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Domination in Bipolar Fuzzy Rough Digraphs with Applications to Decision-Making

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Abstract. Fuzzy Rough Digraphs are insufficient for scenarios involving both positive and negative influences. To address the limitations of Fuzzy Rough Digraphs in modeling conflicting information, this paper introduces the Bipolar Fuzzy Rough Digraph (BFRD) as a new framework for decision-making under uncertainty. We define its fundamental properties, including the strength of paths, connectedness, vertex degree, the Regular BFRD. From these, we establish the concepts of the minimum dominating set and domination number. An algorithm is then developed to apply this framework to practical problems. The model's efficacy is demonstrated through a real-world application: identifying an optimal set of rural areas for establishing medicine supply markets by finding the minimum dominating set. This work provides a robust mathematical tool for solving complex problems involving bipolarity and as a process innovation.

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1. Introduction

The introduction of Fuzzy Set theory by Zadeh [1, 2] provided a revolutionary mathematical framework to handle the inherent vagueness of real-world data, moving beyond the binary limitations of classical set theory. In classical set theory, an element either belongs to a set or does not, represented by binary membership (0 or 1). However, in the real world, many concepts and phenomena are not strictly black or white; they often exhibit shades of gray or degrees of membership. The fuzzy set theory addresses this limitation by allowing elements to belong to a set with a degree of membership that ranges between 0 and 1. The objective of fuzzy set theory is to provide a formal framework that accommodates the inherent fuzziness of real-world data and phenomena. In 1983, Chakraborty and Das [3–5] extended studies of fuzzy relations and fuzzy equivalence relations and other [83–86].

This framework found a natural and powerful application in graph theory, leading to the development of Fuzzy Graphs, a concept pioneered by Kauffmann [6], Yeh and Bang [7], and Rosenfeld [8]. Kauffmann studied the extension of classical graph theory to fuzzy graph theory in 1973. In classical graph theory, edges and vertices are either present or absent, leading to a binary representation. In contrast, fuzzy graph theory allows for edges and vertices to have degrees of membership, indicating the strength or degree of connection between nodes. This innovation spurred extensive research into the structural properties of these graphs, with key contributions on fuzzy groups by Bhattacharya [9], connectedness by Banerjee [10], fuzzy vertex graphs by Koczy [11]. Udupa and Samarasekera [12] used the concept of fuzzy connectedness in image segmentation. Bhutani [13, 14] presented the ideas of the cut node and fuzzy edge node, and node connectivity is presented by Mathew and Sunitha [15, 16]. Binu et al. [17] studied the applications of the connectivity status of fuzzy graphs in network science.

A particularly vital concept that emerged is domination in fuzzy graphs, introduced by Somasundaram and Somasundaram [18], a topic that has been further developed to include domination numbers and independent sets by researchers like Gani and Vadivel [19], Manjusha and Sunitha [20], and Talebi et al. [21]. The idea of domination in fuzzy graphs is used to analyze the control or influence that certain vertices have over others in a fuzzy network, taking into account the uncertainty or ambiguity in the relationships between vertices.

However, standard fuzzy models are limited in scenarios involving conflicting information. To address this, Zhang [22] introduced Bipolar Fuzzy Sets (BFSs), which assign each element both positive and negative membership degrees, indicating the degree to which it belongs and does not belong to the set, respectively. An element's negative degree of membership falls between $[-1, 0]$, where -1 denotes complete non-membership. This concept was quickly extended to graph theory by Akram [23], leading to the creation of Bipolar Fuzzy Graphs. Akram et al. [24–26] examined the uses of BFSs in the theory of graphs, the operations on bipolar fuzzy graphs, and their applications in decision-making problems. Paulik and Ghorai [27] studied the applications of the connectivity index of bipolar fuzzy graphs. Akram et al. [28, 29] examined the various forms of bipolar fuzzy graphs for handling uncertainty. Gong and Hua [30] studied the practical application of fuzzy edge connectivity. Karunambigai et al. [31] and Akram et al. [33] introduced the idea of domination in bipolar fuzzy graphs. Muneera et al. [32] studied the domination in various forms of bipolar fuzzy graphs.

A parallel line of inquiry for managing uncertainty stems from Pawlak's Rough Set theory [34–36], which provides a formal method for approximating concepts from incomplete information. In rough set theory, information is organized into granules or sets of objects. These sets can represent concepts, categories, or classes within a dataset. RSs define lower and upper approximations of sets based on the available information. The lower approximation represents the set of elements that are certainly within the set, while the upper approximation includes elements that may or may not be part of the set. The boundary region of a set in rough set theory consists of elements that are uncertain or indeterminate about their membership status. These elements lie on the boundary between the lower and upper approximations of the set. This approach was shown to be effective for knowledge discovery and data analysis by Kryszkiewicz [37]. To harness the strengths of both paradigms, Dubois and Prade [38, 39] introduced hybrid models like Rough Fuzzy Sets, which were extensively studied and generalized by Yao [40], Radzikowska and Kerre [41], and Yeung et al. [42]. This hybridization was later applied to bipolar fuzzy environments by Yang et al. [43, 44]. After that, researchers [54–57] extended the research of rough graphs and studied different types, including directed rough graphs, and vertex rough graphs, and discussed their properties.

These concepts were then extended to network structures. Rough graph theory extends rough set theory and graph theory to analyze and characterize imprecise or uncertain data in graph-based systems. He et al. [49–52] introduced the notion of rough graphs and their types, including weighted rough graphs and S-rough graphs. Liang et al. [53] studied the type of rough graph, known as an edge rough graph.

Akram and Arshad [58] presented fuzzy rough graph theory. While fuzzy graphs focus on representing uncertainty and vagueness in graph-based data through fuzzy membership values, fuzzy rough graphs extend this concept by incorporating RS approximations to handle uncertainty and approximation simultaneously. Researchers [59, 60] studied properties and extensions of rough fuzzy graphs in the Neutrosophic Sets. The study of directed rough fuzzy graphs was expanded upon by Ahmad and Nawaz [62, 63], who also examined its practical applications in networks related to human trafficking and trade. In their study of vertex degree concepts, Nawaz and Ahmad [64] defined several operations, such as union, cartesian product, and composition of directed rough fuzzy graphs. Akram and Zafar [65] initialized the concept of connectivity between vertices and edges of rough fuzzy directed graphs and discovered its applications in decision-making problems. Ahmad et al. [66, 67] introduced the concepts of strength of connectivity, neighborhood connectivity index, and domination in rough fuzzy directed graphs.

Khan et al. [68, 69] established a non-linear system of variable order of fractional differential equations and a fractal-fractional hybrid model to calculate the impact of fast-moving greenhouse gas emissions on climate change and coastal ecosystems. Kundu et al. [70] studied the complexity of habitat in a discrete predator-prey model. Alzabut et al. [71]

studied a discrete fractional equation model with an initial condition built to investigate tumor-immune interactions.

This brings us to a critical and unaddressed research gap. The literature shows two powerful, but separate, streams of development: Bipolar Fuzzy Graphs and Fuzzy Rough Digraphs. To date, no framework unifies these capabilities. The motivation for this research is to fill this void. The urgency of this integration is underscored by the widespread success of bipolar fuzzy logic in various decision-making domains, as demonstrated in recent studies. The literature provides extensive evidence for the efficacy of Bipolar Fuzzy Sets (BFSs) in advancing Multi-Criteria Decision Making methodologies, as summarized in the work of Akram et al. [72]. This framework's practical utility is demonstrated through its successful integration with established MCDM techniques to solve complex, real-world problems. For example, Akram and Shumaiza [75] developed a bipolar fuzzy PROMETHEE process for green supplier selection, while Akram et al. [76] extended the TOPSIS and ELECTRE-I methods to handle nuanced diagnostic problems.

Beyond direct application, a significant research thrust has focused on optimizing these models for efficiency. Work by Ali et al. has been pivotal in this area, introducing methods for both attribute reduction in bipolar fuzzy relational systems [73] and parameter reduction in bipolar fuzzy soft sets [77], thereby making the decision-making algorithms more computationally manageable. The adaptability of the BFS framework is further highlighted by its application to specialized data structures, such as the analysis of bipolar fuzzy N-soft information by Akram et al. [74].

While the concept of domination in Fuzzy Rough Digraphs (FRDs) has useful applications, it suffers from a critical problem: it cannot effectively model situations involving conflicting positive and negative information. At the same time, bipolar fuzzy sets have been extensively applied to algebraic decision-making to handle precisely this kind of duality. This reveals a significant research gap, as there is currently no mathematical framework that integrates the power of bipolar fuzzy sets with the structural uncertainty of rough digraphs to model networks under conditions of both bipolarity and roughness simultaneously.

The motivation for this research stems directly from this gap. Real-world decision scenarios are frequently characterized by this kind of bipolar uncertainty, and the lack of a suitable model hinders our ability to analyze these problems comprehensively. Therefore, the primary objective of this paper is to introduce and formalize the concept of the Bipolar Fuzzy Rough Digraph (BFRD) to address this need.

The **novelty** of our research is:

- To develop the notions of the strength of a path, strength of connectedness, and domination in BFRDs.
- To develop an effective domination model based on BRFDs.
- To study a type of BFRDs known as Regular Bipolar Fuzzy Rough Digraphs.
- To develop a proposed algorithm and apply it to practical decision-making problems.

The flowchart of domination in digraphs and its extension to the domination in the Bipolar Fuzzy Rough Digraph is given in Figure A.

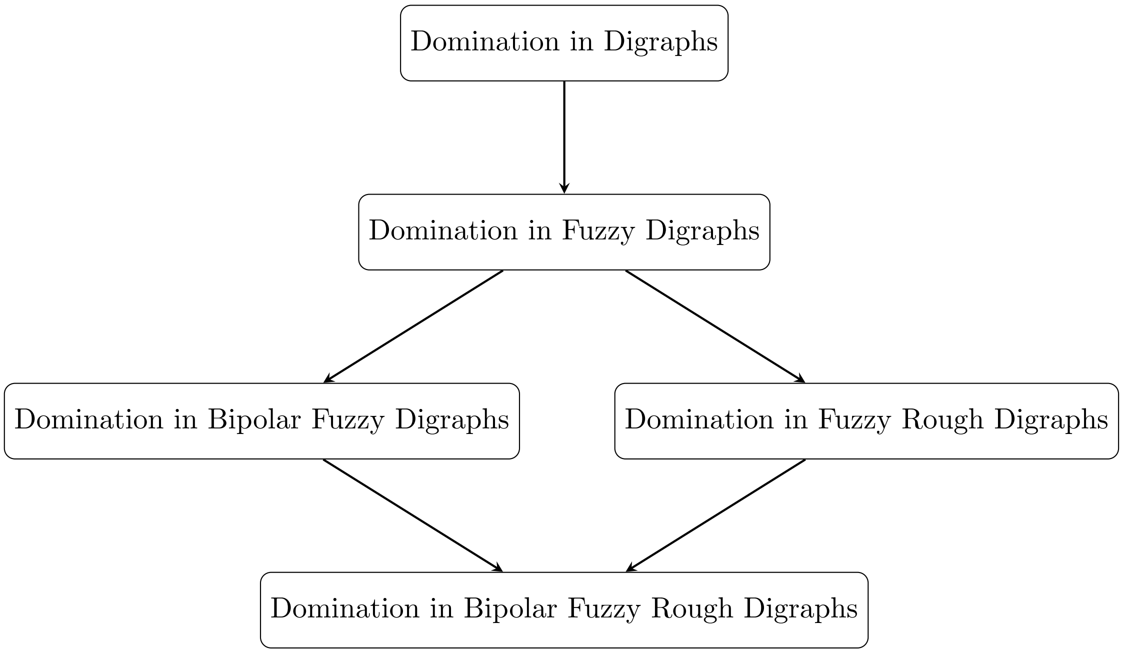


Figure A: Flowchart of domination in digraphs and its types

The list of abbreviations used in this paper is listed in Table 1:

Abbreviations	Name
RS	Rough Set
BFRD	Bipolar Fuzzy Rough Digraph
FRD	Fuzzy Rough Digraph
FS	Fuzzy Set
BFS	Bipolar Fuzzy Set
FRS	Fuzzy Rough Set
BFRS	Bipolar Fuzzy Rough Set
BFTR	Bipolar Fuzzy Tolerance Relation
$\mathbb{DS}$	Dominating Set
$\mathbb{MDS}$	Minimum Dominating Set

Table 1: List of abbreviations

The paper is organized as follows. Section 2 includes the crucial BFRD preliminary information. Section 3 presents findings about domination in BFRDs along with examples, theorems, and some types of BFRDs, including Regular Bipolar Fuzzy Rough Digraphs and Totally Regular Bipolar Fuzzy Rough Digraphs. Section 4 presents an algorithm and practical uses of domination in BFRDs for decision-making. Section 5 presents a succinct summary of the paper.

2. Preliminaries

In this section, we will cover the foundations of domination in Fuzzy Graphs to establish the concept of domination in Bipolar Fuzzy Rough Digraphs.

Definition 1.[39] Consider a fuzzy equivalence relation J on the universal set X and let O be a FS on X, then the lower approximation $\underline{J}O$ and the upper approximation $\overline{J}O$ are given by:

$$\begin{cases} \underline{J}O = \wedge[(1 - J(\alpha, \beta)) \vee O(\beta)] \\ \overline{J}O = \vee[J(\alpha, \beta) \wedge O(\beta)], \quad \forall \alpha \in X \end{cases}$$

Such that $\overline{J}O - \underline{J}O \neq \emptyset$ then the ordered pair $(\underline{J}O, \overline{J}O)$ is called the FRS.

Definition 2.[58] Consider a fuzzy tolerance relation J on the universal set X, B be a fuzzy tolerance relation on $A^* \subseteq X \times X$, O be a FS on X, $JO = (\underline{J}O, \overline{J}O)$ be a FRS on X, and V be a FS on A^* such that:

$$\begin{cases} B(\alpha_1\alpha_2, \beta_1\beta_2) \leq J(\alpha_1\beta_1) \wedge J(\alpha_2\beta_2), \quad \forall\ \alpha_1\alpha_2, \beta_1\beta_2 \in A^* \\ V(\alpha\beta) \leq (\underline{J}O)(\alpha) \wedge (\underline{J}O)(\beta), \forall \alpha\beta \in A^* \end{cases}$$

Then the lower approximation $\underline{B}V$ and the upper approximation $\overline{B}V$ are given by:

$$\begin{cases} (\underline{B}V)(\alpha\beta) = \wedge[(1 - B(\alpha_1\beta_1, \alpha_2\beta_2)) \vee V(\alpha_2\beta_2)] \\ (\overline{B}V)(\alpha\beta) = \vee[B(\alpha_1\beta_1, \alpha_2\beta_2) \wedge V(\alpha_2\beta_2)], \ \forall \alpha_1\beta_1 \in A^* \end{cases}$$

The pair $BV = (\underline{B}V, \overline{B}V)$ is a fuzzy rough relation on X.

