EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6217
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Novel Bright Soliton and Kink Wave Solutions of Nonlinear Evolution Equation

Najla A. Mohammed[1], Naveed Shahid[2], Tahira Sumbal Shaikh[3,*], Nauman Ahmed[2,*], Farwa Ghafoor[3], Muhammad Zafarullah Baber[4,5], Ilyas Khan[6,7,8], Wei Sin Koh[9]

[1] *Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[2] *Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan*
[3] *Department of Mathematics, Lahore College for Women University, Lahore, Pakistan*
[4] *Department of Mathematics, Shanghai University and Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, China*
[5] *Department of Mathematics and Statistics, The University of Lahore Sargodha Campus, Sargodha, Pakistan*
[6] *Department of Mathematical Sciences, Saveetha School of Engineering, SIMATS, Chennai, Tamil Nadu, India*
[7] *Hourani Center for Applied Scientific Research, Al-Ahliyya Amman University, Amman, Jordan*
[8] *Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah, Saudi Arabia*
[9] *INTI International University, Persiaran Perdana BBN Putra Nilai, 71800 Nilai, Negeri Sembilan, Malaysia*

Abstract. This article focuses on the generalized improved Boussinesq (GIB) equation. The GIB equation models nonlinear phenomena in various physical aspects such as shallow water waves, quantum fluid dynamics, and more. The new extended direct algebraic method (NEDAM) is applied to obtain exact solutions for this nonlinear model. Using this approach, we derive kink, anti-kink, solitons, and solitary wave solutions, along with bright, dark, and mixed-form solitons. New families of exponential, hyperbolic, and periodic wave solutions with arbitrary parameters are also constructed. The graphical representations provide deeper insight into the dynamics of nonlinear systems. The obtained results have many potential applications in optical fiber communications, fluid dynamics, and other fields involving wave propagation.

2020 Mathematics Subject Classifications: 35Q51, 35C07, 35R11

Key Words and Phrases: Solitary wave, kink, anti-kink, soliton, dark and bright solitons

---

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6217

Email addresses: tahira.sumbal@lcwu.edu.pk (T. S. Shaikh), nauman.ahmd01@gmail.com (N. Ahmed), i.said@mu.edu.sa (I. Khan)

1. Introduction

Partial differential equations (PDEs) had a central place among various settings of mathematics from the mid-nineteenth century onwards, especially in Riemann's work. Partial differential equations (PDEs) are highly praised due to their practicality in many fields such as acoustics, heat transfer, electro and magnetics, fluid dynamics, elasticity, and quantum mechanics. While these areas may at first glance seem very different, they can all be described in similar mathematical terms via PDEs. Numerous techniques have been developed to examine different facets of solutions and physical phenomena associated with nonlinear wave equations due to their significant mathematical features and broad range of applications. Consequently, studying nonlinear differential equations and determining their precise solutions is a contemporary field of study. This connects with work that began in the 1930s, which involves linking PDEs with the theory of singular integral operators. This has emerged as a fundamental issue in harmonic analysis, particularly with the establishment and generalization of the Calderon-Zygmund theory. The major method for the solution of nonlinear PDEs is Lie group analysis [1]. The mathematics of how to identify them has also been studied (lump exact solutions [2, 3]

As NLPDEs are widely known to appear in bioscience, physics, chemistry, quantum mechanics, fluid dynamics and multiple engineering domains, we aim to detect accurate and convenient solutions in these cases [4–6]. Numerous phenomena, including turbulence and shock wave flow in viscous fluids, have been demonstrated to be modeled by the Burgers' equation. The two-dimensional Burgers' equations were solved analytically by Fletcher using the Hopf-Cole transformation. Numerous numerical strategies have been constructed to solve this system of equations, such as implicit finite-difference schemes, explicit-implicit methods and techniques based on cubic spline functions. Soliman employed partial differential equation similarity reductions to develop a method for tackling the Burgers' issue. The two-dimensional Burgers equation has been solved using high-order accurate techniques. More recently, it is being proposed to achieve precise complex solutions for nonlinear partial differential equations using symbolic computing and direct algebraic approaches [7]. Long waves, such as solitons, move in stable packets at a steady speed without disintegrating. Solitons, sometimes called shallow-water waves, are waves that break apart only after colliding with other solitons. Solitons' robustness and useful applications in physics have attracted the interest of mathematicians, engineers and physicists due to their special property. Solitons appear as partial differential equations that are nonlinear. The notion of solitons is credited to the Scottish naval architect John Scott Russell. He saw a "Great Translation Wave" in the Great Britain Canal's shallow waters in 1834. By constructing a path where the wave could hold its shape and travel a great distance, he aimed to demonstrate the resiliency of the wave. The mathematical community did not think highly of Russell's ideas, and Array, a researcher, rejected them. In his book "Tides and Waves," published in 1845, Array set a hypothesis on long waves and stressed the relationship between height and amplitude and wave speed. This idea denies the existence of the single waves that Russell described [8].

