 = 1.$$

$$Y_3 = \frac{1}{6}.$$

$k = 2$

$$(2+2)(2+1)Y_{2+2} = 6\sum_{m=0}^{2} Y_m Y_{2-m} + \delta(2-1).$$

$$12Y_4 = 6(Y_0Y_2 + Y_1Y_1 + Y_2Y_0) + 0 = 6(0(1) + 1(0)) = 6.$$

$$Y_4 = \frac{1}{2}.$$

$k = 3$

$$(3+2)(3+1)Y_{3+2} = 6\sum_{m=0}^{3} Y_m Y_{3-m} + \delta(3-1).$$

$$20Y_5 = 6(Y_0Y_3 + Y_1Y_2 + Y_2Y_1 + Y_3Y_0) + 0 = 6\left(0 + 1(0) + 0(1) + \frac{1}{6}(0)\right) = 0.$$

$$Y_5 = 0.$$

$$\vdots$$

$$Y_0 = 0, \quad Y_1 = 1, \quad Y_2 = 0, \quad Y_3 = \frac{1}{6}, \quad Y_4 = \frac{1}{2}, \quad Y_5 = 0, \quad Y_6 = \frac{1}{15}, \quad Y_7 = \frac{1}{7},$$

$$Y_8 = \frac{1}{336}, \quad Y_9 = \frac{1}{40}, \quad Y_{10} = \frac{1}{28}, \quad \ldots$$

$$y(x) = Y_0 + Y_1x + Y_2x^2 + Y_3x^3 + Y_4x^4 + Y_5x^5 + Y_6x^6 + Y_7x^7 + Y_8x^8 + Y_9x^9 + \ldots$$

$$y(x) = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^7}{7} + \frac{x^8}{336} + \frac{13\,x^9}{2160} + \frac{x^{10}}{28} + \ldots$$

The error remainder function is evaluated:

$$ER_n = y_n''(x) - 6y_n^2(x) - x.$$

And the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 1 and Figure 1 illustrate the convergence of the solution using MER_n versus n, demonstrating the convergence behavior of the Differential Transform Method (DTM) in solving the first Painlevé equation. The error decreases exponentially as n increases, with the most significant reduction occurring between $n = 1$ and $n = 3$. Beyond $n = 3$, the error stabilizes around 10^{-17}, indicating that further iterations do not significantly improve accuracy due to machine precision limitations. The stabilization of error values for $n = 4$ and $n = 5$ implies that additional iterations beyond this point may be computationally unnecessary. This suggests that DTM achieves high accuracy with a relatively small number of iterations, making it an efficient method for solving nonlinear differential equations.

Table 1: The maximum residual error: MER_n by the DTM where $n = 1, \cdots, 5$.

n	MER_n
1	3.790200×10^{-7}
2	1.116379×10^{-10}
3	1.040834×10^{-16}
4	4.857226×10^{-17}
5	4.857226×10^{-17}

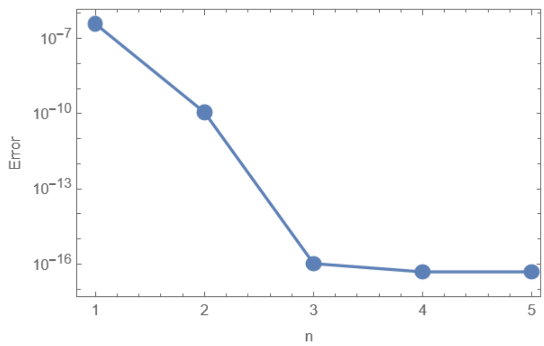

Figure 1: The logarithmic plot of MER $_n$ (DTM, First Painlevé Equation). .

2.1.2. The DTM for solving the Painlevé II differential equation

Rewrite [2]

$$\frac{d^2y}{dx^2} = 2y^3 + xy + \mu, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Assume $y(x)$ can be expressed as a power series:

$$y(x) = \sum_{k=0}^{\infty} Y_k x^k.$$

From the initial conditions:

$$Y_0 = y(0) = 1, \qquad Y_1 = y'(0) = 0.$$

Apply the DTM to the differential equation:

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty}(k+2)(k+1)Y_{k+2}x^k. \tag{12}$$

$$2y^3 = 2\left(\sum_{k=0}^{\infty} Y_k x^k\right)^3 = 2\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m} Y_n Y_{m-n} Y_{k-m}\right)x^k. \tag{13}$$

$$xy = \sum_{k=1}^{k} \delta(k-1)Y_{k-1}x^k. \tag{14}$$

$$\mu x^0 = \mu\sum_{k=0}^{\infty} \delta(k-0)x^k. \tag{15}$$

Substitute (12), (13), (14) and (15) into the equation:

$$\sum_{k=0}^{\infty}(k+2)(k+1)Y_{k+2}x^k = 2\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m} Y_n Y_{m-n} Y_{k-m}\right)x^k \dots$$
$$+\sum_{k=1}^{\infty} \delta(k-1)Y_{k-1}x^k + \mu\sum_{k=0}^{\infty} \delta(k-0)x^k.$$

For each k we get:

$$Y_{k+2} = \frac{1}{(k+2)(k+1)}\left(2\sum_{m=0}^{k}\sum_{n=0}^{m} Y_n Y_{m-n} Y_{k-m} + \delta(k-1)Y_{k-1} + \mu\delta(k-0)\right). \tag{16}$$

$k = 0$

$$Y_2 = \frac{2+\mu}{2}.$$

$k = 1$

$$Y_{1+2} = \frac{1}{(1+2)(1+1)}\left(2\sum_{m=0}^{1}\sum_{n=0}^{m} Y_n Y_{m-n} Y_{1-m} + \delta(1-1)Y_{1-1} + \mu\delta(1-0)\right).$$

$$Y_3 = \frac{1}{6}.$$

$k = 2$

$$Y_{2+2} = \frac{1}{(2+2)(2+1)}\left(2\sum_{m=0}^{2}\sum_{n=0}^{m} Y_n Y_{m-n} Y_{2-m} + \delta(2-1)Y_{2-1} + \mu\delta(2-0)\right).$$

$$Y_4 = \frac{2+\mu}{4},$$

$$Y_5 = \frac{1}{20},$$

$$Y_6 = \frac{1}{20}(2+\mu+(2+\mu)^2),$$

$$Y_7 = \frac{3+\mu}{84},$$

$$\vdots$$

$$y(x) = Y_0 + Y_1 x + Y_2 x^2 + Y_3 x^3 + Y_4 x^4 + Y_5 x^5 + Y_6 x^6 + Y_7 x^7 + \cdots$$

$$y(x) = 1 + \frac{2+\mu}{2}x^2 + \frac{1}{6}x^3 + \frac{2+\mu}{4}x^4 + \frac{1}{20}x^5 + \frac{1}{20}(2+\mu+(2+\mu)^2)x^6 + \frac{3+\mu}{84}x^7 + \cdots$$

The error remainder function is evaluated:

$$ER_n = y_n''(x) - 2y_n^3(x) - xy_n(x) - \mu.$$

And the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 2 and Figure 2 show the convergence of the solution using MER_n versus n demonstrates the convergence behavior of the Differential Transform Method (DTM) in solving the second Painlevé equation. The error decreases significantly from $n = 1$ to $n = 2$, showing a rapid initial improvement in accuracy. However, from $n = 3$ onward, the reduction in error becomes much smaller, indicating a slower rate of convergence. The stabilization of MER_n at approximately 0.001537 for $n = 4$ and $n = 5$ suggests that further iterations do not lead to a significant decrease in error, likely due to numerical limitations or the inherent accuracy of the method. Overall, the DTM achieves rapid initial convergence but reaches a saturation point where further improvements are marginal.