Definition 3. [58] Consider a fuzzy tolerance relation J on the universal set X, B be a fuzzy tolerance relation on $A^* \subseteq X \times X$, O be a FS on X, $JO = (\underline{J}O, \overline{J}O)$ be a FRS on X, and $BV = (\underline{B}V, \overline{B}V)$ be a fuzzy rough relation on X, then the FRD is given by:

$$\mathcal{G} = (O, JO, V, BV)$$

Where $\underline{\mathcal{G}} = (\underline{J}O, \underline{B}V)$ is the lower approximation of graph $\mathcal{G}$ and $\overline{\mathcal{G}} = (\overline{J}O, \overline{B}V)$ is the upper approximation of $\mathcal{G}$ such that:

$$\begin{cases} \underline{B}V(\alpha\beta) \leq \bigwedge[\underline{J}O(\alpha), \underline{J}O(\beta)] \\ \overline{B}V(\alpha\beta) \leq \bigwedge[\overline{J}O(\alpha), \overline{J}O(\beta)], \quad \forall\ \alpha\beta \in A^*. \end{cases}$$

Definition 4.[61] Let $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$ be a FRD on a universal set $\mathcal{X}$ then $\mathcal{P} : \tau_0 \rightarrow$

$\tau_1 \to \cdots \to \tau_n$ is a directed path of length n from $a = \tau_0$ to $b = \tau_n$ in $\underline{\mathcal{O}}$ and $\overline{\mathcal{O}}$ where $\mu_{\underline{\mathcal{O}}}(\tau_{i-1}, \tau_i) > 0$ and $\mu_{\overline{\mathcal{O}}}(\tau_{i-1}, \tau_i) > 0$ for $i = 1, 2, .., n$ then strength of the path is defined as:

$$\left[\mu_{\underline{\mathcal{O}}}(a,b)\right]^n = \wedge \left[\mu_{\underline{\mathcal{O}}}(a,\tau_1), \mu_{\underline{\mathcal{O}}}(\tau_1,\tau_2), \ldots, \mu_{\underline{\mathcal{O}}}(\tau_{n-1},b)\right]$$
$$\left[\mu_{\overline{\mathcal{O}}}(a,b)\right]^n = \wedge \left[\mu_{\overline{\mathcal{O}}}(a,\tau_1), \mu_{\overline{\mathcal{O}}}(\tau_1,\tau_2), \ldots, \mu_{\overline{\mathcal{O}}}(\tau_{n-1},b)\right]$$

Such that the strength of a path in $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$ is defined by:

$$[\mu(a,b)]^n = \left[\mu_{\underline{\mathcal{O}}}(a,b)\right]^n + \left[\mu_{\overline{\mathcal{O}}}(a,b)\right]^n$$

Definition 5.[61] Let $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$ be a FRD on a universal set $\mathcal{X}$. The strength of connectedness from point τ_0 to τ_1 in $\underline{\mathcal{O}}$ and $\overline{\mathcal{O}}$ is defined by:

$$\mathrm{CONN}_{\underline{\mathcal{O}}}(\tau_0,\tau_1) = \sup_{n\in\mathbb{N}}[\mu_{\underline{\mathcal{O}}}(\tau_0,\tau_1)]^n$$
$$\mathrm{CONN}_{\overline{\mathcal{O}}}(\tau_0,\tau_1) = \sup_{n\in\mathbb{N}}[\mu_{\overline{\mathcal{O}}}(\tau_0,\tau_1)]^n$$

Definition 6.[61] An edge $\tau_0\tau_1$ in FRD $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$ is considered as a strong edge in $\underline{\mathcal{O}}$ and $\overline{\mathcal{O}}$ if:

$$\mathrm{CONN}_{\underline{\mathcal{O}}-\tau_0\tau_1}(\tau_0,\tau_1) \le \mu_{\underline{\mathcal{O}}}(\tau_0,\tau_1)$$
$$\mathrm{CONN}_{\overline{\mathcal{O}}-\tau_0\tau_1}(\tau_0,\tau_1) \le \mu_{\overline{\mathcal{O}}}(\tau_0,\tau_1)$$

If an edge $\tau_0\tau_1$ is strong in both $\underline{\mathcal{O}}$ and $\overline{\mathcal{O}}$ then it is strong in $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$.

Definition 7.[67] Let $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$ be a FRD on a universal set $\mathcal{X}$. Let $\tau_0, \tau_1 \in \mathcal{X}$ then vertex τ_0 dominates the vertex τ_1 in $\underline{\mathcal{O}}$ if directed edge from τ_0 to τ_1 is a strong edge in $\underline{\mathcal{O}}$ such that:

$$\mathrm{CONN}_{\underline{\mathcal{O}}-\tau_0\tau_1}(\tau_0,\tau_1) \le \mu_{\underline{\mathcal{O}}}(\tau_0,\tau_1)$$

Similarly, τ_0 dominates the vertex τ_1 in $\overline{\mathcal{O}}$ if directed edge from τ_0 to τ_1 is a strong edge in $\overline{\mathcal{O}}$ such that:

$$\mathrm{CONN}_{\overline{\mathcal{O}}-\tau_0\tau_1}(\tau_0,\tau_1) \le \mu_{\overline{\mathcal{O}}}(\tau_0,\tau_1)$$

If directed edge from τ_0 to τ_1 is a strong edge in both $\underline{\mathcal{O}}$ and $\overline{\mathcal{O}}$ then τ_0 dominates the vertex τ_1 in $\mathcal{O} = (\underline{\mathcal{O}}, \overline{\mathcal{O}})$.

Definition 8. [46] Let X be universal set and $\Upsilon = (\Upsilon^+, \Upsilon^-)$ be a bipolar fuzzy equivalence relation on X. Let $W = (W^+, W^-)$ be a BFS on X. The lower approximation of W under Υ is denoted by $\underline{\Upsilon}W$ and the upper approximation of W under Υ is denoted by $\overline{\Upsilon}W$ such that:

$$\underline{\Upsilon}W = (\underline{\Upsilon}W^+, \underline{\Upsilon}W^-)$$
$$\overline{\Upsilon}W = (\overline{\Upsilon}W^+, \overline{\Upsilon}W^-)$$

$\underline{\Upsilon}W$ and $\overline{\Upsilon}W$ are defined by:

$$\begin{cases} \underline{\Upsilon}W^+(\epsilon_0) = \bigwedge [1 - \Upsilon^+(\epsilon_0, \epsilon_1) \vee W^+(\beta)] \\ \underline{\Upsilon}W^-(\epsilon_0) = \bigvee [-1 - \Upsilon^-(\epsilon_0, \epsilon_1) \wedge W^-(\epsilon_1)] \\ \overline{\Upsilon}W^+(\epsilon_0) = \bigvee [\Upsilon^+(\epsilon_0, \epsilon_1) \wedge W^+(\epsilon_1)] \\ \overline{\Upsilon}W^-(\epsilon_0) = \bigwedge [\Upsilon^-(\epsilon_0, \epsilon_1) \vee W^-(\epsilon_1)], \quad \forall \epsilon_0, \epsilon_1 \in X \end{cases}$$

The pair $(\underline{\Upsilon}W, \overline{\Upsilon}W)$ is called BFRS if $\overline{\Upsilon}W - \underline{\Upsilon}W \neq \emptyset$.

Definition 9. [47] Let X be universal set and $\Upsilon = (\Upsilon^+, \Upsilon^-)$ be a bipolar fuzzy tolerance equivalence relation on X. Let $W = (W^+, W^-)$ be a BFS on X and $(\underline{\Upsilon}W, \overline{\Upsilon}W)$ is a BFRS on X. Let $A \subseteq X \times X$ and $L = (L^-, L^+)$ be a BFTR on A such that:

$$L^+(\alpha\beta, \gamma\theta) \leq \Upsilon^+(\alpha, \gamma) \wedge \Upsilon^+(\beta, \theta)$$
$$L^-(\alpha\beta, \gamma\theta) \geq \Upsilon^-(\alpha, \gamma) \vee \Upsilon^-(\beta, \theta)$$

Let a BFS $\Lambda = (\Lambda^+, \Lambda^-)$ on A such that:

$$\begin{cases} \Lambda^+(\gamma\theta) \leq (\underline{\Upsilon}W^+)(\gamma) \wedge \underline{\Upsilon}W^+(\theta) \\ \Lambda^-(\gamma\theta) \geq (\underline{\Upsilon}W^-)(\gamma) \vee \underline{\Upsilon}W^-(\theta) \end{cases}$$

The lower approximation is denoted by $\underline{L}\Lambda$ and upper approximation is denoted by $\overline{L}\Lambda$ such that:

$$\underline{L}\Lambda = (\underline{L}\Lambda^+, \underline{L}\Lambda^-), \quad \overline{L}\Lambda = (\overline{L}\Lambda^+, \overline{L}\Lambda^-)$$

And are defined by:

$$\begin{cases} \underline{L}\Lambda^+(\alpha\beta) = \bigwedge[(1 - L^+(\alpha\beta, \gamma\theta)) \vee \Lambda^+(\gamma\theta)] \\ \underline{L}\Lambda^-(\alpha\beta) = \bigvee[(-1 - L^-(\alpha\beta, \gamma\theta)) \wedge \Lambda^-(\gamma\theta)] \\ \overline{L}\Lambda^+(\alpha\beta) = \bigvee[L^+(\alpha\beta, \gamma\theta) \wedge \Lambda^+(\gamma\theta)] \\ \overline{L}\Lambda^-(\alpha\beta) = \bigwedge[(L^-(\alpha\beta, \gamma\theta) \vee \Lambda^-(\gamma\theta)], \quad \forall \gamma\theta \in A \end{cases}$$

Then the pair $(\underline{L}\Lambda, \overline{L}, \Lambda)$ is called a Bipolar fuzzy rough relation.

3. Domination in Bipolar Fuzzy Rough Digraphs

In this section, we present the ideas of Bipolar Fuzzy Rough Digraphs, the union of Bipolar Fuzzy Rough Digraph, the strength of a path, the strength of the connectedness, domination, minimum dominating set, domination number, degree of the vertices, and the notions of Regular, Totally Regular Bipolar Fuzzy Rough Digraphs.

Definition 10. Let X be a universal set. If Υ is a BFTR on X, L is a BFTR on $A \subseteq X \times X$, W be a BFS on X, $\Upsilon W = (\underline{\Upsilon} W, \overline{\Upsilon} W)$ is a BFRS on X, and $L\Lambda = (\underline{L}\Lambda, \overline{L}\Lambda)$ is a bipolar fuzzy rough relation on X, then the BFRD is defined by:

$$\Gamma = (W, \Upsilon W, \Lambda, L\Lambda)$$

Where $\underline{\Gamma} = (\underline{\Upsilon} W, \underline{L}\Lambda)$ is the lower approximation of Γ and $\overline{\Gamma} = (\overline{\Upsilon} W, \overline{L}\Lambda)$ is the upper approximation of Γ such that:

$$\begin{cases} \underline{L}\Lambda^+(\alpha\beta) \leq \bigwedge[\underline{\Upsilon} W^+(\alpha), \underline{\Upsilon} W^+(\beta)] \\ \underline{L}\Lambda^-(\alpha\beta) \geq \bigvee[\underline{\Upsilon} W^-(\alpha), \underline{\Upsilon} W^-(\beta)] \\ \overline{L}\Lambda^+(\alpha\beta) \leq \bigwedge[\overline{\Upsilon} W^+(\alpha), \overline{\Upsilon} W^+(\beta)] \\ \overline{L}\Lambda^-(\alpha\beta) \geq \bigvee[\overline{\Upsilon} W^-(\alpha), \overline{\Upsilon} W^-(\beta)], \quad \forall\ \alpha\beta \in A. \end{cases}$$

Example 1. Let $X = \{\epsilon, \eta, \zeta\}$ be a universal set and $\Upsilon = (\Upsilon^+, \Upsilon^-)$ be a BFTR on X defined in Tables 2 and 3.

Υ^+	ϵ	η	ζ
ϵ	1	0.1	0.3
η	0.1	1	0.5
ζ	0.3	0.5	1

Table 2: Υ^+ of relation Υ

Υ^-	ϵ	η	ζ
ϵ	−1	−0.1	−0.4
η	−0.1	−1	−0.2
ζ	−0.4	−0.2	−1

Table 3: Υ^- of relation Υ

Let $W = \{(\epsilon, 0.2, -0.1), (\eta, 0.3, -0.1), (\zeta, 0.1, -0.3)\}$ be a BFS on X. Then the lower approximation of W with respect to Υ is given by:

$$\underline{\Upsilon} W = \{(\epsilon, 0.2, -0.1), (\eta, 0.3, -0.1), (\zeta, 0.1, -0.3)\}$$

The upper approximation $\overline{\Upsilon}$ is given by:

$$\overline{\Upsilon} W = \{(\epsilon, 0.2, -0.3), (\eta, 0.3, -0.2), (\zeta, 0.3, -0.3)\}$$

Since $\overline{\Upsilon} W - \underline{\Upsilon} W \neq \emptyset$, therefore, $(\underline{\Upsilon} W, \overline{\Upsilon} W)$ is a BFRS.
Let $L = (L^+, L^-)$ be a BFTR on $A \subseteq X \times X$ and defined in the Tables 4 and 5:

L^+	$\epsilon\epsilon$	$\epsilon\eta$	$\eta\zeta$	$\epsilon\zeta$
$\epsilon\epsilon$	1	0.1	0.1	0.2
$\epsilon\eta$	0.1	1	0.05	0.02
$\eta\zeta$	0.1	0.05	1	0.03
$\epsilon\zeta$	0.2	0.02	0.03	1

Table 4: L^- of relation L

L^-	$\epsilon\epsilon$	$\epsilon\eta$	$\eta\zeta$	$\epsilon\zeta$
$\epsilon\epsilon$	-1	-0.1	-0.05	-0.3
$\epsilon\eta$	-0.1	-1	-0.1	-0.1
$\eta\zeta$	-0.05	-0.1	-1	-0.05
$\epsilon\zeta$	-0.3	-0.1	-0.05	-1

Table 5: L^- of relation L

Let $\Lambda = (\Lambda^+, \Lambda^-)$ be a BFS on A defined by:

$$\Lambda = \{(\epsilon\epsilon, 0.2. - 0.2), (\epsilon\eta, 0.1, -0.1), (\eta\zeta, 0.2, -0.1), (\epsilon\zeta, 0.1, -0.2)\}$$