In physics and mathematics, solitary waves, or solitons, are wave packets that propagate at the same rate without changing their shape. They use this wave as an example of self-reinforcing. This happens when, in the medium, nonlinear and dispersive effects counteract themselves and cancel each other. Many soliton solutions are obtained for a significant number of weakly nonlinear dispersive partial differential equations that describe physical systems [9–15]. After observing the phenomenon for the first time in Scotland's Union Canal in 1834 and reproducing it in a wave tank, John Scott Russell (1808–1882) named it the "Wave of Translation" [16–20]. A soliton, within optics, signifies an optical field that persists unchanged as it travels, owing to the delicate equilibrium between nonlinear dispersion and linear scattering effects within the medium [21–24].

In recent times, multiple special techniques have been proposed to ensure accurate solutions of nonlinear differential equations. Such methods are the Exp-method, the inverse scattering algorithm [25], the Hirota bilinear method [26], the modified extended tanh function method [27], the homogeneous balancing method [28], the sine-cosine method [29], tanh function method [30], variational iteration method [31]. A direct analytical resolution can be obtained for the nonlinear partial differential equations (NLPDEs) using many methods as the modified simplest equation scheme [32], Kudryashov's method [33], (G'/G) expansion method [34, 35]. Apart from these approaches, a more recent method called the UM scheme is used to analyze exact solutions of NLPDEs in the form of solitary waves. This method yields accurate traveling waves, which are explicit solutions.These explicit solutions have rational and polynomial function solutions, similar to waves. Soliton, elliptic, and solitary waves are various polynomial solutions; periodic and soliton rational solutions are different types of rational solutions. The conformable time-fractionalnonlinear Schrödinger equation (NLSE) and other nonlinear lattice equations (NLEEs) have been studied by several researchers using the UM method. A nonlinear partial differential equation is solved by the new extended direct algebraic method (NEDAM) [36]. The sine-cosine/ sinh-cosh, generalized exponential rational function method, Φ^6-model expansion method are also helpful for obtaining the bright, dark, kink, periodic solitons [37–41].

Numerous techniques have been established to produce the exact solutions of nonlinear partial differential equations. Most of these methods are unable to handle the complexity and nonlinearity of the equations, particularly, nonlinear GIBq equation, or give limited types of solutions. Our research explores the efficacy of an innovative algebraic approach in solving nonlinear evolution equations, thereby expanding the toolkit for analytical solutions. Through the derivation of precise solutions and examination of their characteristics, we gain a deeper understanding of nonlinear dynamics. This study's findings have significant implications for understanding intricate processes in applied sciences, providing a robust framework for investigation and prediction.

In recent years, there has been a significant increase in interestin solving ordinary differential equations and the study of solitary waves. Nonlinear partial differential equations (NLPDEs) have been utilized to simulate different types of phenomena in various application domains. This work aims to present a new approach, namely NEDAM, by which we can accurately soliton solutions for the GIBq problem. It can be beneficial for developing different types of soliton solutions and providing effective and rapid simulations. The NEDAM is the extended form of the traditional direct algebraic method. In this extension, an auxiliary equation in a more generic form is involved, a logarithmic nonlinearity scale factor $\ln B$ and three arbitrary parameters λ, μ and ν. The addition enables this method to be more effective thanproducing a broader class of analytical solutions having periodic, kinks, anti-kinks and exponential forms. We have studied several GIBq problems where this strategy has proven successful. This method yields a wider variety of solutions and is a robust, powerful and potent technique for investigating NLEEs. It also works with computer algebra systems.