Table 2: The maximum residual error: MER_n by the DTM where $n = 1, \cdots, 5$.

n	MER_n
1	0.09390383
2	0.00337824
3	0.00156204
4	0.00153786
5	0.00153758

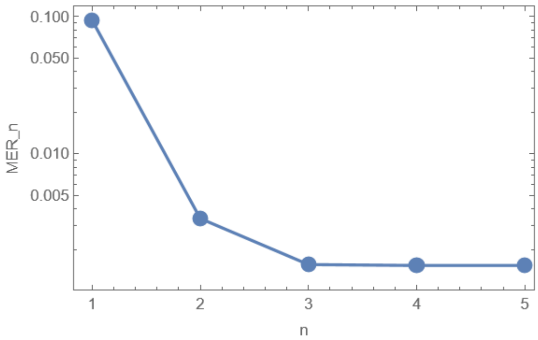

Figure 2: The logarithmic plot of MER_n (DTM, Second Painlevé Equation).

2.2. Modified Adomian decomposition method (MADM)

The Adomian Decomposition Method (ADM), introduced by George Adomian in the 1980s, [7], revolutionized the solution of nonlinear differential equations by providing an efficient decomposition approach without linearization or perturbation [8]. Over the decades, ADM has been widely applied in various fields of applied mathematics and physics due to its ability to handle complex nonlinearities. Building upon this foundation, the Modified Adomian Decomposition Method (MADM) enhances ADM by improving convergence and accuracy, making it particularly effective for solving highly nonlinear problems. In this subsection, we focus on the application of MADM to nonlinear Painlevé equations I and II [28–34].

2.2.1. The MADM for solving Painlevé I Equation

Rewrite (1)

$$y''(x) = 6y^2(x) + x, \quad y(0) = 0, \quad y'(0) = 1. \tag{17}$$

Rewrite in the form $Ly = g(x) - F(y)$:

$$Ly = x + 6y^2.$$

The differential operator L is defined by:

$$L = e^{-\int p(x)\,dx} \frac{d}{dx} \left(e^{\int p(x)\,dx} \frac{d}{dx} \right).$$

$$L = \frac{d^2}{dx^2}.$$

Applying the inverse operator L^{-1}, we get:

$$y(x) = \varphi(x) + L^{-1}(x) + L^{-1}(6y^2).$$

Where satisfies $L\varphi(x) = 0$ from initial conditions.

$$\varphi(x) = y(0) + y'(0)x = x.$$

$$y(x) = x + L^{-1}(x) + L^{-1}(6y^2).$$

Recall that the MADM introduces the solution $y(x)$ and the nonlinear function $F(y)$ by infinite series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

$$F(y) = y^2.$$

$$F(y) = \sum_{n=0}^{\infty} A_n.$$

$$\sum_{n=0}^{\infty} y_n(x) = x + L^{-1}(x) + 6L^{-1} \sum_{n=0}^{\infty} A_n.$$

$$y_0(x) = x + L^{-1}(x).$$

$$y_{n+1}(x) = 6L^{-1}[A_n].$$

$$y_0(x) = x + \int_0^x \int_0^x x \, dx \, dx.$$

$$y_0(x) = x + \frac{x^3}{6}.$$

$$y_1(x) = 6L^{-1}[A_0].$$

Where

$$A_0 = y_0^2.$$

$$A_0 = \left(x + \frac{x^3}{6}\right)^2.$$

Thus

$$A_0 = x^2 + \frac{x^4}{3} + \frac{x^6}{36}.$$

Now compute $L^{-1}[A_0]$.

$$y_1(x) = 6L^{-1}[A_0] = 6 \int_0^x \int_0^x \left(x^2 + \frac{x^4}{3} + \frac{x^6}{36}\right) dx \, dx.$$

Thus

$$y_1(x) = \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}.$$

$$A_1 = 2y_0y_1.$$

$$A_1 = 2\left(x + \frac{x^3}{6}\right)\left(\frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}\right).$$

$$A_1 = x^5 + \frac{3x^7}{10} + \frac{71x^9}{2520} + \frac{x^{11}}{1008}.$$

$$y_2(x) = 6L^{-1}[A_1] = 6\int_0^x\int_0^x (A_1)\,dx\,dx.$$

$$y_2(x) = 6L^{-1}[A_1] = 6\int_0^x\int_0^x \left(x^5 + \frac{3x^7}{10} + \frac{71x^9}{2520} + \frac{x^{11}}{1008}\right)dx\,dx.$$

$$y_2(x) = \frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208}.$$

$$\vdots$$

Then

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

$$y(x) = x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336} + \frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208} + \cdots$$

The remainder function of error is evaluated:

$$ER_n = y_n''(x) - 6y_n^2(x) - x. \tag{18}$$

And the MER_n is:

$$MER_n = \max_{0.01\le x\le 0.1} |ER_n(x)|. \tag{19}$$

In Table 3 and Figure 3 show the convergence of the solution using MER_n versus n demonstrates the rapid convergence of the Modified Adomian Decomposition Method (MADM) for solving the first Painlevé equation, the most significant error reduction occurs between $n = 1$ and $n = 3$, after which the error drops to near $n = 5$. The error decreases exponentially with each iteration, indicating that MADM provides highly accurate results with relatively few terms. The consistent decrease in error on the logarithmic scale confirms the efficiency and robustness of MADM in solving nonlinear differential equations.

Table 3: The maximum residual error: MER_n by the MADM where $n = 1, \cdots, 5$.

n	MER_n
1	0.0000601952
2	3.22501×10^{-8}
3	1.29031×10^{-11}
4	4.40620×10^{-15}
5	6.93889×10^{-18}

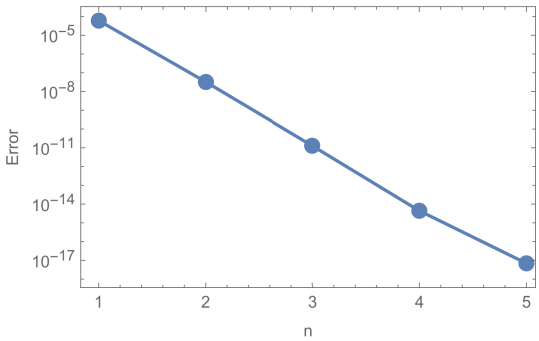

Figure 3: The logarithmic plot of MER_n. (MADM, First Painlevé Equation).

2.2.2. The MADM for solving Painlevé II Equation

Rewrite (2)

$$y''(x) = 2y^3 + xy + \mu. \tag{20}$$

The initial conditions:

$$y(0) = 1, \quad y'(0) = 0.$$

Rewrote (20) in the form: $Ly = g(x) - F(x, y)$.

$$y''(x) - 2y^3 - xy = \mu.$$

$$Ly = \mu - (-2y^3 - xy).$$

$$Ly = \mu + 2y^3 + xy.$$

The differential operator L is defined by

$$L = e^{-\int p(x)\,dx} \frac{d}{dx} \left(e^{\int p(x)\,dx} \frac{d}{dx} \right).$$

$$L = \frac{d^2}{dx^2}.$$

Applying the inverse operator L^{-1}, we get:

$$y(x) = \varphi(x) + L^{-1}(\mu) + L^{-1}(2y^3 + xy).$$

Where satisfies $L\varphi(x) = 0$ from initial conductions:

$$\varphi(x) = y(0) + y'(0)x = 1.$$

$$y(x) = 1 + L^{-1}(\mu) + L^{-1}(2y^3 + xy).$$

Recall that the ADM introduces the solution $y(x)$ and the nonlinear function $F(x, y)$ by infinite series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

$$F(x, y) = 2y^3 + xy.$$

$$F(x, y) = \sum_{n=0}^{\infty} A_n.$$

$$\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-1}(\mu) + L^{-1} \sum_{n=0}^{\infty} A_n.$$

$$y_0(x) = 1 + L^{-1}(\mu).$$

$$y_0(x) = 1 + \int_0^x \int_0^x \mu \, dx \, dx.$$

$$y_0(x) = 1 + \frac{x^2 \mu}{2}.$$

$$y_1(x) = L^{-1}[A_0].$$

$$A_0 = 2y_0^3 + xy_0.$$

$$A_0 = 2 + x + 3x^2\mu + \frac{3x^4\mu^2}{2} + \frac{x^6\mu^3}{4} + \frac{x^3\mu}{2}.$$