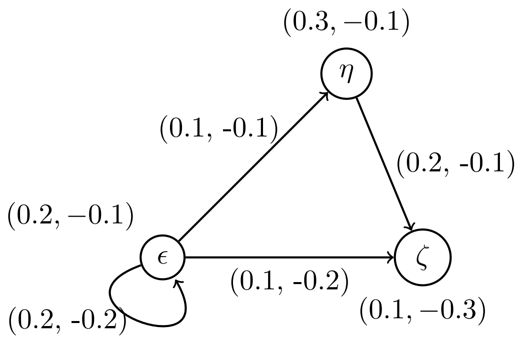
The set of lower approximation of Λ is:

$$\underline{L}\Lambda = \{(\epsilon\epsilon, 0.2, -0.2), (\epsilon\eta, 0.1, -0.1), (\eta\zeta, 0.2, -0.1), (\epsilon\zeta, 0.1, -0.2)\}$$

The set of upper approximation of Λ is:

$$\overline{L}\Lambda = \{(\epsilon\epsilon, 0.2, -0.2), (\epsilon\eta, 0.1, -0.1), (\eta\zeta, 0.2, -0.1), (\epsilon\zeta, 0.2, -0.2)\}$$

The graphs of $\underline{\Gamma}$ and $\overline{\Gamma}$ of Γ are given in Figure 1:



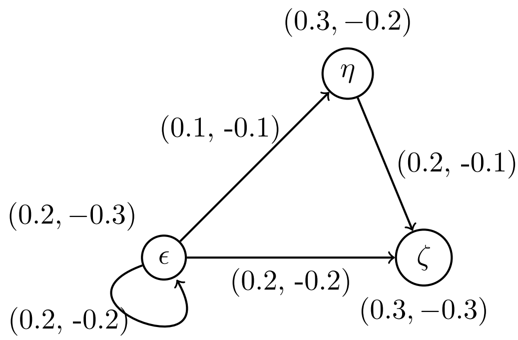


Figure 1: $\overline{\Gamma} = (\underline{\Gamma}, \overline{\Gamma})$

Definition 11. The union of two BFRDs $\Gamma_1 = (\underline{\Gamma}_1, \overline{\Gamma}_1)$ and $\Gamma_2 = (\underline{\Gamma}_2, \overline{\Gamma}_2)$ on universal set X is defined as $\Gamma_1 \cup \Gamma_2 = (\underline{\Gamma}_1 \cup \underline{\Gamma}_2, \overline{\Gamma}_1 \cup \overline{\Gamma}_2)$ where $\underline{\Gamma}_1 \cup \underline{\Gamma}_2 = (\underline{\Upsilon}W_1 \cup \underline{\Upsilon}W_2, \underline{L}\Lambda_1 \cup \underline{L}\Lambda_2)$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = (\overline{\Upsilon}W_1 \cup \overline{\Upsilon}W_2, \overline{L}\Lambda_1 \cup \overline{L}\Lambda_2)$ such that:

$$(\underline{\Upsilon}W_1 \cup \underline{\Upsilon}W_2)(x) = \begin{cases} \underline{\Upsilon}W_1^+(x) \vee \underline{\Upsilon}W_2^+(x) \\ \underline{\Upsilon}W_1^-(x) \wedge \underline{\Upsilon}W_2^-(x), \quad \forall x \in supp(W_1 \cup W_2) \end{cases}$$

$$(\overline{\Upsilon}W_1 \cup \overline{\Upsilon}W_2)(x) = \begin{cases} \overline{\Upsilon}W_1^+(x) \vee \overline{\Upsilon}W_2^+(x) \\ \overline{\Upsilon}W_1^-(x) \wedge \overline{\Upsilon}W_2^-(x), \quad \forall x \in supp(W_1 \cup W_2) \end{cases}$$

$$(\underline{L}\Lambda_1 \cup \underline{L}\Lambda_2)(xy) = \begin{cases} \underline{L}\Lambda_1^+(xy) \vee \underline{L}\Lambda_2^+(xy) \\ \underline{L}\Lambda_1^-(xy) \wedge \underline{L}\Lambda_2^-(xy), \quad \forall xy \in supp(\Lambda_1 \cup \Lambda_2) \end{cases}$$

$$(\overline{L}\Lambda_1 \cup \overline{L}\Lambda_2)(xy) = \begin{cases} \overline{L}\Lambda_1^+(xy) \vee \overline{L}\Lambda_2^+(xy) \\ \overline{L}\Lambda_1^-(xy) \wedge \overline{L}\Lambda_2^-(xy), \quad \forall xy \in supp(\Lambda_1 \cup \Lambda_2) \end{cases}$$

Definition 12. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD on a universal set X then $P : \epsilon_0 \to \epsilon_1 \to \cdots \to \epsilon_n$ is a directed path of length n from $a = \epsilon_0$ to $b = \epsilon_n$ in $\underline{\Gamma}$ and $\overline{\Gamma}$ where $(\underline{L}\Lambda^+(\epsilon_{i-1}, \epsilon_i) > 0, \underline{L}\Lambda^-(\epsilon_{i-1}, \epsilon_i) < 0)$ and $(\overline{L}\Lambda^+(\epsilon_{i-1}, \epsilon_i) > 0, \overline{L}\Lambda^-(\epsilon_{i-1}, \epsilon_i) < 0)$ for $i = 1, 2, .., n$ then:

$$\begin{aligned}
\left[\underline{L}\Lambda^+(a,b)\right]^n &= \wedge \left[\underline{L}\Lambda^+(a,\epsilon_1), \underline{L}\Lambda^+(\epsilon_1,\epsilon_2), \ldots, \underline{L}\Lambda^+(\epsilon_{n-1},b)\right] \\
\left[\underline{L}\Lambda^-(a,b)\right]^n &= \vee \left[\underline{L}\Lambda^-(a,\epsilon_1), \underline{L}\Lambda^-(\epsilon_1,\epsilon_2), \ldots, \underline{L}\Lambda^-(\epsilon_{n-1},b)\right] \\
\left[\overline{L}\Lambda^+(a,b)\right]^n &= \wedge \left[\overline{L}\Lambda^+(a,\epsilon_1), \overline{L}\Lambda^+(\epsilon_1,\epsilon_2), \ldots, \overline{L}\Lambda^+(\epsilon_{n-1},b)\right] \\
\left[\overline{L}\Lambda^-(a,b)\right]^n &= \vee \left[\overline{L}\Lambda^-(a,\epsilon_1), \overline{L}\Lambda^-(\epsilon_1,\epsilon_2), \ldots, \overline{L}\Lambda^-(\epsilon_{n-1},b)\right]
\end{aligned}$$

Such that the strength of a path in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is defined by:

$$\begin{aligned}
\left[L\Lambda^+(a,b)\right]^n &= \left[\underline{L}\Lambda^+(a,b)\right]^n + \left[\overline{L}\Lambda^+(a,b)\right]^n \\
\left[L\Lambda^-(a,b)\right]^n &= \left[\underline{L}\Lambda^-(a,b)\right]^n + \left[\overline{L}\Lambda^-(a,b)\right]^n
\end{aligned}$$

Definition 13. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD on a universal set X. The strength of connectedness from point ϵ_0 to ϵ_1 in $\underline{\Gamma}$ and $\overline{\Gamma}$ is defined by:

$$\begin{aligned}
\text{CONN}^+_{\underline{\Gamma}}(\epsilon_0, \epsilon_1) &= \sup_{n \in \mathbb{N}} [\underline{L}\Lambda^+(\epsilon_0, \epsilon_1)]^n \\
\text{CONN}^-_{\underline{\Gamma}}(\epsilon_0, \epsilon_1) &= \inf_{n \in \mathbb{N}} [\underline{L}\Lambda^-(\epsilon_0, \epsilon_1)]^n \\
\text{CONN}^+_{\overline{\Gamma}}(\epsilon_0, \epsilon_1) &= \sup_{n \in \mathbb{N}} [\overline{L}\Lambda^+(\epsilon_0, \epsilon_1)]^n \\
\text{CONN}^-_{\overline{\Gamma}}(\epsilon_0, \epsilon_1) &= \inf_{n \in \mathbb{N}} [\overline{L}\Lambda^-(\epsilon_0, \epsilon_1)]^n
\end{aligned}$$

and it is denoted by:

$$\begin{aligned}
\text{CONN}_{\underline{\Gamma}}(\epsilon_0, \epsilon_1) &= (\text{CONN}^+_{\underline{\Gamma}}(\epsilon_0, \epsilon_1), \text{CONN}^-_{\underline{\Gamma}}(\epsilon_0, \epsilon_1)) \\
\text{CONN}_{\overline{\Gamma}}(\epsilon_0, \epsilon_1) &= (\text{CONN}^+_{\overline{\Gamma}}(\epsilon_0, \epsilon_1), \text{CONN}^-_{\overline{\Gamma}}(\epsilon_0, \epsilon_1))
\end{aligned}$$

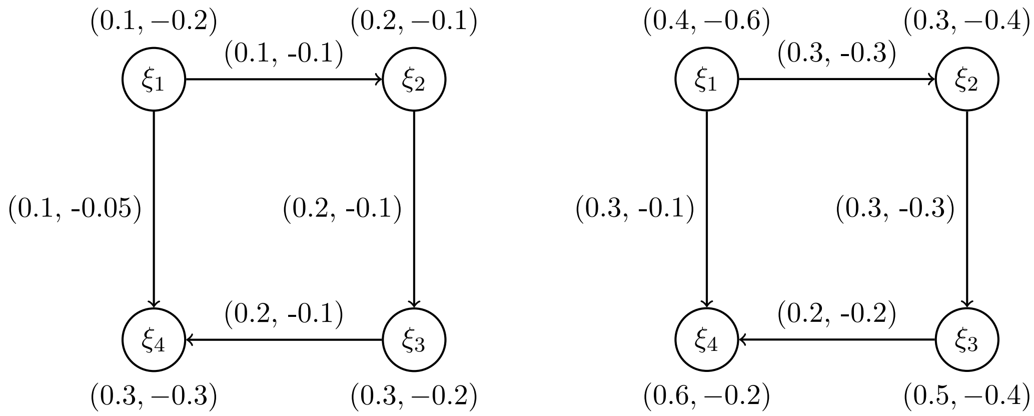
Definition 14. An edge $\epsilon_0\epsilon_1$ in BFRD $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is considered as a strong edge in $\underline{\Gamma}$ and $\overline{\Gamma}$ if:

$$\text{CONN}^+_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0, \epsilon_1) \le \underline{L}\Lambda^+(\epsilon_0, \epsilon_1)$$

$$\mathrm{CONN}^{-}_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \underline{L}\Lambda^{-}(\epsilon_0,\epsilon_1)$$
$$\mathrm{CONN}^{+}_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \leq \overline{L}\Lambda^{+}(\epsilon_0,\epsilon_1)$$
$$\mathrm{CONN}^{-}_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \overline{L}\Lambda^{-}(\epsilon_0,\epsilon_1)$$

If an edge $\epsilon_0\epsilon_1$ is strong in both $\underline{\Gamma}$ and $\overline{\Gamma}$ then it is strong in $\Gamma = (\underline{\Gamma},\overline{\Gamma})$.

Example 2. Let $\Gamma = (\underline{\Gamma},\overline{\Gamma})$ be a BFRD defined on the universal set $X = \{\xi_1,\xi_2,\xi_3,\xi_4\}$ given in Figure 2:



$\underline{\Gamma} = (\underline{\Upsilon}W, \underline{L}\Lambda)$ $\overline{\Gamma} = (\overline{\Upsilon}W, \overline{L}\Lambda)$

Figure 2: $\overline{\Gamma} = (\underline{\Gamma},\overline{\Gamma})$

$$\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi_1\xi_2}(\xi_1,\xi_2) = 0 \leq 0.1 = \underline{L}\Lambda^{+}(\xi_1,\xi_2)$$
$$\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi_1\xi_2}(\xi_1,\xi_2) = 0 \geq -0.1 = \underline{L}\Lambda^{-}(\xi_1,\xi_2)$$
$$\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi_2\xi_3}(\xi_2,\xi_3) = 0 \leq 0.2 = \underline{L}\Lambda^{+}(\xi_2,\xi_3)$$
$$\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi_2\xi_3}(\xi_2,\xi_3) = 0 \geq -0.1 = \underline{L}\Lambda^{-}(\xi_2,\xi_3)$$
$$\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi_3\xi_4}(\xi_3,\xi_4) = 0 \leq 0.2 = \underline{L}\Lambda^{+}(\xi_3,\xi_4)$$
$$\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi_3\xi_4}(\xi_3,\xi_4) = 0 \geq -0.1 = \underline{L}\Lambda^{-}(\xi_3,\xi_4)$$
$$\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi_1\xi_4}(\xi_1,\xi_4) = 0.1 \leq 0.1 = \underline{L}\Lambda^{+}(\xi_1,\xi_4)$$
$$\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi_1\xi_4}(\xi_1,\xi_4) = -0.1 \leq -0.05 = \underline{L}\Lambda^{-}(\xi_1,\xi_4)$$
$$\mathrm{CONN}^{+}_{\overline{\Gamma}-\xi_1\xi_2}(\xi_1,\xi_2) = 0 \leq 0.3 = \overline{L}\Lambda^{+}(\xi_1,\xi_2)$$
$$\mathrm{CONN}^{-}_{\overline{\Gamma}-\xi_1\xi_2}(\xi_1,\xi_2) = 0 \geq -0.3 = \overline{L}\Lambda^{-}(\xi_1,\xi_2)$$
$$\mathrm{CONN}^{+}_{\overline{\Gamma}-\xi_1\xi_2}(\xi_2,\xi_3) = 0 \leq 0.3 = \overline{L}\Lambda^{+}(\xi_2,\xi_3)$$
$$\mathrm{CONN}^{-}_{\overline{\Gamma}-\xi_2\xi_3}(\xi_2,\xi_3) = 0 \geq -0.3 = \overline{L}\Lambda^{-}(\xi_2,\xi_3)$$

$$\text{CONN}^{+}_{\overline{\Gamma}-\xi_3\xi_4}(\xi_3,\xi_4) = 0 \leq 0.2 = \overline{L}\Lambda^{+}(\xi_3,\xi_4)$$
$$\text{CONN}^{-}_{\overline{\Gamma}-\xi_3\xi_4}(\xi_3,\xi_4) = 0 \geq -0.2 = \overline{L}\Lambda^{-}(\xi_3,\xi_4)$$
$$\text{CONN}^{+}_{\overline{\Gamma}-\xi_1\xi_4}(\xi_1,\xi_4) = 0.2 \leq 0.3 = \overline{L}\Lambda^{+}(\xi_1,\xi_4)$$
$$\text{CONN}^{-}_{\overline{\Gamma}-\xi_1\xi_4}(\xi_1,\xi_4) = -0.2 \leq -0.1 = \overline{L}\Lambda^{-}(\xi_1,\xi_4)$$

The strong edges in $\underline{\Gamma}$ are $(\xi_1,\xi_2),(\xi_2,\xi_3)$, and (ξ_3,ξ_4). The strong edges in $\overline{\Gamma}$ are $(\xi_1,\xi_2),(\xi_2,\xi_3)$, and (ξ_3,ξ_4). Therefore $(\xi_1,\xi_2),(\xi_2,\xi_3)$, and (ξ_3,ξ_4) are strong edges in $\Gamma = (\underline{\Gamma},\overline{\Gamma})$.