Nonlinear evolution equations are pivotal in modeling complex phenomena across various scientific disciplines. While traditional methods like inverse scattering transform and Hirota's bilinear method have been instrumental in solving these equations, they often come with limitations in applicability and complexity. In response, our study leverages the extended direct algebraic method, a novel approach that has shown promise in deriving exact solutions for nonlinear systems. By applying this method to nonlinear evolution equations, we aim to uncover new insights into the dynamics of these systems. Our research not only demonstrates the efficacy of the extended direct algebraic method but also contributes to the existing body of knowledge by providing previously unexplored solutions. This advancement has significant implications for understanding intricate processes in applied sciences, offering a robust framework for investigation and prediction. Through this work, we seek to expand the toolkit for analytical solutions and enhance our understanding of nonlinear dynamics [42, 43].

The new extended direct algebraic method's applicability to nonlinear partial differential equations (NLPDEs) is notable, yet not all-encompassing. Its effectiveness is pronounced for equations with polynomial nonlinearity and single spatial dimensions. However, equations with highly complex nonlinearity, non-polynomial terms, fractional derivatives, or high-dimensional systems may pose challenges, potentially limiting the method's efficacy. Recognizing these boundaries enables researchers to judiciously apply this method and explore alternative approaches when confronted with complex NLPDEs. The extended hyperbolic function method's efficacy can be further contextualized by comparing it with other symbolic calculation methods, such as the Hirota bilinear method. The comparison in Table 1 would highlight the strengths and limitations of each approach, providing a more comprehensive understanding of their applicability to nonlinear partial differential equations.

The new extended direct algebraic method is employed in this study due to its efficacy in tackling nonlinear evolution equations. Its systematic approach enables efficient derivation of exact solutions, making it a valuable tool for understanding complex phenomena. Although not universally applicable, this method's flexibility and simplicity render it suitable for a broad range of nonlinear equations. By leveraging its strengths, this research aims to uncover novel insights into the behavior of the equation under investigation, ultimately contributing to the advancement of knowledge in this field [44–47].

Our research is driven by the quest for robust analytical frameworks to tackle nonlinear evolution equations, which underpin complex phenomena in diverse scientific realms. The pursuit of exact solutions is paramount, as they offer profound insights into system behavior and serve as benchmarks for numerical validation. By harnessing the potential

Method	Applicability	Solution Form	Complexity	Advantages	Limitations
Proposed Method	NLPDEs with polynomial nonlinearity	Exact hyperbolic solutions	Moderate	Simple, efficient for certain NLPDEs	Limited for highly nonlinear equations
Hirota Bilinear Method	Wide range of NLPDEs	Exact bilinear solutions	High	Effective for multi-soliton solutions	Requires careful variable transformation
Extended Tanh Method	NLPDEs with polynomial nonlinearity	Exact tanh solutions	Moderate	Straightforward, suitable for certain NLPDEs	Limited for high-degree nonlinearity
Other Methods	Specific types of NLPDEs	Exact solutions in various	High	Powerful tool for certain NLPDEs	Requires advanced techniques, limited applicability

Table 1: Comparison table.

of the extended direct algebraic method, we seek to unlock novel solutions and propel advancements in this field, ultimately enriching our understanding of nonlinear dynamics. Nonlinear evolution equations govern a wide range of physical phenomena, including wave dynamics in optics, fluid flow, and plasma physics. The intricate behavior exhibited by these systems, such as soliton formation and chaotic dynamics, stems from the interplay between nonlinearity and dispersion. By investigating exact solutions to these equations, we can gain a deeper understanding of the underlying physics and unlock new avenues for research and applications in fields like materials science and engineering.