Now compute $L^{-1}[A_0]$:

$$y_1(x) = L^{-1}[A_0] = \int_0^x \int_0^x \left(2 + x + 3x^2\mu + \frac{3x^4\mu^2}{2} + \frac{x^6\mu^3}{4} + \frac{x^3\mu}{2} \right) dx \, dx.$$

Performing the integration:

$$y_1(x) = x^2 + \frac{x^3}{6} + \frac{x^4\mu}{4} + \frac{x^5\mu}{40} + \frac{x^6\mu^2}{20} + \frac{x^8\mu^3}{224}.$$

$$A_1 = y_1(6y_0^2 + x).$$

$$\begin{aligned} A_1 \quad = \quad & 2x^3 + \frac{x^4}{2} + \frac{x^5}{30} + \frac{3x^5\mu}{2} + \frac{7x^6\mu}{30} + \frac{x^7\mu}{280} + \frac{33x^7\mu^2}{70} \cdots \\ & + \frac{9x^8\mu^2}{160} + \frac{131x^9\mu^3}{1680} + \frac{47x^{10}\mu^3}{11200} + \frac{57x^{11}\mu^4}{6160} + \frac{3x^{13}\mu^5}{5824}. \end{aligned}$$

$$y_2(x) = L^{-1}[A_1] = \int_0^x \int_0^x (A_1)\, dx\, dx.$$

$$\begin{aligned} y_2(x) &= \frac{x^4}{2} + \frac{x^5}{10} + \frac{x^6}{180} + \frac{x^6\mu}{4} + \frac{x^7\mu}{30} + \frac{x^8\mu}{2240} + \frac{33\, x^8\mu^2}{560} \dots \\ &+ \frac{x^9\mu^2}{160} + \frac{131x^{10}\mu^3}{16800} + \frac{47x^{11}\mu^3}{123200} + \frac{19x^{12}\mu^4}{24640} + \frac{3x^{14}\mu^5}{81536}. \end{aligned}$$

$$\vdots$$

$$y_{n+1}(x) = L^{-1}[A_n].$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots$$

$$\begin{aligned} y(x) &= 1 + \frac{x^2\mu}{2} + x^2 + \frac{x^3}{6} + \frac{x^4\mu}{4} + \frac{x^5\mu}{40} + \frac{x^6\mu^2}{20} + \frac{x^8\mu^3}{224} + \frac{x^4}{2} \dots \\ &+ \frac{x^5}{10} + \frac{x^6}{180} + \frac{x^6\mu}{4} + \frac{x^7\mu}{30} + \frac{x^8\mu}{2240} + \frac{33\, x^8\mu^2}{560} + \frac{x^9\mu^2}{160} \dots \\ &+ \frac{131x^{10}\mu^3}{16800} + \frac{47x^{11}\mu^3}{123200} + \frac{19x^{12}\mu^4}{24640} + \frac{3x^{14}\mu^5}{81536} + \cdots \end{aligned}$$

The remainder function of error is evaluated:

$$ER_n = y_n''(x) - 2y_n^3(x) - xy_n(x) - \mu.$$

And the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 4 and Figure 4 show the convergence of the solution using MER_n, the error decreases with increasing n. The results indicate a rapid reduction in error, with the most significant improvement occurring between $n = 1$ and $n = 3$. By $n = 5$, the error reaches an extremely low magnitude, around 10^{-10}, confirming the high accuracy of the method. This behavior highlights the computational efficiency of MADM, as it provides highly accurate solutions to nonlinear differential equations with minimal computational effort.

Table 4: The maximum residual error: MER_n by the MADM, where $n = 1, \ldots, 5$.

n	MER_n
1	0.06341250000
2	0.00095060500
3	0.00001041000
4	9.77952×10^{-8}
5	8.40300×10^{-10}

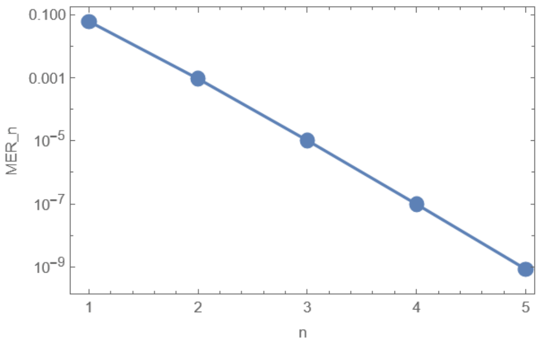

Figure 4: The logarithmic plot of MER_n. (MADM, Second Painlevé Equation).

2.3. Laplace decomposition Method (LDM)

The Laplace Decomposition Method (LDM) is an analytical technique that combines the Laplace transform and the Adomian Decomposition Method (ADM) to solve nonlinear differential equations efficiently. The Laplace transform, introduced by Pierre-Simon Laplace in the 18th century, [9, 10] is widely used for transforming differential equations into algebraic equations, simplifying their solutions. Integrating Adomian decomposition methods and Laplace transform offers a reliable approach to solving nonlinear ordinary differential equations, particularly those encountered in mathematical physics and engineering applications [35–38].

In this subsection, we apply LDM to obtain analytical solutions for Painlevé equations, by leveraging the strengths of both the Laplace transform and the Adomian Decomposition Method. LDM provides a systematic and effective framework for deriving solutions in a series form with rapid convergence. We aim to explore the effectiveness of LDM in solving these equations and contribute to the broader understanding of analytical techniques for nonlinear differential equations.

2.3.1. The LDM for solving Painlevé equation I

Rewrite (1)

$$y''(x) = 6y^2 + x. \tag{21}$$

With the initial conditions:

$$y(0) = 0, \quad y'(0) = 1. \tag{22}$$

Now, apply the Laplace transform on both sides of (21).
With initial conditions:

$$\mathcal{L}[y''] = \mathcal{L}[6y^2] + \mathcal{L}[x].$$

$$s^2 Y(s) + y(0)s - y'(0) = \mathcal{L}[6y^2] + \mathcal{L}[x]. \tag{23}$$

Substituting (22) into (23):

$$Y(s) = \frac{1}{s^2} + \frac{1}{s^4} + \frac{6}{s^2}\mathcal{L}[y^2]. \tag{24}$$

Taking the inverse of the Laplace transform

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^4} + \frac{6}{s^2}\mathcal{L}[y^2]\right]. \tag{25}$$

$$y(x) = x + \frac{x^3}{3!} + \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[y^2]\right]. \tag{26}$$

Equation (26) can be written as

$$y(x) = y_0(x) + \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[A_n]\right], \qquad n \geq 0 \tag{27}$$

$$y_0(x) = x + \frac{x^3}{3!}.$$

From (27), we conclude that

$$y_{n+1}(x) = \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[A_n]\right], \qquad n \geq 0 \tag{28}$$