Definition 15. Let $\Gamma = (\underline{\Gamma},\overline{\Gamma})$ be a BFRD on a universal set X. Let $\epsilon_0,\epsilon_1 \in X$ then vertex ϵ_0 dominates the vertex ϵ_1 in $\underline{\Gamma}$ if directed edge from ϵ_0 to ϵ_1 is a strong edge in $\underline{\Gamma}$ such that:

$$\text{CONN}^{+}_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \leq \underline{L}\Lambda^{+}(\epsilon_0,\epsilon_1)$$
$$\text{CONN}^{-}_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \underline{L}\Lambda^{-}(\epsilon_0,\epsilon_1)$$

Similarly, ϵ_0 dominates the vertex ϵ_1 in $\overline{\Gamma}$ if directed edge from ϵ_0 to ϵ_1 is a strong edge in $\overline{\Gamma}$ such that:

$$\text{CONN}^{+}_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \leq \overline{L}\Lambda^{+}(\epsilon_0,\epsilon_1)$$
$$\text{CONN}^{-}_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \overline{L}\Lambda^{-}(\epsilon_0,\epsilon_1)$$

ϵ_0 dominates the vertex ϵ_1 in $\Gamma = (\underline{\Gamma},\overline{\Gamma})$ if directed edge from ϵ_0 to ϵ_1 is a strong edge both in $\underline{\Gamma}$ and $\overline{\Gamma}$.

Definition 16. The fuzzy cardinality of set of points $\mathcal{V}$ in $\underline{\Gamma}$ or order of $\underline{\Gamma}$ is defined as:

$$|\mathcal{V}(\underline{\Gamma})| = \sum_{\epsilon_i \in \mathcal{V}} \frac{1 + \underline{\Upsilon}W^{+}(\epsilon_i) + \underline{\Upsilon}W^{-}(\epsilon_i)}{2}$$

The fuzzy cardinality of set of points $\mathcal{V}$ in $\overline{\Gamma}$ or order of $\overline{\Gamma}$ is defined as:

$$|\mathcal{V}(\overline{\Gamma})| = \sum_{\epsilon_i \in \overline{\mathcal{V}}} \frac{1 + \overline{\Upsilon}W^{+}(\epsilon_i) + \overline{\Upsilon}W^{-}(\epsilon_i)}{2}$$

Example 3. Let $\Gamma = (\underline{\Gamma},\overline{\Gamma})$ be a BFRD defined on the universal set $X = \{\eta,\xi,\epsilon\}$ given in Figure 1:
The fuzzy cardinality of vertices in $\underline{\Gamma}$ are:

$$|\mathcal{V}(\underline{\Gamma})| = \sum_{\epsilon_i \in \mathcal{V}} \frac{1 + \underline{\Upsilon}W^{+}(\epsilon_i) + \underline{\Upsilon}W^{-}(\epsilon_i)}{2}$$
$$|\mathcal{V}(\underline{\Gamma})| = \frac{1 + \underline{\Upsilon}W^{+}(\eta) + \underline{\Upsilon}W^{-}(\eta)}{2} + \frac{1 + \underline{\Upsilon}W^{+}(\xi) + \underline{\Upsilon}W^{-}(\xi)}{2} + \frac{1 + \underline{\Upsilon}W^{+}(\epsilon) + \underline{\Upsilon}W^{-}(\epsilon)}{2}$$

$$|\mathcal{V}(\underline{\Gamma})| = \frac{1+0.3+(-0.1)}{2} + \frac{1+(0.1)+(-0.3)}{2} + \frac{1+(0.2)+(-0.1)}{2}$$
$$|\mathcal{V}(\underline{\Gamma})| = 1.55$$

$$|\mathcal{V}(\overline{\Gamma})| = \sum_{\epsilon_i \in \mathcal{V}} \frac{1+\overline{\Upsilon}W^+(\epsilon_i)+\overline{\Upsilon}W^-(\epsilon_i)}{2}$$
$$|\mathcal{V}(\overline{\Gamma})| = \frac{1+\overline{\Upsilon}W^+(\eta)+\overline{\Upsilon}W^-(\eta)}{2} + \frac{1+\overline{\Upsilon}W^+(\xi)+\overline{\Upsilon}W^-(\xi)}{2} + \frac{1+\overline{\Upsilon}W^+(\epsilon)+\overline{\Upsilon}W^-(\epsilon)}{2}$$
$$|\mathcal{V}(\overline{\Gamma})| = \frac{1+0.3+(-0.2)}{2} + \frac{1+(0.3)+(-0.3)}{2} + \frac{1+(0.2)+(-0.2)}{2}$$
$$|\mathcal{V}(\overline{\Gamma})| = 1.55$$

Definition 17. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD. A subset $\underline{\gamma}F$ of $\underline{\Upsilon}W$ is said to be the lower dominating set of $\underline{\Gamma}$ if for every $\epsilon_0 \in \underline{\Upsilon}W - \underline{\gamma}F$ there exists $\epsilon_1 \in \underline{\gamma}F$ such that ϵ_1 dominates ϵ_0.
A subset $\overline{\gamma}F$ of $\overline{\Upsilon}W$ is said to be the upper dominating set of $\overline{\Gamma}$ if for every $\epsilon_0 \in \overline{\Upsilon}W - \overline{\gamma}F$ there exists $\epsilon_1 \in \overline{\gamma}F$ such that ϵ_1 dominates ϵ_0. A set of vertices is called a $\mathbb{DS}$ in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ if it is both upper and lower dominating set.

Definition 18. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD. A $\mathbb{DS}$ in $\underline{\Gamma}$ is called a lower minimal dominating set if there are no proper subsets of it in $\underline{\Gamma}$. A $\mathbb{DS}$ in $\overline{\Gamma}$ is called an upper minimal dominating set if there are no proper subsets of it in $\overline{\Gamma}$. A $\mathbb{DS}$ that is both lower and upper minimal dominating set is called a minimal dominating set in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$.
A minimal dominating set for which the sum of fuzzy cardinalities in $\underline{\Gamma}$ and $\overline{\Gamma}$ is least among all minimal dominating sets of $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is called a $\mathbb{MDS}$ $D(\Gamma)$.

$$|D(\underline{\Gamma})| + |D(\overline{\Gamma})| = \sum_{\epsilon_i \in \mathcal{V}} \frac{1+\underline{\Upsilon}W^+(\epsilon_i)+\underline{\Upsilon}W^-(\epsilon_i)}{2} + \sum_{\epsilon_i \in \mathcal{V}} \frac{1+\overline{\Upsilon}W^+(\epsilon_i)+\overline{\Upsilon}W^-(\epsilon_i)}{2}$$

Definition 19. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD. The fuzzy cardinality of a minimal dominating set $D(\underline{\Gamma})$ in $\underline{\Gamma}$ is called lower domination number $\Omega_D(\underline{\Gamma})$. The fuzzy cardinality of a minimal dominating set $D(\overline{\Gamma})$ in $\overline{\Gamma}$ is called upper domination number $\Omega_D(\overline{\Gamma})$. The sum of the lower and upper domination number of a $\mathbb{MDS}$ $D(\Gamma)$ is called the domination number of BFRD $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$:

$$\Omega_D(\Gamma) = \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma})$$

Example 4. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD defined on the universal set $X = \{\xi, \kappa, \varepsilon, \delta\}$ given in Figure 3:

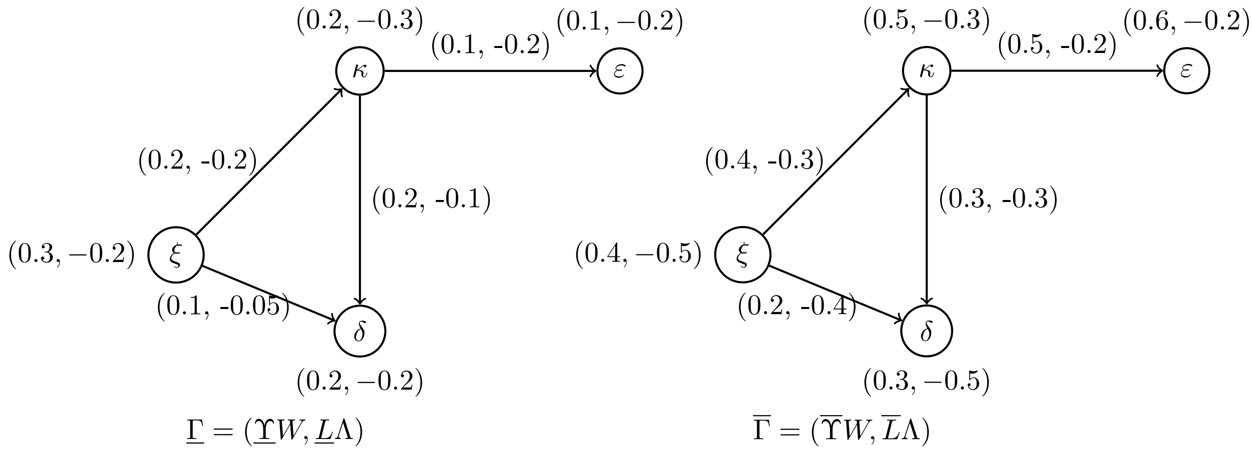


Figure 3: $\overline{\Gamma} = (\underline{\Gamma}, \overline{\Gamma})$

$$
\begin{aligned}
\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi\kappa}(\xi,\kappa) &= 0 \leq 0.2 = \underline{L}\Lambda^{+}(\xi,\kappa)\\
\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi\kappa}(\xi,\kappa) &= 0 \geq -0.2 = \underline{L}\Lambda^{-}(\xi,\kappa)\\
\mathrm{CONN}^{+}_{\underline{\Gamma}-\kappa\epsilon}(\kappa,\epsilon) &= 0 \leq 0.1 = \underline{L}\Lambda^{+}(\kappa,\epsilon)\\
\mathrm{CONN}^{-}_{\underline{\Gamma}-\kappa\epsilon}(\kappa,\epsilon) &= 0 \geq -0.2 = \underline{L}\Lambda^{-}(\kappa,\epsilon)\\
\mathrm{CONN}^{+}_{\underline{\Gamma}-\kappa\delta}(\kappa,\delta) &= 0 \leq 0.2 = \underline{L}\Lambda^{+}(\kappa,\delta)\\
\mathrm{CONN}^{-}_{\underline{\Gamma}-\kappa\delta}(\kappa,\delta) &= 0 \geq -0.1 = \underline{L}\Lambda^{-}(\kappa,\delta)\\
\mathrm{CONN}^{+}_{\underline{\Gamma}-\xi\delta}(\xi,\delta) &= 0.2 \geq 0.1 = \underline{L}\Lambda^{+}(\xi,\delta)\\
\mathrm{CONN}^{-}_{\underline{\Gamma}-\xi\delta}(\xi,\delta) &= -0.1 \leq -0.05 = \underline{L}\Lambda^{-}(\xi,\delta)\\
\mathrm{CONN}^{+}_{\overline{\Gamma}-\xi\kappa}(\xi,\kappa) &= 0 \leq 0.4 = \overline{L}\Lambda^{+}(\xi,\kappa)\\
\mathrm{CONN}^{-}_{\overline{\Gamma}-\xi,\kappa}(\xi,\kappa) &= 0 \geq -0.3 = \overline{L}\Lambda^{-}(\xi,\kappa)\\
\mathrm{CONN}^{+}_{\overline{\Gamma}-\kappa\epsilon}(\kappa,\epsilon) &= 0 \leq 0.5 = \overline{L}\Lambda^{+}(\kappa,\epsilon)\\
\mathrm{CONN}^{-}_{\overline{\Gamma}-\kappa\epsilon}(\kappa,\epsilon) &= 0 \geq -0.2 = \overline{L}\Lambda^{-}(\kappa,\epsilon)\\
\mathrm{CONN}^{+}_{\overline{\Gamma}-\kappa\delta}(\kappa,\delta) &= 0 \leq 0.3 = \overline{L}\Lambda^{+}(\kappa,\delta)\\
\mathrm{CONN}^{-}_{\overline{\Gamma}-\kappa\delta}(\kappa,\delta) &= 0 \geq -0.3 = \overline{L}\Lambda^{-}(\kappa,\delta)\\
\mathrm{CONN}^{+}_{\overline{\Gamma}-\xi\delta}(\xi,\delta) &= 0.3 \geq 0.2 = \overline{L}\Lambda^{+}(\xi,\delta)\\
\mathrm{CONN}^{-}_{\overline{\Gamma}-\xi\delta}(\xi,\delta) &= -0.3 \leq -0.4 = \overline{L}\Lambda^{-}(\xi,\delta)
\end{aligned}
$$

Therefore $\xi\kappa, \kappa\epsilon$, and $\kappa\delta$ are strong edges but $\xi\delta$ is not a strong edge. The lower minimum dominating set in $\underline{\Gamma}$ is $D = \{\kappa, \xi\}$ and the upper minimum dominating set in $\overline{\Gamma}$ is $D = \{\kappa, \xi\}$. The fuzzy cardinality of $\mathbb{MDS}$ in $\underline{\Gamma}$ is:

$$\begin{aligned}\Omega_D(\underline{\Gamma}) =& \frac{1 + \underline{\Upsilon}W^+(\kappa) + \underline{\Upsilon}W^-(\kappa)}{2} + \frac{1 + \underline{\Upsilon}W^+(\xi) + \underline{\Upsilon}W^-(\xi)}{2} \\ =& \frac{1 + 0.2 + (-0.3)}{2} + \frac{1 + 0.3 + (-0.2)}{2} = 1\end{aligned}$$

The fuzzy cardinality of $\mathbb{MDS}$ in $\overline{\Gamma}$ is:

$$\begin{aligned}\Omega_D(\overline{\Gamma}) =& \frac{1 + \overline{\Upsilon}W^+(\kappa) + \overline{\Upsilon}W^-(\kappa)}{2} + \frac{1 + \overline{\Upsilon}W^+(\xi) + \overline{\Upsilon}W^-(\xi)}{2} \\ =& \frac{1 + 0.5 + (-0.3)}{2} + \frac{1 + 0.4 + (-0.5)}{2} = 1.05\end{aligned}$$

The domination number of BFRD is:

$$\Omega_D(\Gamma) = \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma}) = 1 + 1.05 = 2.05$$

Minimum dominating set Bounds. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD. Let $\epsilon_0\epsilon_1$ be a strong edge in Γ, $\forall \epsilon_1 \in \Gamma$ then the $\mathbb{MDS}$ is:

$$D = \{\epsilon_0\}$$

Therefore, a $\mathbb{MDS}$ has a minimum of 1 vertex.
A $\mathbb{MDS}$ is always a subset of a vertex set with n vertices X such that:

$$D \subseteq X$$

Therefore, a $\mathbb{MDS}$ has a maximum of n vertices.