1.1. Generalized Improved Boussinesq (GIBE) Equation

The Generalized Improved Boussinesq Equation (GIBE) is

$$u_{tt}-au_{xx}-u_{xxtt}-bu_{xxt}=\left(u^2\right)_{xx}+cu^3+du_{xx}u, \tag{1}$$

$$x\in[a,\ b],\ \ t\in[0,\ T],T\ >\ 0, \tag{2}$$

where $u=u(x,t)$ is a wave profile with x demonstrating the spatial component and t indicating the temporal component, respectively. Equation (1) represents the generalized form of the classical Boussinesq framework that covers the dispersive and highly nonlinear behaviours of wave phenomena. In this equation, term u_{xxtt} represents the dispersion of high order, u_{xxt} reflects the spatiotemporal dissipative behavior, the terms $\left(u^2\right)_{xx}$ and u^3 account for nonlinear effects and the nonlinear interaction term is modeled by the term

u_{xx} u. Moreover, the constants a, b, c and d are the parameters representing the linear diffusion coefficient, rate of spatiotemporal dissipation and the rate of nonlinear reactions, respectively.

1.2. Overview of NEDAM

Here, we provide a high-level overview of the proposed method.

1. **Step 1:** Suppose NLPDE

$$R(W, W_t, W_x, W_{tt}, W_{xt}, W_{xx}, \dots) = 0, \tag{3}$$

where $W = W(x,t)$symbolizes anunidentified function and Rsignifies a polynomial of $W(x,t)$ with its arguments.

1. **Step 2:** The real variables x and t are combined to generate the compound variable

$$W\left(x,t\right) = \phi\left(\zeta\right), \zeta = jx - \rho t, \tag{4}$$

where ρ indicates the wave velocity and j indicates the wave number. Eq. 3 becomes an ordinary differential equation (ODE) when the travelingwave transformation Eq.(4) is employed

$$\pi\left(\phi, \phi', \phi'', \phi''', \dots\right) = 0, \tag{5}$$

where π is a polynomial of ϕ with its derivatives.

1. **Step 3:** We assume that the following polynomial, which may be used to represent the trial solution to Eq. (5)

$$\phi\left(\zeta\right) = \sum_{i=0}^{N} \alpha_i \Omega^i\left(\zeta\right), \alpha_{N\neq 0}, \tag{6}$$

where $\alpha_i (0 \leq i \leq N)$ are unknown constants and $\Omega\left(\zeta\right)$ is a real-valued functionthat satisfies the auxiliary ODE,

$$\Omega'\left(\zeta\right) = \ln\left(B\right)\ \left(\mu + \lambda\Omega\left(\zeta\right) + \nu\Omega^2\left(\zeta\right)\right), B\ \neq\ 0,\ 1, \tag{7}$$

where μ, ν *and* λ are constants. The solution to Eq. (7) can be expressed as follows:

1. When λ^2 $-4\mu\nu$<0 $and \nu$≠0 then,

$$\Omega_1\left(\zeta\right) = -\frac{\lambda}{2\nu} + \frac{\sqrt{-\left(\lambda^2 - 4\mu\nu\right)}}{2\nu} \tan_B\left(\frac{\sqrt{-\left(\lambda^2 - 4\mu\nu\right)}}{2}\zeta\right), \tag{8}$$

$$\Omega_2\left(\zeta\right) = -\frac{\lambda}{2\nu} - \frac{\sqrt{-\left(\lambda^2 - 4\mu\nu\right)}}{2\nu} \cot_B\left(\frac{\sqrt{-\left(\lambda^2 - 4\mu\nu\right)}}{2}\zeta\right), \tag{9}$$

$$\Omega_3(\zeta)=-\frac{\lambda}{2\nu}+\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{2\nu}\left(\tan_B\left(\sqrt{-(\lambda^2-4\mu\nu)}\zeta\right)\pm\sqrt{pq}\sec_B\left(\sqrt{-(\lambda^2-4\mu\nu)}\zeta\right)\right),\quad(10)$$

$$\Omega_4(\zeta)=-\frac{\lambda}{2\nu}-\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{2\nu}\left(\cot_B\left(\sqrt{-(\lambda^2-4\mu\nu)}\zeta\right)\pm\sqrt{pq}\csc_B\left(\sqrt{-(\lambda^2-4\mu\nu)}\zeta\right)\right),\quad(11)$$

$$\Omega_5(\zeta)=-\frac{\lambda}{2\nu}+\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{4\nu}\left(\tan_B\left(\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{4}\zeta\right)-\cot_B\left(\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{4}\zeta\right)\right).\quad(12)$$