The terms becomes

$$y_1(x) = \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[A_0]\right].$$

$$y_2(x) = \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[A_1]\right].$$

$$y_3(x) = \mathcal{L}^{-1}\left[\frac{6}{s^2}\mathcal{L}[A_2]\right].$$

$$\vdots$$

Therefore, from (28), the other remaining terms of the function $y(x)$ can be easily calculated as follows:

$$y_0(x) = x + \frac{x^3}{3!}.$$

$$y_1(x) = \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336}.$$

$$y_2(x) = \frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208}.$$

$$y_3(x) = \frac{x^{10}}{28} + \frac{23x^{12}}{3080} + \frac{5219x^{14}}{8408400} + \frac{3551x^{16}}{1441440000} + \frac{95x^{18}}{224550144}.$$

$$y_4(x) = \frac{3x^{13}}{364} + \frac{131x^{15}}{64680} + \frac{19867x^{17}}{95295200} + \frac{163469x^{19}}{14378364000} + \frac{163451x^{21}}{491203440000} + \frac{31x^{23}}{7101398304}.$$

$$\vdots$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{29}$$

$$\begin{aligned} y(x) \quad = \quad & x + \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^6}{15} + \frac{x^8}{336} + \frac{x^7}{7} + \frac{x^9}{40} + \frac{71x^{11}}{46200} + \frac{x^{13}}{26208} + \frac{x^{10}}{28} \cdots \\ & + \frac{23x^{12}}{3080} + \frac{5219x^{14}}{8408400} + \frac{3551x^{16}}{1441440000} + \frac{95x^{18}}{224550144} + \frac{3x^{13}}{364} + \frac{131x^{15}}{64680} \cdots \\ & + \frac{19867x^{17}}{95295200} + \frac{163469x^{19}}{14378364000} + \frac{163451x^{21}}{491203440000} + \frac{31x^{23}}{7101398304} + \cdots \end{aligned}$$

The remainder function of error is evaluated:

$$ER_n = y_n''(x) - 6y_n^2(x) - x.$$

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 5 and Figure 5 show the convergence of the solution using MER_n against n for the first Painlevé equation by LDM, the plot demonstrates the rapid decrease in error with increasing n, confirming the convergence of the Laplace Decomposition Method (LDM). The nearly linear trend in the logarithmic scale indicates an exponential decay of the maximum residual error, showcasing the method's high accuracy and efficiency.

Table 5: The maximum residual error: MER_n by the LDM, where $n = 1, \ldots, 5$.

n	MER_n
1	0.0000601952
2	3.22501×10^{-8}
3	1.29031×10^{-11}
4	4.4062×10^{-15}
5	6.93889×10^{-18}

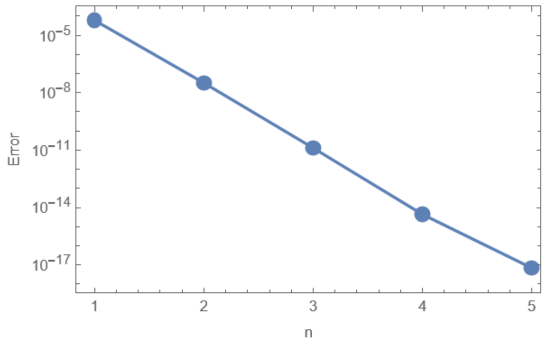

Figure 5: logarithmic plot of MER_n (LDM, First Painlevé Equation).

2.3.2. The LDM for solving Painlevé equation II

Rewrite (2)

$$y''(x) = 2y^3 + xy + \mu. \tag{30}$$

The initial conditions:

$$y(0) = 1, \quad y'(0) = 0. \tag{31}$$

Now, apply Laplace transform on both sides of (30).
With initial conditions:

$$\mathcal{L}[y''] = \mathcal{L}[2y^3 + xy + \mu].$$

$$y'' = s^2 Y(s) - sy(0) - y'(0). \tag{32}$$

Substituting (31) into (32), we have:

$$s^2 Y(s) - s = \mathcal{L}[2y^3] + \mathcal{L}[xy] + \mathcal{L}[\mu]. \tag{33}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s^2}\mathcal{L}[2y^3] + \frac{1}{s^2}\mathcal{L}[xy] + \frac{\mu}{s^3}. \tag{34}$$

Equation (34) becomes:

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s^2}\mathcal{L}[2y^3] + \frac{1}{s^2}\mathcal{L}[xy] + \frac{\mu}{s^3}\right].$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{\mu}{s^3} + \frac{1}{s^2}\mathcal{L}[2y^3 + xy]\right]. \tag{35}$$

$$y(x) = 1 + \frac{x^2\mu}{2} + \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_n]\right]. \tag{36}$$

$$y_0(x) = 1 + \frac{x^2\mu}{2}.$$

From (36) we conclude that:

$$y_{n+1}(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_n]\right], \quad n \geq 0.$$

Then

$$y_1(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_0]\right].$$

$$A_0 = F(y_0).$$

$$F(y) = 2y^3 + xy.$$

$$y_1(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[2y_0^3 + xy_0]\right].$$

$$y_1(x) = \mathcal{L}^{-1}\left[\frac{2}{s^3} + \frac{1}{s^4} + \frac{3(2!)\,\mu}{s^5} + \frac{3!\,\mu}{2\,s^6} + \frac{3(4!)\,\mu^2}{2\,s^7} + \frac{6!\,\mu^3}{4\,s^9}\right].$$

$$y_1(x) = x^2 + \frac{x^3}{6} + \frac{x^4\mu}{4} + \frac{x^5\mu}{40} + \frac{x^6\mu^2}{20} + \frac{x^8\mu^3}{224}.$$

$$\begin{aligned} A_1 \;&=\; 2x^3 + \frac{x^4}{2} + \frac{x^5}{30} + \frac{3x^5\mu}{4} + \frac{7x^6\mu}{30} + \frac{x^7\mu}{280} + \frac{33x^7\mu^2}{70}\ldots \\ &+\frac{9x^8\mu^2}{160} + \frac{131x^9\mu^3}{1680} + \frac{47x^{10}\mu^3}{11200} + \frac{57x^{11}\mu^4}{6160} + \frac{3x^{13}\mu^5}{5824}. \end{aligned}$$

$$y_2(x) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_1]\right].$$

$$\begin{aligned} y_2(x) \;&=\; \frac{x^4}{2} + \frac{x^5}{10} + \frac{x^6}{180} + \frac{x^6\mu}{4} + \frac{x^7\mu}{30} + \frac{x^8\mu}{2240} + \frac{33\,x^8\mu^2}{560}\ldots \\ &+\frac{x^9\mu^2}{160} + \frac{131x^{10}\mu^3}{16800} + \frac{47x^{11}\mu^3}{123200} + \frac{19x^{12}\mu^4}{24640} + \frac{3x^{14}\mu^5}{81536}. \end{aligned}$$

$$\vdots$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

$$\begin{aligned} y(x) \;&=\; 1 + \frac{x^2\mu}{2} + x^2 + \frac{x^3}{6} + \frac{x^4\mu}{4} + \frac{x^5\mu}{40} + \frac{x^6\mu^2}{20} + \frac{x^8\mu^3}{224} + \frac{x^4}{2}\ldots \\ &+\frac{x^5}{10} + \frac{x^6}{180} + \frac{x^6\mu}{4} + \frac{x^7\mu}{30} + \frac{x^8\mu}{2240} + \frac{33x^8\mu^2}{560} + \frac{x^9\mu^2}{160}\ldots \\ &+\frac{131x^{10}\mu^3}{16800} + \frac{47x^{11}\mu^3}{123200} + \frac{19x^{12}\mu^4}{24640} + \frac{3x^{14}\mu^5}{81536} + \cdots \end{aligned}$$

The remainder function of error is evaluated:

$$ER_n = y_n''(x) - 2y_n^3(x) - xy_n(x) - \mu.$$

And the MER_n is:

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 6 and Figure 6 show the convergence of the solution using MER_n against n for the second Painlevé equation by LDM. The data shows a clear trend of rapid error reduction as n increases. Initially, at $n = 1$, the error is relatively large at 0.0634125. However, as n increases to 2 and 3, the error drops significantly, and at $n = 5$, it reaches an extremely small value of 8.40300×10^{-10}. This exponential decrease in error suggests that the LDM exhibits fast convergence, making it a highly effective method for solving the second Painlevé equation with high accuracy.