Remark. Every $\mathbb{MDS}$ is a minimal dominating set in a BFRD.

Theorem 1. $\Omega(\Gamma = (\underline{\Gamma}, \overline{\Gamma})) \leq |V(\Gamma)|$ where $|V(\Gamma)| = |V(\underline{\Gamma})| + |V(\overline{\Gamma})|$ is the order of Bipolar Fuzzy Rough Digraphs.
Proof: By definition of order of BFRDs:

$$|V(\Gamma)| = |V(\underline{\Gamma})| + |V(\overline{\Gamma})|$$

Where

$$\begin{cases} |\mathcal{V}(\underline{\Gamma})| = \sum_{\epsilon_i \in \underline{\mathcal{V}}} \frac{1 + \underline{\Upsilon}W^+(\epsilon_i) + \underline{\Upsilon}W^-(\epsilon_i)}{2} \\ |\mathcal{V}(\overline{\Gamma})| = \sum_{\epsilon_i \in \overline{\mathcal{V}}} \frac{1 + \overline{\Upsilon}W^+(\epsilon_i) + \overline{\Upsilon}W^-(\epsilon_i)}{2} \end{cases}$$

such that $|\mathcal{V}(\underline{\Gamma})|, |\mathcal{V}(\overline{\Gamma})| > 0$. In BFRD the $\mathbb{MDS}$ $D(\Gamma)$ of $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is a subset $\mathcal{V}$ of $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$.
Case 1: If $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ has no strong edge then $D(\Gamma) = \mathcal{V}$:

$$\begin{aligned}
\Omega_D(\underline{\Gamma}) =& |\mathcal{V}(\underline{\Gamma})| = \sum_{\epsilon_i \in \underline{\mathcal{V}}} \frac{1 + \underline{\Upsilon} W^+(\epsilon_i) + \underline{\Upsilon} W^-(\epsilon_i)}{2} \\
\Omega_D(\overline{\Gamma}) =& |\mathcal{V}(\overline{\Gamma})| = \sum_{\epsilon_i \in \overline{\mathcal{V}}} \frac{1 + \overline{\Upsilon} W^+(\epsilon_i) + \overline{\Upsilon} W^-(\epsilon_i)}{2} \\
\Omega_D(\Gamma) =& \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma}) \\
\Omega_D(\Gamma) =& |\mathcal{V}(\underline{\Gamma})| + |\mathcal{V}(\overline{\Gamma})| \\
\Omega_D(\Gamma) =& |\mathcal{V}(\Gamma)|
\end{aligned}$$

Case 2: If there is at least one strong edge in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ which is strong edge in $\underline{\Gamma}$ and $\overline{\Gamma}$ then $D(\Gamma) \subset \mathcal{V}$:

$$\begin{aligned}
\Omega_D(\underline{\Gamma}) <& |V(\underline{\Gamma})| \\
\Omega_D(\overline{\Gamma}) <& |V(\overline{\Gamma})| \\
\Omega_D(\Gamma) =& \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma}) \\
\Omega_D(\Gamma) <& |V(\underline{\Gamma})| + |V(\overline{\Gamma})| \\
\Omega_D(\Gamma) <& |V(\Gamma)|
\end{aligned}$$

Hence $\Omega(\Gamma = (\underline{\Gamma}, \overline{\Gamma})) \leq |V(\Gamma)|$ is proved.

Definition 20. The fuzzy edge cardinality of set of edges $\underline{\mathcal{E}}$ in $\underline{\Gamma}$ is defined as:

$$|\mathcal{E}(\underline{\Gamma})| = \sum_{\epsilon_i \epsilon_j \in \underline{\mathcal{E}}} \frac{1 + \underline{L}\Lambda^+(\epsilon_i \epsilon_j) + \underline{L}\Lambda^-(\epsilon_i \epsilon_j)}{2}$$

The fuzzy cardinality of set of edges $\overline{\mathcal{E}}$ in $\overline{\Gamma}$ is defined as:

$$|\mathcal{E}(\overline{\Gamma})| = \sum_{\epsilon_i \epsilon_j \in \overline{\mathcal{E}}} \frac{1 + \overline{L}\Lambda^+(\epsilon_i \epsilon_j) + \overline{L}\Lambda^-(\epsilon_i \epsilon_j)}{2}$$

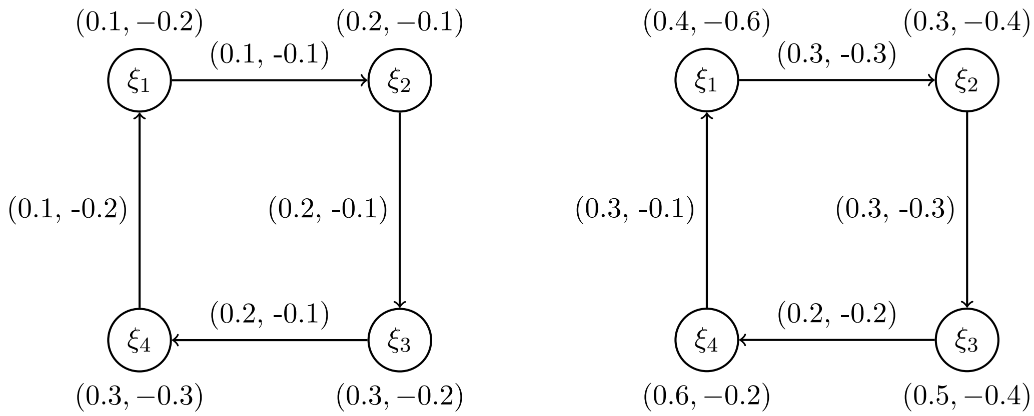
Theorem 2. Let $\Gamma_1 = (\underline{\Gamma}_1, \overline{\Gamma}_1)$ and $\Gamma_2 = (\underline{\Gamma}_2, \overline{\Gamma}_2)$ be two Bipolar Fuzzy Rough Digraphs such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are set of vertices of Γ_1 and Γ_2 respectively then $\Omega(\Gamma_1 \cup \Gamma_2) = \Omega(\Gamma_1) + \Omega(\Gamma_2)$.
Proof: Let D_1 be the $\mathbb{MDS}$ of Γ_1 and D_2 be the $\mathbb{MDS}$ of Γ_2 such that $\Omega_{D_1}(\Gamma_1)$ is domination number of Γ_1 and $\Omega_{D_2}(\Gamma_2)$ is domination number of Γ_2. Since $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, therefore $D_1 + D_2$ is the $\mathbb{MDS}$ of $\Gamma_1 \cup \Gamma_2$ such that $\Omega_{D_1+D_2}(\Gamma_1 \cup \Gamma_2)$ is domination number of $\Gamma_1 \cup \Gamma_2$, then:

$$\Omega_{D_1+D_2}(\Gamma_1 \cup \Gamma_2) = \Omega_{D_1}(\Gamma_1) + \Omega_{D_2}(\Gamma_2)$$

It follows that:

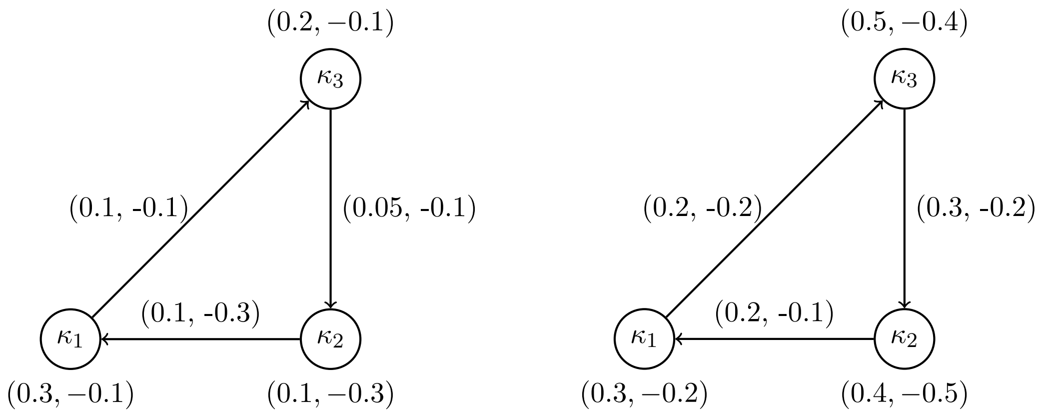
$$\Omega(\Gamma_1 \cup \Gamma_2) = \Omega(\Gamma_1) + \Omega(\Gamma_2)$$

Example 5. Let $\Gamma_1 = (\underline{\Gamma}_1, \overline{\Gamma}_1)$ and $\Gamma_2 = (\underline{\Gamma}_2, \overline{\Gamma}_2)$ be two BFRDs given in Figures 4 and 5:



$\underline{\Gamma_1} = (\underline{\Upsilon} W_1, \underline{L}\Lambda_1)$ $\overline{\Gamma_1} = (\overline{\Upsilon} W_1, \overline{L}\Lambda_1)$

Figure 4: $\overline{\Gamma} = (\underline{\Gamma}, \overline{\Gamma})$



$\underline{\Gamma_2} = (\underline{\Upsilon} W_2, \underline{L}\Lambda_2)$ $\overline{\Gamma_2} = (\overline{\Upsilon} W_2, \overline{L}\Lambda_2)$

Figure 5: $\overline{\Gamma} = (\underline{\Gamma}, \overline{\Gamma})$

Then the lower and upper approximations of union of graphs $\Gamma_1 \cup \Gamma_2$ are given in Figure 6 and 7:

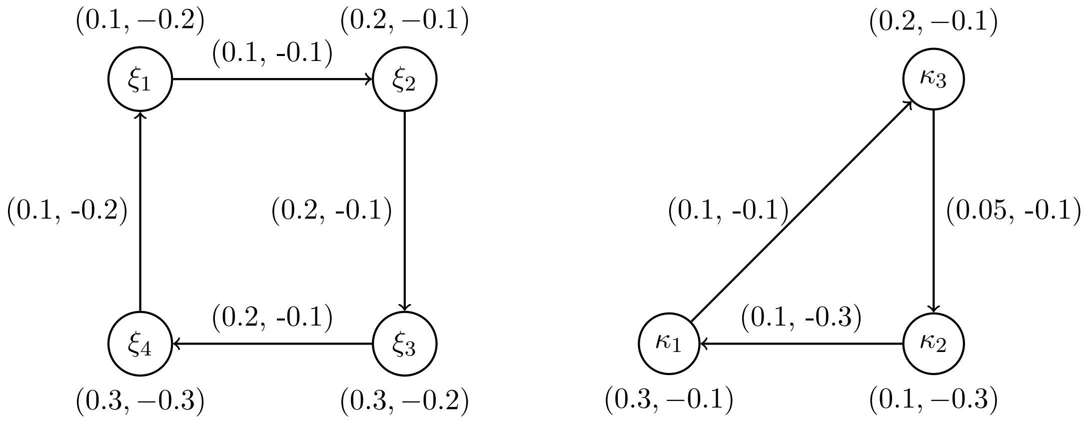


Figure 6: $\underline{\Gamma_1} \cup \underline{\Gamma_2}$

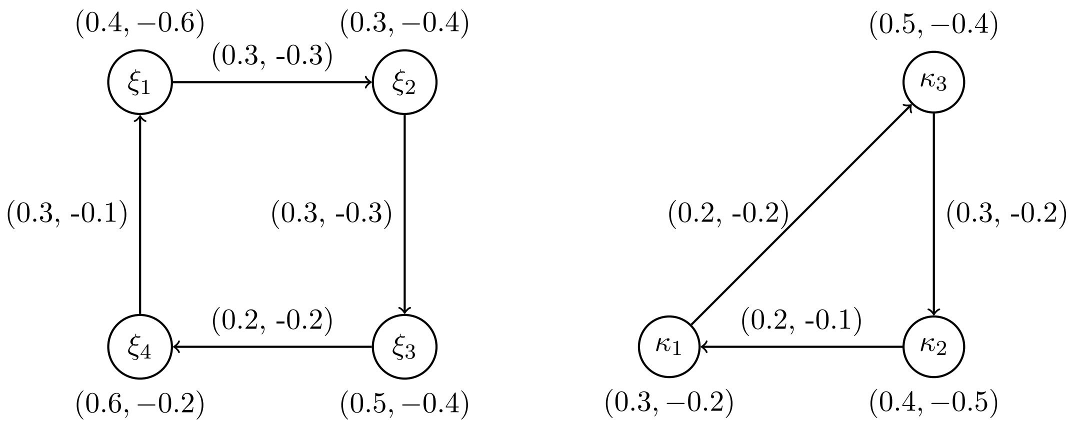


Figure 7: $\overline{\Gamma_1} \cup \overline{\Gamma_2}$

The minimal dominating sets of Γ_1 are $D_1 = \{\xi_2, \xi_4\}$ and $D_2 = \{\xi_1, \xi_3\}$. The sum of fuzzy cardinalities of $D_1 = \{\xi_2, \xi_4\}$ in $\underline{\Gamma}_1$ and $\overline{\Gamma}_1$ is:

$$\Omega_{D_1}(\underline{\Gamma}_1) + \Omega_{D_1}(\overline{\Gamma}_1) = \frac{1 + 0.2 + (-0.1)}{2} + \frac{1 + 0.3 + (-0.3)}{2} + \frac{1 + 0.3 + (-0.4)}{2} + \frac{1 + 0.6 + (-0.2)}{2}$$

$$\Omega_{D_1}(\underline{\Gamma}_1) + \Omega_{D_1}(\overline{\Gamma}_1) = 2.2$$

The sum of fuzzy cardinalities of $D_2 = \{\xi_1, \xi_3\}$ in $\underline{\Gamma}_1$ and $\overline{\Gamma}_1$ is:

$$\Omega_{D_2}(\underline{\Gamma}_1) + \Omega_{D_1}(\overline{\Gamma}_1) = \frac{1 + 0.1 + (-0.2)}{2} + \frac{1 + 0.3 + (-0.2)}{2} + \frac{1 + 0.4 + (-0.6)}{2} + \frac{1 + 0.5 + (-0.4)}{2}$$

$$\Omega_{D_2}(\underline{\Gamma}_1) + \Omega_{D_1}(\overline{\Gamma}_1) = 1.95$$

$D_2 = \{\xi_1, \xi_3\}$ has minimum sum of fuzzy cardinalities, therefore $\{\xi_1, \xi_3\}$ is a $\mathbb{MDS}$ and the domination number is $\Omega(\Gamma_1) = 1.95$.