2. When $\lambda^2 - 4\mu\nu > 0$ *and* $\nu \neq 0$ then,

$$\Omega_6(\zeta)=-\frac{\lambda}{2\nu}-\frac{\sqrt{(\lambda^2-4\mu\nu)}}{2\nu}\tanh_B\left(\frac{\sqrt{(\lambda^2-4\mu\nu)}}{2}\zeta\right),\quad(13)$$

$$\Omega_7(\zeta)=-\frac{\lambda}{2\nu}-\frac{\sqrt{(\lambda^2-4\mu\nu)}}{2\nu}\coth_B\left(\frac{\sqrt{(\lambda^2-4\mu\nu)}}{2}\zeta\right),\quad(14)$$

$$\Omega_8(\zeta)=-\frac{\lambda}{2\nu}-\frac{\sqrt{(\lambda^2-4\mu\nu)}}{2\nu}\left(\tanh_B\left(\sqrt{(\lambda^2-4\mu\nu)}\ \zeta\right)\pm i\sqrt{pq}\operatorname{sech}_B\left(\sqrt{(\lambda^2-4\mu\nu)}\ \zeta\right)\right),\quad(15)$$

$$\Omega_9(\zeta)=-\frac{\lambda}{2\nu}-\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{2\nu}\left(\coth_B\left(\sqrt{(\lambda^2-4\mu\nu)}\ \zeta\right)\pm\sqrt{pq}\operatorname{csch}_B\left(\sqrt{(\lambda^2-4\mu\nu)}\ \zeta\right)\right),\quad(16)$$

$$\Omega_{10}(\zeta)=-\frac{\lambda}{2\nu}+\frac{\sqrt{-(\lambda^2-4\mu\nu)}}{4\nu}\left(\tanh_B\left(\frac{\sqrt{(\lambda^2-4\mu\nu)}}{4}\ \zeta\right)+\coth_B\left(\frac{\sqrt{(\lambda^2-4\mu\nu)}}{4}\ \zeta\right)\right).\quad(17)$$

3. When $\mu\nu > 0$ *and* $\lambda = 0$ then,

$$\Omega_{11}(\zeta)=\sqrt{\frac{\mu}{\nu}}tan_B\left(\sqrt{\mu\nu}\ \zeta\right),\quad(18)$$

$$\Omega_{12}(\zeta) = -\sqrt{\frac{\mu}{\nu}} cot_B \left(\sqrt{\mu\nu}\ \zeta\right), \tag{19}$$

$$\Omega_{13}(\zeta) = \sqrt{\frac{\mu}{\nu}} \left(tan_B \left(2\sqrt{\mu\nu}\ \zeta\right) \pm \sqrt{pq} sec_B \left(2\sqrt{\mu\nu}\ \zeta\right)\right), \tag{20}$$

$$\Omega_{14}(\zeta) = -\sqrt{\frac{\mu}{\nu}} \left(cot_B \left(2\sqrt{\mu\nu}\ \zeta\right) \pm \sqrt{pq} csc_B \left(2\sqrt{\mu\nu}\ \zeta\right)\right), \tag{21}$$

$$\Omega_{15}(\zeta) = \frac{1}{2}\sqrt{\frac{\mu}{\nu}} \left(tan_B \left(\frac{\sqrt{\mu\nu}}{2}\ \zeta\right) -\ cot_B \left(\frac{\sqrt{\mu\nu}}{2}\ \zeta\right)\right). \tag{22}$$

4. When $\mu\nu < 0$ *and* $\lambda = 0$ then,

$$\Omega_{16}(\zeta) = -\sqrt{-\frac{\mu}{\nu}} tanh_B \left(\sqrt{-\mu\nu}\ \zeta\right), \tag{23}$$

$$\Omega_{17}(\zeta) = -\sqrt{-\frac{\mu}{\nu}} coth_B \left(\sqrt{-\mu\nu}\ \zeta\right), \tag{24}$$

$$\Omega_{18}(\zeta) = -\sqrt{-\frac{\mu}{\nu}} \left(tanh_B \left(2\sqrt{-\mu\nu}\ \zeta\right) \pm i\sqrt{pq} sech_B \left(2\sqrt{-\mu\nu}\ \zeta\right)\right), \tag{25}$$