Table 6: The maximum residual error: MER_n by the LDM, where n = 1, . . . , 5.

n	MER_n
1	0.0634125
2	0.000950605
3	0.00001041
4	9.77952×10^{-8}
5	8.40300×10^{-10}

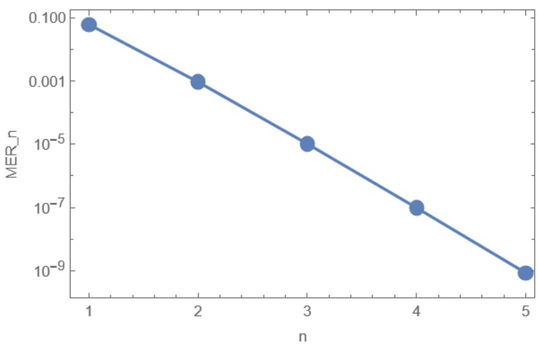

Figure 6: logarithmic plot of MER_n (LDM, Second Painlevé Equation).

2.4. Natural Decomposition Method (NDM)

The Natural Decomposition Method (NDM) was developed in the early 21st century as an advancement in analytical techniques for solving nonlinear differential equations. Research on NDM began appearing in mathematical literature in the 2000s and 2010s as an improvement over existing decomposition methods [11, 12]. It was introduced to enhance the accuracy and convergence of solutions for highly nonlinear problems without requiring linearization or perturbation. Since its introduction, NDM has been applied to various fields, including fluid dynamics, mathematical physics, and engineering. In this subsection we use NDM to solving Painlevé equations I and II. [39–41].

2.4.1. The NDM for solving the Painlevé I differential equation

Rewrite [1]

$$y''(x) = 6y^2 + x. \tag{37}$$

$$y(0) = 0, \quad y'(0) = 1. \tag{38}$$

By applying the N-transform on both sides of (37):

$$N^+\left[y''(x)\right] = N^+\left[6y^2 + x\right]. \tag{39}$$

Using the properties of N-transform we obtain that

$$\frac{s^2}{t^2}y(s,t) - \frac{s}{t^2}y(0) - \frac{1}{t}y'(0) = N^+[6y^2] + \frac{t}{s^2}. \tag{40}$$

Substituting (38) into (40), we have

$$y(s,t) = \frac{t}{s^2} + \frac{t^3}{s^4} + \frac{t^2}{s^2}N^+[6y^2]. \tag{41}$$

Now, applying the inverse of N-transform, we have

$$N^{-1}[y(s,t)] = N^{-1}\left[\frac{t}{s^2} + \frac{t^3}{s^4} + \frac{6t^2}{s^2}N^+[y^2]\right]. \tag{42}$$

Equation (42) becomes

$$y(x) = x + \frac{x^3}{3!} + N^{-1}\left[\frac{6t^2}{s^2}N^+[y^2]\right]. \tag{43}$$

Equation (43) can be written as

$$y(x) = x + \frac{x^3}{3!} + N^{-1}\left[\frac{6t^2}{s^2}N^+[A_n]\right]. \tag{44}$$

$$y_0(x) = x + \frac{x^3}{3!}.$$

From (44), we can conclude that

$$y_{n+1}(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_n]\right], \quad n \geq 0. \tag{45}$$

The terms become:

$$y_1(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_0]\right].$$

$$y_2(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_1]\right].$$

$$\vdots$$

Therefore, from (45), the other remaining terms of the function $y(x)$ can be easily calculated as follows:

$$\begin{aligned}
y_1(x) &= \frac{1}{2}x^4 + \frac{1}{15}x^6 + \frac{1}{336}x^8.\\
y_2(x) &= \frac{1}{7}x^7 + \frac{1}{40}x^9 + \frac{71}{46200}x^{11} + \frac{1}{26200}x^{13}.\\
y_3(x) &= \frac{1}{28}x^{10} + \frac{23}{3080}x^{12} + \frac{5219}{8408400}x^{14} + \frac{3551}{144144000}x^{16} + \frac{95}{224550144}x^{18}.\\
y_4(x) &= \frac{3}{364}x^{13} + \frac{131}{64680}x^{15} + \frac{19867}{95295200}x^{17} + \frac{163469}{14378364000}x^{19}\dots\\
&\quad + \frac{163451}{491203440000}x^{21} + \frac{131}{7101398304}x^{23}.
\end{aligned}$$

In this way, we obtained the approximate result of the function $y(x)$, which is given as

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{46}$$

$$\begin{aligned}
y(x) &= x + \frac{x^3}{6} + \frac{1}{2}x^4 + \frac{1}{15}x^6 + \frac{1}{336}x^8 + \frac{1}{7}x^7 + \frac{1}{40}x^9 + \frac{71}{64200}x^{11} + \frac{1}{26208}x^{13}\dots\\
&\quad + \frac{1}{28}x^{10} + \frac{23}{3080}x^{12} + \frac{5219}{8408400}x^{14} + \frac{3551}{144144000}x^{16} + \frac{95}{224550144}x^{18}\dots\\
&\quad + \frac{3}{364}x^{13} + \frac{131}{64680}x^{15} + \frac{19867}{95295200}x^{17} + \frac{163469}{14378364000}x^{19}\dots\\
&\quad + \frac{163451}{491203440000}x^{21} + \frac{131}{7101398304}x^{23} + \cdots
\end{aligned}$$

The remainder function of error is evaluated:

$$ER_n = y_n''(x) - 6y_n^2(x) - x.$$

$$MER_n = \max_{0.01 \le x \le 0.1} |ER_n(x)|.$$

In Table 7 and Figure 7 show the convergence of the solution using MER_n from $n = 1$ to $n = 5$ using the NDM. A significant reduction in error as n increases. Initially, for $n = 1$, the error is relatively large, but it decreases drastically for $n = 2$ and continues to drop to 6.93889×10^{-18} for $n = 5$. This exponential decay in error indicates that each additional term in the LDM approximation significantly improves the accuracy of the solution. The results confirm that LDM provides a highly effective approach for solving the second Painlevé equation, with higher-order approximations yielding more precise solutions.

Table 7: The maximum residual error: MER_n by the LDM, where n = 1, . . . , 5.

n	MER_n
1	0.0000601952
2	3.22501×10^{-8}
3	1.29031×10^{-11}
4	4.4062×10^{-15}
5	6.93889×10^{-18}

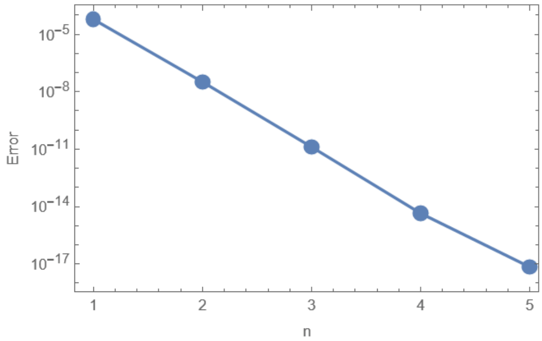

Figure 7: The logarithmic plot of MER_n (NDM. First Painlevé Equation).