The minimal dominating sets are $D_1 = \{\kappa_1, \kappa_2\}, D_2 = \{\kappa_1, \kappa_3\}$, and $D_3 = \{\kappa_2, \kappa_3\}$. The sum of fuzzy cardinalities of $D_1 = \{\kappa_1, \kappa_2\}$ in $\underline{\Gamma}_2$ and $\overline{\Gamma}_2$ is:

$$\begin{aligned}\Omega_{D_1}(\underline{\Gamma}_2) + \Omega_{D_2}(\overline{\Gamma}_2) =& \frac{1+0.3+(-0.1)}{2} + \frac{1+0.1+(-0.3)}{2} + \frac{1+0.3+(-0.2)}{2} + \frac{1+0.4+(-0.5)}{2} \\ \Omega_{D_1}(\underline{\Gamma}_2) + \Omega_{D_1}(\overline{\Gamma}_2) =& 2\end{aligned}$$

The sum of fuzzy cardinalities of $D_2 = \{\kappa_1, \kappa_3\}$ in $\underline{\Gamma}_2$ and $\overline{\Gamma}_2$ is:

$$\begin{aligned}\Omega_{D_2}(\underline{\Gamma}_2) + \Omega_{D_2}(\overline{\Gamma}_2) =& \frac{1+0.3+(-0.1)}{2} + \frac{1+0.2+(-0.1)}{2} + \frac{1+0.3+(-0.2)}{2} + \frac{1+0.5+(-0.4)}{2} \\ \Omega_{D_2}(\underline{\Gamma}_2) + \Omega_{D_2}(\overline{\Gamma}_2) =& 2.25\end{aligned}$$

The sum of fuzzy cardinalities of $D_3 = \{\kappa_2, \kappa_3\}$ in $\underline{\Gamma}_2$ and $\overline{\Gamma}_2$ is:

$$\begin{aligned}\Omega_{D_3}(\underline{\Gamma}_2) + \Omega_{D_3}(\overline{\Gamma}_2) =& \frac{1+0.2+(-0.1)}{2} + \frac{1+0.1+(-0.3)}{2} + \frac{1+0.5+(-0.4)}{2} + \frac{1+0.4+(-0.5)}{2} \\ \Omega_{D_3}(\underline{\Gamma}_2) + \Omega_{D_3}(\overline{\Gamma}_2) =& 1.95\end{aligned}$$

$D_3 = \{\kappa_2, \kappa_3\}$ has minimum sum of fuzzy cardinalities, therefore $\{\kappa_2, \kappa_3\}$ is a $\mathbb{MDS}$ and the domination number is $\Omega(\Gamma_2) = 1.95$. By using the same procedure, the minimal dominating set of $\Gamma_1 \cup \Gamma_2$ is $\{\xi_1, \xi_3, \kappa_2, \kappa_3\}$ and the domination number is $\Omega(\Gamma_1 \cup \Gamma_2) = 3.9$. Notice that $\Omega(\Gamma_1 \cup \Gamma_2) = \Omega(\Gamma_1) + \Omega(\Gamma_2)$.

Definition 21. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD, the degree of a vertex $\epsilon_0 \in \mathcal{V}$ in Γ is defined as:

$$\begin{aligned}d_{\underline{\Gamma}}(\epsilon_0) =& (d^+_{\underline{\Gamma}}(\epsilon_0), d^-_{\underline{\Gamma}}(\epsilon_0)) \\ =& (((\sum_{\epsilon_0 \neq \epsilon_1} \underline{L}\Lambda^+(\epsilon_0, \epsilon_1) + \sum_{\epsilon_0 \neq \epsilon_1} \underline{L}\Lambda^+(\epsilon_1, \epsilon_0) + \sum_{\epsilon_0 = \epsilon_1} \underline{L}\Lambda^+(\epsilon_0, \epsilon_1)), \\ & (\sum_{\epsilon_0 \neq \epsilon_1} \underline{L}\Lambda^-(\epsilon_0, \epsilon_1) + \sum_{\epsilon_0 \neq \epsilon_1} \underline{L}\Lambda^-(\epsilon_1, \epsilon_0)) + \sum_{\epsilon_0 = \epsilon_1} \underline{L}\Lambda^-(\epsilon_1, \epsilon_0))) \\ d_{\overline{\Gamma}}(\epsilon_0) =& (d^+_{\overline{\Gamma}}(\epsilon_0), d^-_{\overline{\Gamma}}(\epsilon_0)) \\ =& (((\sum_{\epsilon_0 \neq \epsilon_1} \overline{L}\Lambda^+(\epsilon_0, \epsilon_1) + \sum_{\epsilon_0 \neq \epsilon_1} \overline{L}\Lambda^+(\epsilon_1, \epsilon_0) + \sum_{\epsilon_0 = \epsilon_1} \overline{L}\Lambda^+(\epsilon_0, \epsilon_1)), \\ & (\sum_{\epsilon_0 \neq \epsilon_1} \overline{L}\Lambda^-(\epsilon_0, \epsilon_1) + \sum_{\epsilon_0 \neq \epsilon_1} \overline{L}\Lambda^-(\epsilon_1, \epsilon_0)) + \sum_{\epsilon_0 = \epsilon_1} \overline{L}\Lambda^-(\epsilon_1, \epsilon_0)))\end{aligned}$$

such that:

$$d_\Gamma(\epsilon_0) = d_{\underline{\Gamma}}(\epsilon_0) + d_{\overline{\Gamma}}(\epsilon_0) = (d^+_{\underline{\Gamma}}(\epsilon_0) + d^+_{\overline{\Gamma}}(\epsilon_0), d^-_{\underline{\Gamma}}(\epsilon_0) + d^-_{\overline{\Gamma}}(\epsilon_0))$$

Example 6. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD defined in Figure 1 then the degree of ϵ is given by:

$$\begin{aligned}
d_{\underline{\Gamma}}(\epsilon) =&(d^{+}_{\underline{\Gamma}}(\epsilon), d^{-}_{\underline{\Gamma}}(\epsilon))\\
=&\big((\underline{L}\Lambda^{+}(\epsilon,\epsilon) + \underline{L}\Lambda^{+}(\epsilon,\eta) + \underline{L}\Lambda^{+}(\epsilon,\xi)), (\underline{L}\Lambda^{-}(\epsilon,\epsilon) + \underline{L}\Lambda^{-}(\epsilon,\eta) + \underline{L}\Lambda^{-}(\epsilon,\xi))\big)\\
=&(0.4, -0.5)\\
d_{\overline{\Gamma}}(\epsilon) =&(d^{+}_{\overline{\Gamma}}(\epsilon), d^{-}_{\overline{\Gamma}}(\epsilon))\\
=&\big((\overline{L}\Lambda^{+}(\epsilon,\epsilon) + \overline{L}\Lambda^{+}(\epsilon,\eta) + \overline{L}\Lambda^{+}(\epsilon,\xi)), (\overline{L}\Lambda^{-}(\epsilon,\epsilon) + \overline{L}\Lambda^{-}(\epsilon,\eta) + \overline{L}\Lambda^{-}(\epsilon,\xi))\big)\\
=&(0.5, -0.5)\\
d_{\Gamma}(\epsilon) =&(d^{+}_{\underline{\Gamma}}(\epsilon) + d^{+}_{\overline{\Gamma}}(\epsilon), d^{-}_{\underline{\Gamma}}(\epsilon) + d^{-}_{\overline{\Gamma}}(\epsilon))\\
=&(0.9, -1.0)
\end{aligned}$$

Definition 22. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD, the total degree of a vertex $\epsilon_0 \in \mathcal{V}$ in $\underline{\Gamma}$ is defined as:

$$\begin{aligned}
td_{\underline{\Gamma}}(\epsilon_0) =&(td^{+}_{\underline{\Gamma}}(\epsilon_0), td^{-}_{\underline{\Gamma}}(\epsilon_0))\\
td^{+}_{\underline{\Gamma}}(\epsilon_0) =&d^{+}_{\underline{\Gamma}}(\epsilon_0) + \underline{\Upsilon}W^{+}(\epsilon_0)\\
td^{-}_{\underline{\Gamma}}(\epsilon_0) =&d^{-}_{\underline{\Gamma}}(\epsilon_0) + \underline{\Upsilon}W^{-}(\epsilon_0)\\
td_{\overline{\Gamma}}(\epsilon_0) =&(td^{+}_{\overline{\Gamma}}(\epsilon_0), td^{-}_{\overline{\Gamma}}(\epsilon_0))\\
td^{+}_{\overline{\Gamma}}(\epsilon_0) =&d^{+}_{\overline{\Gamma}}(\epsilon_0) + \overline{\Upsilon}W^{+}(\epsilon_0)\\
td^{-}_{\overline{\Gamma}}(\epsilon_0) =&d^{-}_{\overline{\Gamma}}(\epsilon_0) + \overline{\Upsilon}W^{-}(\epsilon_0)
\end{aligned}$$

such that:

$$td_{\Gamma}(\epsilon_0) = td_{\underline{\Gamma}}(\epsilon_0) + td_{\overline{\Gamma}}(\epsilon_0) = (td^{+}_{\underline{\Gamma}}(\epsilon_0) + td^{+}_{\overline{\Gamma}}(\epsilon_0), td^{-}_{\underline{\Gamma}}(\epsilon_0) + td^{-}_{\overline{\Gamma}}(\epsilon_0))$$

Example 7. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD on universal set $X = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ defined below in Figure 8:

The total degree of vertex ξ_3 is given by:

$$\begin{aligned}
td^{+}_{\underline{\Gamma}}(\xi_3) =&0.1 + 0.1 + 0.1 + 0.2 = 0.5\\
td^{-}_{\underline{\Gamma}}(\xi_3) =&-0.1 + (-0.1) + (-0.1) + (-0.2) = -0.5\\
td^{+}_{\overline{\Gamma}}(\xi_3) =&0.2 + 0.2 + 0.1 + 0.3 = 0.8\\
td^{-}_{\overline{\Gamma}}(\xi_3) =&-0.2 + (-0.1) + (-0.2) + (-0.4) = -0.9
\end{aligned}$$

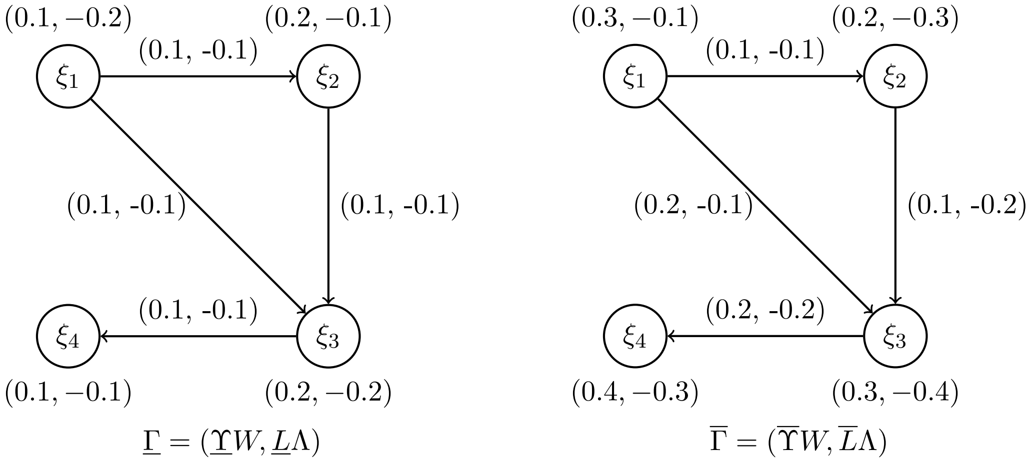


Figure 8: $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$

$$td_\Gamma(\xi_3) = (0.5 + 0.8, -0.5 + (-0.9)) = (1.3, -1.4)$$

Definition 23. A BFRD $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is said to be Regular if $d_\Gamma(\epsilon_0) = (\alpha, \beta) \ \ \forall \epsilon_0 \in \mathcal{V}$, where α, β are real numbers.

Definition 24. A BFRD $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is said to be Totally Regular if $td_\Gamma(\epsilon_0) = (\alpha, \beta) \ \ \forall \epsilon_0 \in \mathcal{V}$, where α, β are real numbers.

Example 8. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a BFRD on universal set $X = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ defined in Figure 9:

Notice that the graph Γ is both regular and totally Regular Bipolar Fuzzy Rough Digraph.