$$\Omega_{19}(\zeta) = -\sqrt{-\frac{\mu}{\nu}} \left(coth_B \left(2\sqrt{-\mu\nu}\ \zeta\right) \pm \sqrt{pq} csch_B \left(2\sqrt{-\mu\nu}\ \zeta\right)\right), \tag{26}$$

$$\Omega_{20}(\zeta) = -\frac{1}{2}\sqrt{-\frac{\mu}{\nu}} \left(tanh_B \left(\frac{\sqrt{-\mu\nu}}{2}\ \zeta\right) -\ coth_B \left(\frac{\sqrt{-\mu\nu}}{2}\ \zeta\right)\right). \tag{27}$$

5. When $\lambda = 0$ *and* $\nu = \mu$ then,

$$\Omega_{21}(\zeta) = tan_B(\mu\zeta), \tag{28}$$

$$\Omega_{22}(\zeta) = -cot_B(\mu\zeta), \tag{29}$$

$$\Omega_{23}(\zeta) = tan_B(2\mu\zeta) \pm \sqrt{pq} sec_B(2\mu\zeta), \tag{30}$$

$$\Omega_{24}(\zeta) = -cot_B(2\mu\zeta) \pm \sqrt{pq} csc_B(2\mu\zeta), \tag{31}$$

$$\Omega_{25}(\zeta) = \frac{1}{2}\left(tan_B \left(\frac{\mu}{2}\ \zeta\right) -\ cot_B \left(\frac{\mu}{2}\ \zeta\right)\right). \tag{32}$$

6. When $\lambda = 0$ *and* $\nu = -\mu$ then,

$$\Omega_{26}(\zeta) = -tanh_B(\mu\zeta), \tag{33}$$

$$\Omega_{27}(\zeta) = -coth_B(\mu\zeta), \tag{34}$$

$$\Omega_{28}(\zeta) = -tanh_B(2\mu\zeta) \pm i\sqrt{pq} sech_B(2\mu\zeta), \tag{35}$$

$$\Omega_{29}(\zeta) = -coth_B(2\mu\zeta) \pm \sqrt{pq} csch_B(2\mu\zeta), \tag{36}$$

$$\Omega_{30}(\zeta) = -\frac{1}{2}\left(tanh_B \left(\frac{\mu}{2}\ \zeta\right) +\ cot_B \left(\frac{\mu}{2}\ \zeta\right)\right). \tag{37}$$

7. When $\lambda^2 = 4\mu\nu$ then,

$$\Omega_{31}(\zeta) = \frac{-2\mu(\lambda\zeta\ln(B)+2)}{\lambda^2\zeta\ln(B\ )}. \tag{38}$$

8. When $\lambda = \chi, \mu = r\chi\,(r \neq 0)$ *and* $\nu = 0$ then,

$$\Omega_{32}(\zeta) = B^{\chi\zeta} - r. \tag{39}$$

9. When $\lambda = \nu = 0$ then,

$$\Omega_{33}(\zeta) = \mu\zeta\ln(B)\ . \tag{40}$$

10. When $\lambda = \mu = 0$ then,

$$\Omega_{34}(\zeta) = \frac{-1}{\nu\zeta\ln(B)}. \tag{41}$$

11. When $\lambda \neq 0,\ \mu = 0$ then,

$$\Omega_{35}(\zeta) = -\frac{p\lambda}{\nu(\cosh_B(\lambda\zeta) - \sinh_B(\lambda\zeta) + p)}, \tag{42}$$

$$\Omega_{36}(\zeta) = -\frac{\lambda(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta))}{\nu(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta) + q)}. \tag{43}$$

12. When $\lambda = \chi, \nu = r\chi\,(r \neq 0)$ *and* $\mu = 0$ then,

$$\Omega_{37}(\zeta) = \frac{pB^{\chi\zeta}}{p - rqB^{\chi\zeta}}. \tag{44}$$

The consequent claims state generalized hyperbolic and trigonometric functions in terms of the previous results.