2.4.2. The NDM for solving the Painlevé II differential equation

Rewrite (2)

$$y''(x) = 2y^3 + xy + \mu. \tag{47}$$

$$y(0) = 1, \quad y'(0) = 0. \tag{48}$$

By apply N-transform on both sides of (47):

$$N^+[y''(x)] = N^+[2y^3 + xy + \mu]. \tag{49}$$

By using the properties of N-transform we obtain:

$$\frac{s^2}{t^2}y(s,t) - \frac{s}{t^2}y(0) - \frac{1}{t}y'(0) = N^+[2y^3 + xy] + \frac{\mu}{s^2}. \tag{50}$$

Substituting (48) into (50), we have

$$y(s,t) = \frac{1}{s} + \frac{\mu t^2}{s^4} + \frac{t^2}{s^2}N^+[2y^3 + xy]. \tag{51}$$

Now, apply the inverse of N-transform we have

$$N^{-1}[y(s,t)] = N^{-1}\left[\frac{1}{s} + \frac{\mu t^2}{s^4} + \frac{t^2}{s^2}N^+[2y^3 + xy]\right]. \tag{52}$$

$$y(x) = 1 + \frac{\mu x^2}{2} + N^{-1}\left[\frac{t^2}{s^2}N^+[2y^3 + xy]\right]. \tag{53}$$

Equation (53) can be written as

$$y(x) = 1 + \frac{\mu x^2}{2} + N^{-1}\left[\frac{t^2}{s^2}N^+[A_n]\right], \quad n \geq 0 \tag{54}$$

$$y_0(x) = 1 + \frac{\mu x^2}{2}.$$

From (54) we can conclude that

$$y_{n+1}(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_n]\right], \quad n \geq 0. \tag{55}$$

The terms becomes

$$y_1(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_0]\right].$$

$$y_2(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_1]\right].$$

Therefore, from (55) the other remaining terms of the function $y(x)$ can be easily calculated as follows:

$$A_0 = 2 + 3x^2\mu + \frac{3}{2}x^4\mu^2 + \frac{x^6\mu^3}{4} + x + \frac{x^3\mu}{2}.$$

$$y_1(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_0]\right].$$

$$y_1(x) = x^2 + \frac{x^4\mu}{4} + \frac{x^6\mu^2}{20} + \frac{x^8\mu^3}{224} + \frac{x^3}{6} + \frac{x^5\mu}{40}.$$

$$\begin{aligned} A_1 \quad = \quad & 2x^3 + \frac{x^4}{2} + \frac{x^5}{30} + \frac{3x^5\mu}{2} + \frac{7x^6\mu}{30} + \frac{x^7\mu}{280} + \frac{33x^7\mu^2}{70}\dots \\ & + \frac{9x^8\mu^2}{160} + \frac{131x^9\mu^3}{1680} + \frac{47x^{10}\mu^3}{11200} + \frac{57x^{11}\mu^4}{6160} + \frac{3x^{13}\mu^5}{5824}. \end{aligned}$$

$$y_2(x) = N^{-1}\left[\frac{t^2}{s^2}N^+[A_1]\right].$$

$$\begin{aligned} y_2(x) \quad & = N^{-1}\left\{\tfrac{t^2}{s^2}N^+\left[2x^3 + \tfrac{x^4}{2} + \tfrac{x^5}{30} + \tfrac{3x^5\mu}{2} + \tfrac{7x^6\mu}{30} + \tfrac{x^7\mu}{280}\dots\right.\right. \\ & \left.\left. + \tfrac{33x^7\mu^2}{70} + \tfrac{9x^8\mu^2}{160} + \tfrac{131x^9\mu^3}{1680} + \tfrac{47x^{10}\mu^3}{11200} + \tfrac{57x^{11}\mu^4}{6160} + \tfrac{3x^{13}\mu^5}{5824}\right]\right\}. \end{aligned}$$

$$y_2(x) = \frac{x^4}{2} + \frac{x^5}{10} + \frac{x^6}{180} + \frac{x^6\mu}{4} + \frac{x^7\mu}{30} + \frac{x^8\mu}{2240} + \frac{33x^8\mu^2}{560}\dots$$

$$+\frac{x^9\mu^2}{160}+\frac{131x^{10}\mu^3}{16800}+\frac{47x^{11}\mu^3}{123200}+\frac{19x^{12}\mu^4}{24640}+\frac{3x^{14}\mu^5}{81536}.$$

$$\vdots$$

$$y(x)=\sum_{n=0}^{\infty}y_n(x).$$

$$\begin{aligned} y(x) \quad = \quad & 1+\frac{x^2\mu}{2}+x^2+\frac{x^3}{6}+\frac{x^4\mu}{4}+\frac{x^5\mu}{40}+\frac{x^6\mu^2}{20}+\frac{x^8\mu^3}{224}\cdots \\ & +\frac{x^4}{2}+\frac{x^5}{10}+\frac{x^6}{180}+\frac{x^6\mu}{4}+\frac{x^7\mu}{30}+\frac{x^8\mu}{2240}+\frac{33x^8\mu^2}{560}\cdots \\ & +\frac{x^9\mu^2}{160}+\frac{131x^{10}\mu^3}{16800}+\frac{47x^{11}\mu^3}{123200}+\frac{19x^{12}\mu^4}{24640}+\frac{3x^{14}\mu^5}{81536}+\cdots \end{aligned}$$

The remainder function of error is evaluated:

$$ER_n=y_n''(x)-2y_n^3(x)-xy_n(x)-\mu.$$

And the MER_n is:

$$\mathrm{MER}_n=\max_{0.01\le x\le 0.1}|ER_n(x)|.$$

In Table 8 and Figure 8 show the convergence of the solution using MER_n for the second Painlevé equation by NDM. Shows the exponential decay of error and nearly linear downward trend on the logarithmic scale. The values indicate a rapid reduction in error as n increases, demonstrating improved accuracy with higher-order approximations. Initially, for $n=1$, the error is relatively large. However, as n increases, the error decreases significantly, reaching a much smaller value at $n=5$. This trend suggests that the NDM effectively enhances the accuracy of the solution.

Table 8: The maximum residual error: MER_n by the LDM, where n = 1, . . . , 5.

n	MER_n
1	0.0634125
2	0.000950605
3	0.00001041
4	9.77952×10^{-8}
5	8.40300×10^{-10}

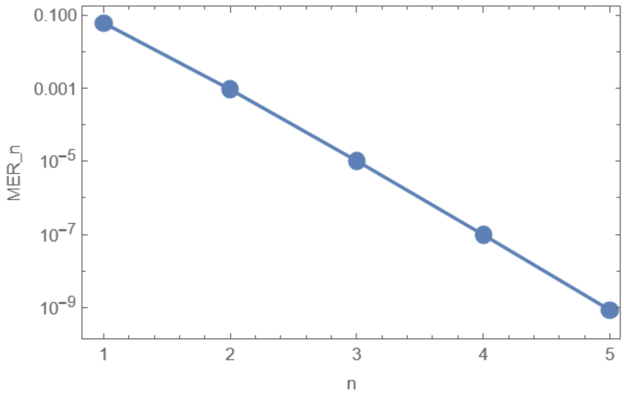

Figure 8: The logarithmic plot of MER_n (NDM. Second Painlevé Equation).

3. Numerical methods for solving Painlevé Equations I, II

The numerical methods provide practical means for approximating solutions where analytical methods fall short. The Finite Difference Method discretizes the continuous domain and approximates derivatives using difference quotients, which is particularly useful for handling boundary value problems. [13, 14] The RK4 method, on the other hand, is well-known for its robustness and high accuracy in solving initial value problems, making it an excellent choice for exploring the dynamical behavior of these nonlinear equations [42].

In this section, we focus exclusively on numerical approaches, specifically detailing the implementation of the Finite Difference Method (FDM) and the fourth-order Runge-Kutta Method (RK4) for solving the first and second Painlevé equations. We aim to provide clear insights into their strengths and limitations when applied to the Painlevé equations I and II.

3.1. Finite difference method (FDM)

The finite difference method (FDM) is a numerical approach for solving ordinary differential equations (ODEs), originally studied by Richardson in 1910. FDM became widely used in the mid-20th century for solving engineering and physics problems. With advancements in computing, it remains a popular method due to its simplicity, efficiency, and accuracy. [15, 16] The finite difference method works by discretizing the domain and approximating derivatives using finite differences. This transforms the ODE into a system of algebraic equations, making it easier to solve numerically. In this subsection, we use FDM to solve Painlevé equations [43, 44].