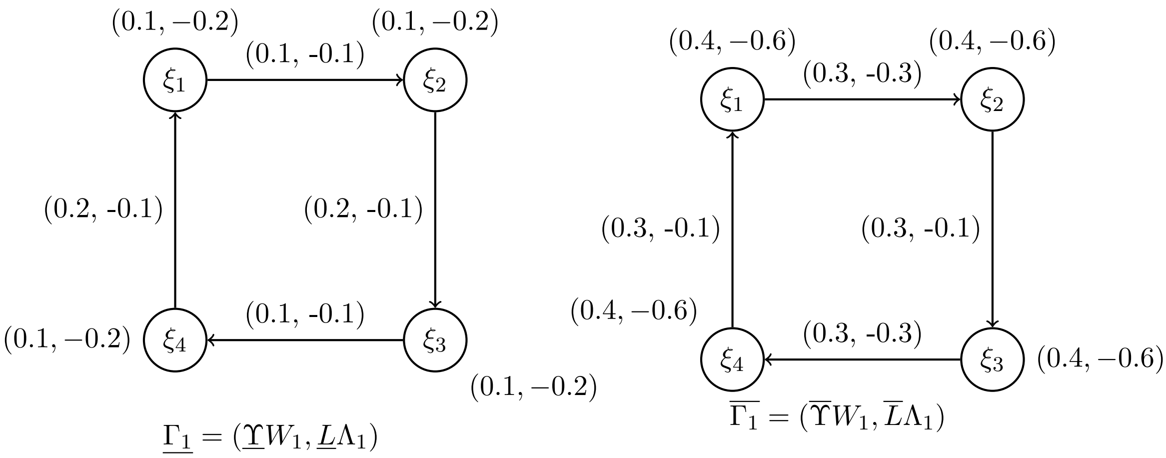


Figure 9: $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$

Theorem 3. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a Bipolar Fuzzy Rough Digraphs and $\Upsilon W = (\Upsilon W^+, \Upsilon W^-)$ be a constant function in $\underline{\Gamma}$ and $\overline{\Gamma}$ then Γ is Regular iff it is Totally Regular bipolar fuzzy rough digraph.
Proof. Let $\Upsilon W = (\Upsilon W^+, \Upsilon W^-)$ be a constant function such that $\Upsilon W^+(\epsilon_0) = \rho_1$ and $\Upsilon W^-(\epsilon_0) = \rho_2, \quad \forall \epsilon_0 \in \mathcal{V}$. Consider Γ be a Regular Bipolar Fuzzy Rough Digraph then:

$$d_\Gamma^+(\epsilon_0) = \alpha, \quad d_\Gamma^-(\epsilon_0) = \beta \;\; \forall \epsilon_0 \in \mathcal{V}$$

$$\begin{aligned} td_\Gamma^+(\epsilon_0) =& d_\Gamma^+(\epsilon_0) + \Upsilon W^+(\epsilon_0), \quad td_\Gamma^-(\epsilon_0) = d_\Gamma^-(\epsilon_0) + \Upsilon W^-(\epsilon_0) \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies td_\Gamma^+(\epsilon_0) =& \alpha + \rho_1, \quad td_\Gamma^-(\epsilon_0) = \beta + \;\rho_2 \;\; \forall \epsilon_0 \in \mathcal{V} \end{aligned}$$

Hence Γ is a Totally Regular Bipolar Fuzzy Rough Digraph.
Conversely, suppose that Γ is a Totally Regular Bipolar Fuzzy Rough Digraph such that:

$$\begin{aligned} td_\Gamma^+(\epsilon_0) =& \mu_1, \quad td_\Gamma^-(\epsilon_0) = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & d_\Gamma^+(\epsilon_0) + \Upsilon W^+(\epsilon_0) = \mu_1, \quad d_\Gamma^-(\epsilon_0) + \Upsilon W^-(\epsilon_0) = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & d_\Gamma^+(\epsilon_0) + \rho_1 = \mu_1, \quad d_\Gamma^-(\epsilon_0) + \rho_2 = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & d_\Gamma^+(\epsilon_0) = \mu_1 - \rho_1, \quad d_\Gamma^-(\epsilon_0) = \mu_2 - \rho_2 \;\; \forall \epsilon_0 \in \mathcal{V} \end{aligned}$$

Hence Γ is a Regular Bipolar Fuzzy Rough Digraph.

Theorem 4. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a Regular as well as Totally Regular Bipolar Fuzzy Rough Digraph then $\Upsilon W = (\Upsilon W^+, \Upsilon W^-)$ is a constant function in $\underline{\Gamma}$ and $\overline{\Gamma}$.
Proof. Let $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ be a Regular and Totally Regular Bipolar Fuzzy Rough Digraph such that:

$$\begin{aligned} d_\Gamma^+(\epsilon_0) =& \alpha, \quad d_\Gamma^-(\epsilon_0) = \beta \;\; \forall \epsilon_0 \in \mathcal{V} \\ td_\Gamma^+(\epsilon_0) =& \mu_1, \quad td_\Gamma^-(\epsilon_0) = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & d_\Gamma^+(\epsilon_0) + \Upsilon W^+(\epsilon_0) = \mu_1, \quad d_\Gamma^-(\epsilon_0) + \Upsilon W^-(\epsilon_0) = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & \alpha + \Upsilon W^+(\epsilon_0) = \mu_1, \quad \beta + \Upsilon W^-(\epsilon_0) = \mu_2 \;\; \forall \epsilon_0 \in \mathcal{V} \\ \implies & \Upsilon W^+(\epsilon_0) = \mu_1 - \alpha, \quad \Upsilon W^-(\epsilon_0) = \mu_2 - \beta \;\; \forall \epsilon_0 \in \mathcal{V} \end{aligned}$$

Hence $\Upsilon W = (\Upsilon W^+, \Upsilon W^-)$ is a constant function.

4. Applications

4.1. Selecting a minimum set of rural areas to set up medicine market

Rural areas often face greater challenges in accessing medical facilities due to a combination of geographical, infrastructural, economic, and demographic factors. As a result, residents in rural areas may experience difficulties in obtaining essential medications needed to manage acute and chronic health conditions. Lack of access to medicines can lead to delayed treatment, exacerbation of illnesses, and preventable health complications. Moreover, rural populations are often disproportionately affected by certain health issues, such as chronic diseases, infectious diseases, and maternal and child health concerns. Providing access to medicines in rural areas helps to improve health outcomes, reduce morbidity and mortality rates, and enhance the overall quality of life for rural residents. Additionally, ensuring medication supply in rural communities contributes to economic development, as healthy individuals are better able to participate in the workforce and contribute to local economies. Consider a set of rural areas $X = \{R_1, R_2, R_3, R_4, R_5, R_6\}$. A medicine company wants to set up a supply market in a minimum number of rural areas such that medicines can be supplied to each rural area of the set to meet the needs of each rural area. A vertex's positive membership value in the graph indicates how much the rural area has the supply of medicines in the set of rural areas and the negative membership value represents how much the rural area needs the supply of medicines in the set of rural areas. Every directed edge depicts the transport of medications between rural areas. The directed edge's positive membership value from one rural area to another represents the positive supply relationship from one rural area to the other. The directed edge's negative membership value from one rural area to another represents a certain level of uncertainty or ambiguity in the supply relationship due to factors like transportation challenges, stock availability, and demand fluctuations. Let $\Upsilon = (\Upsilon^+, \Upsilon^-)$ be a BFTR on X defined in the Tables 6 and 7.

Υ^+	R_1	R_2	R_3	R_4	R_5	R_6
R_1	1	0.2	0.3	0.5	0.1	0.4
R_2	0.2	1	0.1	0.2	0.3	0.5
R_3	0.3	0.1	1	0.3	0.4	0.2
R_4	0.5	0.2	0.3	1	0.1	0.2
R_5	0.1	0.3	0.4	0.1	1	0.5
R_6	0.4	0.5	0.2	0.2	0.5	1

Table 6: Υ^+ of relation Υ

Υ^-	R_1	R_2	R_3	R_4	R_5	R_6
R_1	-1	-0.1	-0.2	-0.4	-0.3	-0.25
R_2	-0.1	-1	-0.3	-0.2	-0.5	-0.4
R_3	-0.2	-0.3	-1	-0.3	-0.1	-0.6
R_4	-0.4	-0.2	-0.3	-1	-0.35	-0.45
R_5	-0.3	-0.5	-0.1	-0.35	-1	-0.2
R_6	-0.25	-0.4	-0.6	-0.45	-0.2	-1

Table 7: Υ^- of relation Υ

Let $W = \{(R_1, 0.2, -0.1), (R_2, 0.3, -0.3), (R_3, 0.1, -0.3), (R_4, 0.4, -0.2), (R_5, 0.3, -0.5), (R_6, 0.4, -0.4)\}$ be a BFS on X. The lower approximation of W with respect to Υ is:

$$\underline{\Upsilon}W = \{(R_1, 0.2, -0.1), (R_2, 0.3, -0.3), (R_3, 0.1, -0.3), (R_4, 0.4, -0.2), (R_5, 0.3, -0.5), (R_6, 0.4, -0.4)\}$$

The upper approximation of W with respect to Υ is:

$$\overline{\Upsilon}W = \{(R_1, 0.4, -0.3), (R_2, 0.4, -0.5), (R_3, 0.3, -0.4), (R_4, 0.4, -0.4), (R_5, 0.4, -0.5), (R_6, 0.4, -0.4)\}$$

Let $L = (L^+, L^-)$ be a BFTR on $A \subseteq X \times X$ and defined by the Tables 8 and 9:

L^+	R_1R_6	R_2R_1	R_2R_5	R_3R_4	R_5R_6	R_2R_6	R_5R_1
R_1R_6	1	0.2	0.2	0.1	0.1	0.2	0.05
R_2R_1	0.2	1	0.1	0.075	0.3	0.4	0.3
R_2R_5	0.2	0.1	1	0.1	0.3	0.5	0.1
R_3R_4	0.1	0.075	0.1	1	0.2	0.1	0.05
R_5R_6	0.1	0.3	0.3	0.2	1	0.1	0.4
R_2R_6	0.2	0.4	0.5	0.1	0.1	1	0.3
R_5R_1	0.05	0.3	0.1	0.05	0.4	0.3	1

Table 8: L^+ of relation L

L^-	R_1R_6	R_2R_1	R_2R_5	R_3R_4	R_5R_6	R_2R_6	R_5R_1
R_1R_6	-1	-0.1	-0.05	-0.2	-0.3	-0.1	-0.25
R_2R_1	-0.1	-1	-0.3	-0.2	-0.25	-0.1	-0.5
R_2R_5	-0.05	-0.3	-1	-0.3	-0.2	-0.1	-0.3
R_3R_4	-0.2	-0.2	-0.3	-1	-0.1	-0.3	-0.05
R_5R_6	-0.3	-0.25	-0.2	-0.1	-1	-0.5	-0.25
R_2R_6	-0.1	-0.1	-0.1	-0.3	-0.5	-1	-0.1
R_5R_1	-0.25	-0.5	-0.3	-0.05	-0.25	-0.1	-1

Table 9: L^+ of relation L

Let $\Lambda = (\Lambda^+, \Lambda^-)$ be a BFS on A defined by:

$$\Lambda = \{(R_1R_6, 0.2. -0.1), (R_2R_1, 0.1, -0.1), (R_2R_5, 0.3, -0.3), (R_3R_4, 0.1, -0.2),\\ (R_5R_6, 0.1, -0.2), (R_2R_6, 0.3, -0.2), (R_5R_1, 0.2, -0.1)\}$$

The set of lower approximations of Λ is:

$$\underline{L}\Lambda = \{(R_1R_6, 0.2, -0.1), (R_2R_1, 0.1, -0.1), (R_2R_5, 0.3, -0.3), (R_3R_4, 0.1, -0.2),\\ (R_5R_6, 0.1, -0.2), (R_2R_6, 0.3, -0.2), (R_5R_1, 0.2, -0.1)\}$$

The set of upper approximation of Λ is:

$$\overline{L}\Lambda = \{(R_1R_6, 0.2, -0.2), (R_2R_1, 0.3, -0.3), (R_2R_5, 0.3, -0.3), (R_3R_4, 0.1, -0.3),\\ (R_5R_6, 0.3, -0.2), (R_2R_6, 0.3, -0.2), (R_5R_1, 0.3, -0.3)\}$$

Using this information, the lower and upper approximations of the BFRDs are shown in Figures 10 and 11 respectively.

Notice that R_5R_6 is not a strong arc in $\underline{\Gamma}$ but it is a strong arc in $\overline{\Gamma}$. Therefore, all the arcs except R_5R_6 are strong in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$. The strong arcs show that a maximum amount of medicines can be supplied between two rural areas most cost-effectively.

The upper and lower minimum dominating set is $D = \{R_2, R_3\}$, therefore the $\mathbb{MDS}$ of

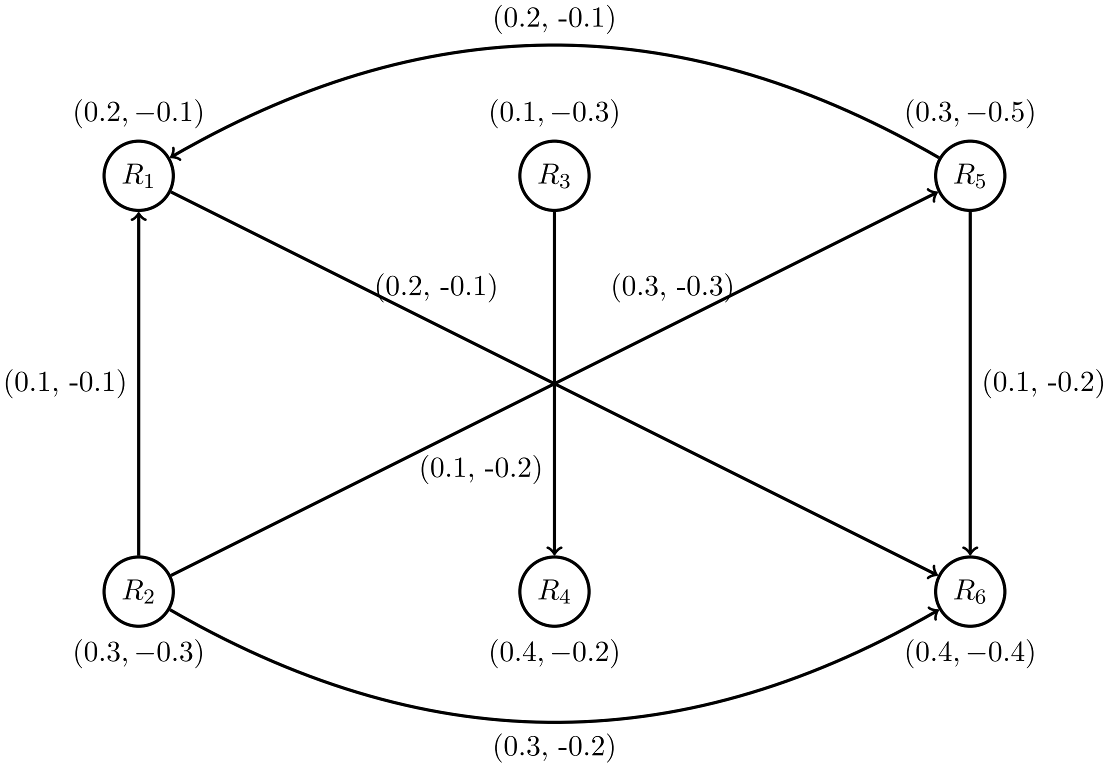


Figure 10: $\underline{\Gamma} = (\underline{\Upsilon}W, \underline{L}\Lambda)$

$\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ is $D = \{R_2, R_3\}$. The lower domination number is:

$$\begin{aligned}\Omega_D(\underline{\Gamma}) =& \frac{1 + (0.3) + (-0.3)}{2} + \frac{1 + (0.1) + (-0.3)}{2} \\ =& 0.9\end{aligned}$$