3.1.1. The FDM method for solving the Painlevé I differential equation

Rewrite (1)

$$y''(x) = 6y^2 + x. \tag{56}$$

$$y(0) = 0, \quad y'(0) = 1.$$

Divide the interval $(0, 1)$ into four sub-intervals such that $h = \frac{1}{4}$ and the pivot points are at

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad \text{and} \quad x_4 = 1.$$

Then the second differential equation is approximated as

$$y''(x) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

Substitute into (56)

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = 6y_i^2 + x_i.$$

$$y_{i+1} - 2y_i + y_{i-1} = h^2(6y_i^2 + x_i). \tag{57}$$

By using a forward difference approximation for the derivative to compute the initial value for y_1.

$$y'(x) = \frac{1}{h}\left[y(x + h) - y(x)\right].$$

Since

$$y_0 = y(x_0) = y(0).$$

$$y_1 = y(x_1) = y\left(\frac{1}{4}\right).$$

$$y'(0) = \frac{y(h) - y_0}{h}.$$

$$1 = \frac{y_1 - 0}{h}.$$

$$y_1 = h.$$

Thus we have

$$y_0 = 0, \quad y_1 = \frac{1}{4}.$$

$$y_{i+1} - 2y_i + y_{i-1} = h^2(6y_i^2 + x_i).$$

For $i = 1$,

$$y_2 - 2y_1 + y_0 = \left(\frac{1}{4}\right)^2 (6y_1^2 + x_1).$$

$$y_2 = \frac{69}{128}.$$

For $i = 2$,

$$y_3 - 2y_2 + y_1 = \frac{3y_2^2}{8} + \frac{x_2}{16}.$$

$$y_3 = \frac{126923}{131072}.$$

Their solution gives

$$y_1 = \frac{1}{4}, \quad y_2 = \frac{69}{128}, \quad y_3 = \frac{126923}{131072}.$$

$$\vdots$$

3.1.2. The FDM for solving the Painlevé II differential equation

Rewrite (2)

$$y''(t) = 2y^3 + xy + \mu. \tag{58}$$

$$y(0) = 1, \quad y'(0) = 0.$$

Divide the interval $(0, 1)$ into four sub-intervals such that $h = \frac{1}{4}$ and the pivot points are at

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad \text{and} \quad x_4 = 1.$$

Then the second differential equation is approximated as

$$y''(x) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

Substitute into (58):

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = 2y_i^3 + x_i y_i + \mu,$$

$$y_{i+1} - 2y_i + y_{i-1} = h^2 \left(2y_i^3 + x_i y_i + \mu\right).$$

By using a forward difference approximation for the derivative to compute the initial value for y_1,

$$y'(x) = \frac{1}{h}\left[y(x+h) - y(x)\right].$$

$$y'(0) = \frac{y(h) - y_0}{h}.$$

$$0 = \frac{y_1 - 1}{h}.$$

$$y_1 = 1.$$

Thus we have

$$y_{i+1} - 2y_i + y_{i-1} = h^2(2y_i^3 + x_i y_i + \mu).$$

For $i = 1$

$$y_2 - 2y_1 + y_0 = h^2(2y_1^3 + x_1 y_1 + \mu).$$

$$y_2 = \frac{73}{64} + \frac{\mu}{16}.$$

For $i = 2$

$$y_3 - 2y_2 + y_1 = h^2(2y_2^3 + x_2y_2 + \mu).$$

$$y_3 - 2\left(\frac{73}{64} + \frac{\mu}{16}\right) + 1 = \left(\frac{1}{4}\right)\left(2\left(\frac{73}{64} + \frac{\mu}{16}\right)^3 + \frac{1}{2}\left(\frac{73}{64} + \frac{\mu}{16}\right) + \mu\right).$$

$$y_3 = \frac{1135513}{524288} + \frac{66163\mu}{131072} + \frac{219\mu^2}{32768} + \frac{\mu^3}{8192}.$$

Their solution gives:

$$y_1 = 1, \quad y_2 = \frac{73}{64} + \frac{\mu}{16}, \quad y_3 = \frac{1135513}{524288} + \frac{66163\mu}{131072} + \frac{219\mu^2}{32768} + \frac{\mu^3}{8192}.$$

3.2. Runge-Kutta Method (RK4)

The Runge-Kutta method is a powerful numerical technique for solving ordinary differential equations (ODEs), including nonlinear equations like the Painlevé equations. Developed by Carl Runge and Wilhelm Kutta in 1990, [45, 46], it provides higher accuracy than the Euler method without requiring higher-order derivatives. The fourth-order Runge-Kutta method (RK4) is particularly popular due to its balance between computational efficiency and precision. By computing intermediate slopes at each step, RK4 improves stability and accuracy; in this subsection, we use it to solve the Painlevé equations [47–50].

3.2.1. The RK4 for solving the Painlevé I differential equation

Rewrite (1)

$$y''(x) = 6y^2 + x. \tag{59}$$

with the initial conditions:

$$y(0) = 0, \quad y'(0) = 1. \tag{60}$$

Rewrite the second-order equation as a system of first-order equations:

$$\frac{dy}{dx} = z = f(x, y, z).$$

$$y' = z,$$
$$z' = y'' = 6y^2 + x = \phi(x, y, z).$$

Here, $f(x, y, z)$ represents the first derivative of y, while $\phi(x, y, z)$ defines the right-hand side of the second equation. With initial conditions at $x_0 = 0, \quad y_0 = 0, \quad z_0 = 1$, we proceed. Use step size $h = \frac{1}{4}$ and calculate k_1, k_2, k_3, k_4 and l_1, l_2, l_3, l_4. Runge-Kutta formulas for y, we use:

$$k_1 = hf(x_0, y_0, z_0) = \frac{1}{4}.$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = \frac{1}{4},$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = \frac{263}{1024},$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = \frac{135}{512}.$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{391}{1536}.$$

$$l_1 = h\phi(x_0, y_0, z_0) = 0.$$

$$l_2 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = \frac{7}{128}.$$

$$l_3 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = \frac{7}{128}.$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = \frac{338579}{2097152}.$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = \frac{797331}{8388608}.$$

Hence,

$$y = y_0 + k = \frac{391}{1536} \approx 0.25456.$$

$$y' = z = z_0 + l = 1 + \frac{797331}{8388608} \approx 1.06337.$$

3.2.2. The RK4 for solving the Painlevé II differential equation

Rewrite (2)

$$y''(t) = 2y^3 + xy + \mu. \tag{61}$$

With the initial conditions:

$$y(0) = 1, \quad y'(0) = 0. \tag{62}$$

Rewrite the second-order equation as a system of first-order equations:

$$\frac{dy}{dx} = z = f(x, y, z), \qquad y' = z.$$

$$z' = y'' = 2y^3 + xy + \mu = \varphi(x, y, z).$$

Here, $f(x, y, z)$ represents the first derivative of y, while $\varphi(x, y, z)$ defines the right-hand side of the second equation. With initial conditions at $x_0 = 0$, $y_0 = 1$, and $z_0 = 0$.

Use step size $h = \frac{1}{4}$ and calculate k_1, k_2, k_3, k_4 and l_1, l_2, l_3, l_4. Range Kutta formulas for y, we use:

$$k_1 = hf(x_0, y_0, z_0) = 0.$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = \frac{2+\mu}{32}.$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right) = \frac{1+256\mu}{1024}.$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3, z_0 + l_3\right) = \frac{966280 + 160140\mu + 294\mu^2 + \mu^3}{524288}.$$

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = \frac{1032840 + 455052\mu + 294\mu^2 + \mu^3}{3145728}.$$

$$l_1 = h\varphi(x_0, y_0, z_0) = \frac{2+\mu}{4}.$$

$$l_2 = h\varphi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = \frac{1+256\mu}{1024}.$$

$$l_3 = h\varphi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = \frac{966280 + 160140\mu + 294\mu^2 + \mu^3}{524288}.$$

$$l_4 = h\varphi\left(x_0 + h, y_0 + k_3, z_0 + l_3\right) = \frac{1025}{16384} + \frac{1}{2}\left(\frac{1025}{1024} + \frac{\mu}{4}\right)^3 + \frac{17\mu}{64}.$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = \frac{10204941 + 4299784960\,\mu + 203931648\,\mu^2 + 16785408\mu^3}{12884901888}.$$

Hence,

$$y = y_0 + k = \frac{4178568 + 455052\mu + 294\mu^2 + \mu^3}{3145728},$$

$$y' = z = z_0 + l = \frac{10204941 + 4299784960\,\mu + 203931648\,\mu^2 + 16785408\mu^3}{12884901888}.$$

4. Comparative Analysis of Analytical and Numerical Methods

The study of nonlinear differential equations, particularly Painlevé equations, requires robust analytical and numerical approaches to obtain accurate solutions. Analytical methods such as the Differential Transform Method (DTM), Modified Adomian Decomposition Method (MADM), Laplace Decomposition Method (LDM), and Natural Decomposition Method (NDM) provide series solutions, while numerical techniques such as the Finite Difference Method (FDM) and Runge-Kutta Method (RK4) offer direct computational approximations. This section presents a rigorous comparison of these methods [1, 2].

4.1. Comparison between analytical methods

To assess the accuracy and convergence of DTM, MADM, LDM, and NDM, we compute approximate solutions for the Painlevé I and II equations at selected points $x = 0, 0.25, 0.5, 0.75, 1$. This is presented for Painlevé equation I in Table 9 and for Painlevé equation II in Table 10. A graphical comparison of the approximate solutions obtained from DTM, MADM, LDM, and NDM for Painlevé equation I is presented in Figure 9, and for Painlevé equation II in Figure 10.

Table 9: Provides the computed values of $y(x)$ at selected points for Painlevé equation I

x	DTM	MADM	LDM	NDM
0.00	0.000000	0.000000	0.000000	0.000000
0.25	0.254582309975	0.2545823823838	0.2545823823838	0.2545823823838
0.50	0.5542582826348	0.5542898995536	0.5542898995536	0.5542898995536
0.75	1.009901884624	1.0113271032061	1.0113271032061	1.0113271032061
1.00	1.882208994709	1.9011904761905	1.9011904761905	1.9011904761905

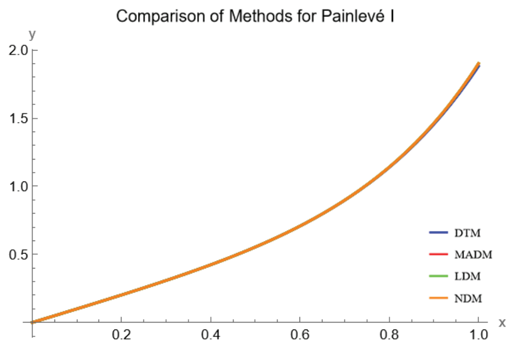

Figure 9: Convergence methods for Painlevé Equation I.

Table 10: Provides the computed values of $y(x)$ at selected points for Painlevé equation II.

x	DTM	MADM	LDM	NDM
0.00	1.0000	1.0000	1.0000	1.0000
0.25	1.099332682292	1.097355143229	1.097355143229	1.097355143229
0.50	1.444270833333	1.412239583333	1.412239583333	1.412239583333
0.75	2.163232421875	1.999096679688	1.999096679688	1.999096679688
1.00	3.466666666667	2.941666666667	2.941666666667	2.941666666667

4.2. Comparison between numerical methods

In this subsection the numerical methods considered include the Finite Difference Method (FDM) and the Runge-Kutta Method (RK4) [47].

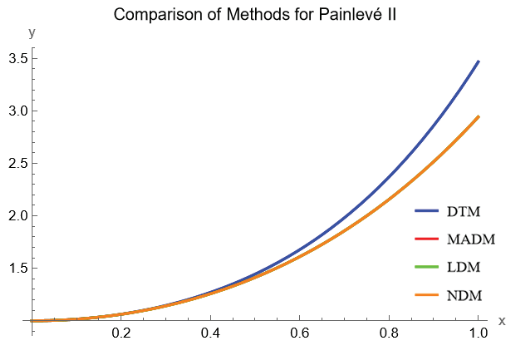

Figure 10: Convergence behavior of these methods for Painlevé equation II.

The computed values for Painlevé I at selected points are presented in Table 11, which compares the approximate solutions obtained using FDM and RK4. The graphical comparison of these numerical methods is further illustrated in Figure 11. For Painlevé II, the computed values using FDM and RK4 are presented in Table 12, and the graphical comparison is shown in Figure 12.

Table 11: Comparison of the approximate solution by FDM and RK4, for Painlevé equation I at selected points.

x	FDM	RK4
0.00	0.00000	0.00000
0.25	0.250000000000	0.254455682891
0.50	0.539062500000	0.554123359580
0.75	0.968345642090	1.013010257333
1.00	1.955078125000	1.955172884313

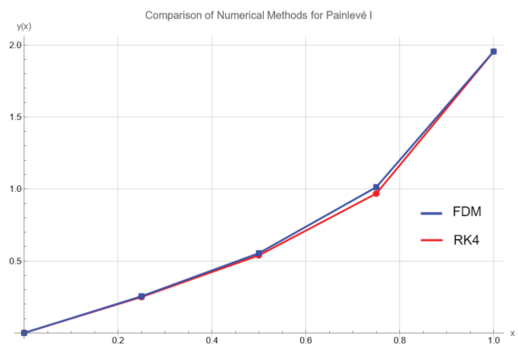

Figure 11: The accuracy of numerical methods for Painlevé equation I.

Table 12: Comparison of the approximate solution by FDM and RK4, for Painlevé equation II at selected points.

x	FDM	RK4
0.00	1.00000	1.00000
0.25	1.00000	1.099484364191691
0.50	1.203125000000000	1.458011474879827
0.75	2.677408218383789	2.377499005219746
1.00	6.738824991141589	5.909391830792607

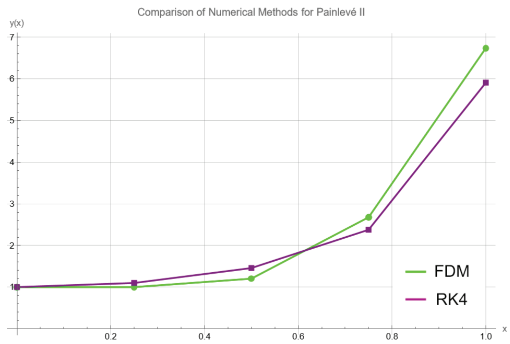

Figure 12: The accuracy of numerical methods for Painlevé equation II.

5. Conclusion

This paper explored various analytical and numerical methods for solving the Painlevé equations I and II, which are significant in the field of nonlinear differential equations due to their integrability and applications in mathematical physics. The analytical methods examined, including the Differential Transform Method, the Modified Adomian Decomposition Method, the Laplace Decomposition Method, and the Natural Decomposition Method, showcased their potential for handling the inherent complexities of nonlinear equations with high precision.

On the other hand, numerical methods, specifically the Finite Difference Method and the fourth-order Runge-Kutta Method (RK4), demonstrated their effectiveness in providing approximate solutions when exact solutions were challenging to derive.

Comparing the two approaches, analytical methods are advantageous for gaining insights into the structural properties of solutions, while numerical methods are more versatile for practical computations. This combination of techniques forms a comprehensive toolbox for tackling nonlinear ordinary differential equations like the Painlevé equations. Future work will focus on analytic and numerical methods' approaches for solving systems of differential equations.

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Conflict of Interest

The authors declare that there is no conflict of interest to disclose.

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