The upper domination number is:

$$\begin{aligned}\Omega_D(\overline{\Gamma}) =& \frac{1 + (0.4) + (-0.5)}{2} + \frac{1 + (0.1) + (-0.3)}{2} \\ =& 0.85\end{aligned}$$

The domination number is:

$$\Omega_D(\Gamma) = \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma}) = 0.9 + 0.85 = 1.75$$

The $\mathbb{MDS}$ is the optimum set of the minimum number of rural areas that can supply the medicines to the other rural areas in the most cost-effective manner. The domination number of BFRD shows that the net cost to supply medicines to the rural areas of the

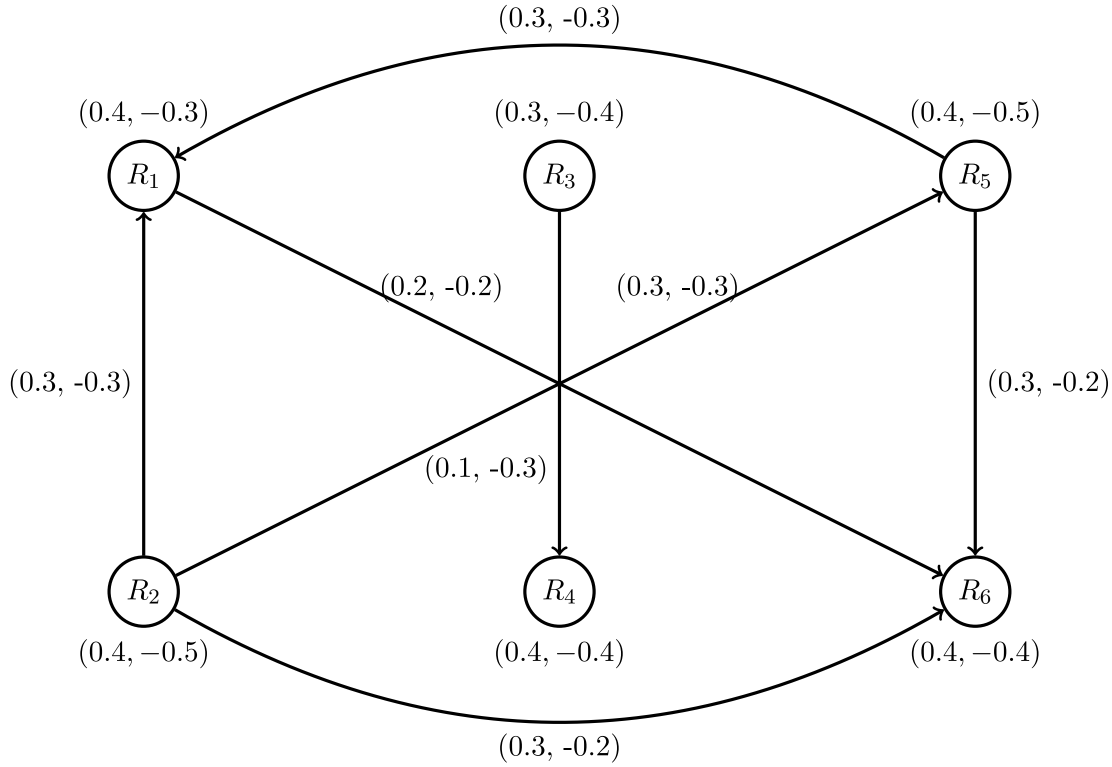


Figure 11: $\overline{\Gamma} = (\overline{\Upsilon}W, \overline{L}\Lambda)$

$\mathbb{MDS}$.

Our analysis of the rural medicine supply network yielded a clear and actionable strategic plan. The algorithm identified the $\mathbb{MDS}$ as $D = \{R_2, R_3\}$ with an overall domination number of 1.75. It means that he selection of $\{R_2, R_3\}$ as the optimal set of supply hubs is not arbitrary; it is a direct result of their strategic position within the Bipolar Fuzzy Rough Digraph. These two locations were identified as the strongest positive dominators, meaning they possess the most effective combination of high supply capacity which means they have a strong positive capacity to serve other areas and their connections to other areas are characterized by high positive weights (representing efficiency) and low negative weights (representing minimal uncertainty or risk).

In practical terms, R_2 might be located at a major highway intersection, ensuring reliable transport, while R_3 might have superior storage infrastructure. Our model prioritizes these locations over others that might be more geographically central but have less reliable supply lines.

By identifying a minimum dominating set of just two locations, the company can avoid the significant expense of establishing depots in three, four, or more areas. This centralization drastically reduces costs related to infrastructure, staffing, inventory management, and bulk transportation, thereby maximizing the return on investment.

In summary, the results of our BFRD model go beyond simply identifying important nodes; they provide a multi-faceted, evidence-based strategy for optimizing a supply chain, managing risk, and minimizing operational costs. The algorithm of the proposed method is given in Table 10.

4.2. A comparison of Bipolar Fuzzy Rough Digraph and Fuzzy Rough Digraph

To clarify the incremental contribution of our work, this section provides a direct comparison between the analytical process of existing Fuzzy Rough Domination (FRD) models and our proposed Bipolar Fuzzy Rough Domination (BFRD) methodology.

4.2.1. The Process and Limitations of Existing Fuzzy Rough Digraph Domination

Existing models for domination in Fuzzy Rough Digraphs follow a single-channel process. They model a network using one fuzzy value per relationship to represent its strength. The analysis then computes a single lower and upper approximation of the graph to identify a minimum dominating set.
The critical limitation of this process is the ambiguity of its output. While it can identify a set of influential nodes, it provides no information about the nature of that influence. A node identified as "dominant" could be a positive leader or a negative bottleneck, but the model cannot distinguish between the two, leaving decision-makers with an incomplete picture.

4.2.2. The Incremental Contribution of the Proposed BFRD Algorithm

Our proposed methodology, detailed in the algorithm Table 10, introduces a dual-channel analytical process that provides a far richer and more actionable output. The incremental contribution is embedded in the specific steps of our algorithm:

(i) Unlike FRD models, our algorithm begins by capturing both positive and negative influences explicitly, using Bipolar Fuzzy sets for vertices W and edges Λ (Steps 2-6). This establishes a two-dimensional foundation for the entire analysis from the outset.

(ii) The core novelty lies in the computation of separate lower $(\underline{\Upsilon}W, \underline{L}\Lambda)$ and upper $(\overline{\Upsilon}W, \overline{L}\Lambda)$ approximations for both the positive and negative relationships (Steps 7-8). This constructs a multi-layered Bipolar Fuzzy Rough Digraph $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$ that preserves the distinction between positive and negative ties, rather than collapsing them into a single measure of strength.

(iii) Crucially, the domination analysis is performed on these distinct layers. Our algorithm finds minimal dominating sets independently in the lower approximation $\underline{\Gamma}$ (representing certain influence) and the upper approximation $\overline{\Gamma}$ (representing potential influence) (Steps 11-13). This multi-step process allows us to identify nodes based on the specific nature and certainty of their influence—a level of granularity impossible with existing methods.

(iv) The final minimum dominating set (D_m) and domination number (Ω_D) (Steps 14-16) are derived from this richer, multi-layered analysis. The final interpretation (Step 17) is therefore not just an identification of "who is influential," but a precise, evidence-based recommendation on which nodes form the optimal positive-influence core of the network.

To understand the practical difference, consider a scenario of evaluating a project team's performance. Team dynamics are complex, influenced by positive factors like collaboration and expertise, as well as negative factors like conflict and poor communication.
A standard Fuzzy Rough Digraph (FRD) can capture the strength of the relationships. For example, it might assign the connection between Team Member A and Team Member B a high value of 0.8, indicating a strong bond. However, the FRD cannot distinguish the nature of this bond. Is it a strong, positive collaboration, or a tense, negative rivalry that forces them to interact? The model's inability to handle this bipolarity leaves the analysis incomplete.
This is where the Bipolar Fuzzy Rough Digraph (BFRD) provides a far more insightful analysis. The BFRD can represent the same relationship using two distinct values:

- A positive membership of 0.8 to reflect their strong collaboration.
- A negative membership of -0.3 to account for some minor communication issues.

This dual representation offers a much clearer and more realistic understanding of the team's dynamics, capturing both the supportive and conflicting aspects of their relationship.

1.	Consider the set X of vertices $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$
2.	Consider the Bipolar Fuzzy Vertex set W on X
3.	Consider the set A of edges $q_1, q_2, \ldots, q_u$ where $q_r = \epsilon_s \epsilon_t$ for some $1 \leq s, t \leq n$
4.	Consider the Bipolar Fuzzy Edge set Λ on A
5.	Insert the Bipolar Fuzzy tolerance relation $L = (L^+, L^-)$ on $A \subseteq X \times X$
6.	Insert the Bipolar fuzzy tolerance relation Υ on X
7.	Compute lower approximation $\underline{\Upsilon}W$ and upper approximation $\overline{\Upsilon}W$ by using definition: $\begin{cases} \underline{\Upsilon}W^+(\alpha) = \bigwedge\left[1 - \Upsilon^+(\alpha,\beta) \vee W^+(\beta)\right], \ \underline{\Upsilon}W^-(\alpha) = \bigvee\left[-1 - \Upsilon^-(\alpha,\beta) \wedge W^-(\beta)\right] \\ \overline{\Upsilon}W^+(\alpha) = \bigvee\left[\Upsilon^+(\alpha,\beta) \wedge W^+(\beta)\right], \ \overline{\Upsilon}W^-(\alpha) = \bigwedge\left[\Upsilon^-(\alpha,\beta) \vee W^-(\beta)\right], \quad \forall \alpha, \beta \in X \end{cases}$
8.	Compute lower approximation $\underline{L}\Lambda$ and upper approximation $\overline{L}\Lambda$ by using definitions: $\begin{cases} \underline{L}\Lambda^+(\alpha\beta) = \bigwedge[(1 - L^+(\alpha\beta,\gamma\theta)) \vee \Lambda^+(\gamma\theta)], \ \underline{L}\Lambda^-(\alpha\beta) = \bigvee[(-1 - L^-(\alpha\beta,\gamma\theta)) \wedge \Lambda^-(\gamma\theta)] \\ \overline{L}\Lambda^+(\alpha\beta) = \bigvee[L^+(\alpha\beta,\gamma\theta) \wedge \Lambda^+(\gamma\theta)], \ \overline{L}\Lambda^-(\alpha\beta) = \bigwedge[(L^-(\alpha\beta,\gamma\theta) \vee \Lambda^-(\gamma\theta)], \quad \forall \gamma\theta \in A \end{cases}$
9.	Draw Bipolar Fuzzy Rough Digraph $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$
10.	Find out strong arcs in lower approximation $\underline{\Gamma}$ and in upper approximation $\overline{\Gamma}$ by using the definition: $\begin{cases} \mathrm{CONN}^+_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \leq \underline{L}\Lambda^+(\epsilon_0,\epsilon_1), \mathrm{CONN}^-_{\underline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \underline{L}\Lambda^-(\epsilon_0,\epsilon_1) \\ \mathrm{CONN}^+_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \leq \overline{L}\Lambda^+(\epsilon_0,\epsilon_1), \mathrm{CONN}^-_{\overline{\Gamma}-\epsilon_0\epsilon_1}(\epsilon_0,\epsilon_1) \geq \overline{L}\Lambda^-(\epsilon_0,\epsilon_1) \end{cases}$
11.	Find the minimal dominating sets $D_i, i = 1, 2, \ldots, k$ in $\underline{\Gamma}$
12.	Find the minimal dominating sets $D_i, i = 1, 2, \ldots, k$ in $\overline{\Gamma}$
13.	Find the minimal dominating sets $D_i, i = 1, 2, \ldots, k$ in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$
14.	Find the minimum dominating sets D_m in $\Gamma = (\underline{\Gamma}, \overline{\Gamma})$
14.	Find the lower domination number $\Omega_D(\underline{\Gamma})$ of a minimum dominating set
15.	Find the upper domination number $\Omega_D(\overline{\Gamma})$ of a minimum dominating set
16.	Find the domination number $\Omega_D(\Gamma) = \Omega_D(\underline{\Gamma}) + \Omega_D(\overline{\Gamma})$ of BFRD
17.	The minimum dominating set is the optimum set of the minimum number of rural areas that can supply the medicines to the other rural areas in the most cost-effective manner

Table 10: Algorithm for finding the optimum set of vertices

5. Conclusion

This research addressed a critical limitation in existing Fuzzy Rough Digraph (FRD) models: their inability to handle the conflicting positive and negative preferences inherent in many real-world decision problems. To overcome this, we introduced the Bipolar Fuzzy Rough Digraph (BFRD), a novel framework designed to model and analyze complex networks under conditions of bipolar uncertainty.

The primary motivation for this research was the critical inability of existing Fuzzy Rough Digraphs (FRDs) to model conflicting information in decision-making scenarios. Our work addresses this gap through the principal novelty of introducing the Bipolar Fuzzy Rough Digraph (BFRD), a new mathematical structure designed to handle uncertainty with both positive and negative preferences.

Our proposed method centers on a new domination theory developed specifically for BFRDs. We established the foundational concepts of the domination number, vertex degree, and Regular BFRDs, which together provide the mathematical tools to identify key influential nodes within networks characterized by dualistic properties. The primary significance of this work lies in its ability to provide a more robust and realistic model for decision analysis. By formally incorporating both supporting and opposing factors, the BFRD framework enables a more nuanced and accurate assessment of complex systems, which is a crucial advancement for decision-support systems dealing with real-world ambiguity.

The practical utility of our theoretical framework was validated through a real-world application. We developed a novel algorithm based on our domination model to solve a critical logistics problem: identifying an optimal set of rural areas for medicine distribution. The results were clear and impactful: our algorithm successfully identified the minimum dominating set of the BFRD, which directly corresponds to the smallest number of supply hubs needed to efficiently serve the entire network. This tangible outcome confirms that our BFRD-based method is not just a theoretical construct but an effective, practical tool that can be used to generate cost-effective solutions for complex optimization challenges. Looking ahead, this work opens several avenues for future research. The immediate next steps will involve extending this concept to more complex, multi-layered structures, such as Bipolar Fuzzy Soft Graphs and Intuitionistic Bipolar Fuzzy Rough Soft Digraphs. Furthermore, there is significant potential in developing dynamic BFRD models to analyze evolving networks and in exploring the integration of our domination-based algorithms with machine learning techniques for predictive analysis.

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