$$\sinh_B(\zeta) = \frac{pB^{\zeta} - pB^{-\zeta}}{2}, \cosh_B(\zeta) = \frac{pB^{\zeta} + pB^{-\zeta}}{2},$$

$$\tanh_B(\zeta) = \frac{pB^{\zeta} - pB^{-\zeta}}{pB^{\zeta} + pB^{-\zeta}}, \coth_B(\zeta) = \frac{pB^{\zeta} + pB^{-\zeta}}{pB^{\zeta} - pB^{-\zeta}},$$

$$\mathrm{sech}_B(\zeta) = \frac{2}{pB^{\zeta} + pB^{-\zeta}}, \mathrm{csch}_B(\zeta) = \frac{2}{pB^{\zeta} - pB^{-\zeta}},$$

$$\sin_B(\zeta) = \frac{pB^{i\zeta} - pB^{-i\zeta}}{2}, \cos_B(\zeta) = \frac{pB^{i\zeta} + pB^{-i\zeta}}{2},$$

$$\tan_B(\zeta) = -i\frac{pB^{i\zeta} - pB^{-i\zeta}}{pB^{i\zeta} + pB^{-i\zeta}}, \cot_B(\zeta) = i\frac{pB^{i\zeta} + pB^{-i\zeta}}{pB^{i\zeta} - pB^{-i\zeta}},$$

$$\sec_B(\zeta) = \frac{2}{pB^{i\zeta} + pB^{-i\zeta}}, \csc_B(\zeta) = \frac{2i}{pB^{i\zeta} - pB^{-i\zeta}},$$

where ζ is an independent variable and $p, q > 0$.

1. Step 4: To calculate N for Eq. (6), we apply the homogeneous balancing procedure from Eq. (5).

2. Step 5: To create the strategic equations, just insert Eq. (6) and its derivatives into Eq. (5) and solve for coefficients of the same power of $\Omega(\zeta)$ to zero. When Mathematica solves the strategic equations, we have to solve the system to obtain the values of the unknowns.

1.3. Exact Wave Solution

The above equation (1) can be solved using the wave transformation $u(x,\ t) = \ Q(\zeta)$, where $\zeta = x - kt$ and $k \neq 0$.
By putting the relevant derivatives of above equation into Eq.(1), we get

$$k^2Q''(\zeta) - aQ''(\zeta) - k^2Q''''(\zeta) + \ bkQ'''(\zeta) = \ \left(Q^2\right)''(\zeta) + cQ^3(\zeta) + dQ''(\zeta)\,Q(\zeta)\,.$$

Also, we may write above Eq. as

$$k^2Q'' - a\mathrm{Q}^{''} - k^2Q'''' + \ bkQ''' - \ \left(Q^2\right)'' - cQ^3 - dQ''Q = 0. \tag{45}$$

Application of NEDAM
The new extended direct algebraic method's applicability to nonlinear partial differential equations (NLPDEs) is notable, yet not all-encompassing. Its effectiveness is pronounced for equations with polynomial nonlinearity and single spatial dimensions. However, equations with highly complex nonlinearity, non-polynomial terms, fractional derivatives, or high-dimensional systems may pose challenges, potentially limiting the method's efficacy. Recognizing these boundaries enables researchers to judiciously apply this method and explore alternative approaches when confronted with complex NLPDEs.
The suggested method is employed for appraising novel solutions to Eq. (1). Applying the homogeneous balance principle, which Eq. (45) yields, $n = 2$. So, the solution to Eq. (45) is assumed to be formulated as:

$$\mathrm{Q}(\zeta) = b_0 + b_1\Omega(\zeta) + b_2\Omega^2(\zeta)\,. \tag{46}$$

By entering Eq. (46) and its derivatives into Eq. (45) and specifying coefficients of comparable powers of $\mathit{\Omega}(\zeta)$ to zero, we can quickly get the strategic equations.
We have

$$\Omega'(\zeta) = \ln(B)(\mu + \lambda\Omega(\zeta) + \nu\Omega^2(\zeta)), B \ \neq \ 0,\ 1.$$

The system of algebraic equations may be solved by using the softwareMathematica to find the constant values.
Family-1.

$$\mathrm{b}_0 = \frac{-6\lambda\sqrt{\mathrm{k}^4\nu^2\left(\lambda^2 - 4\mu\nu\right)\ln^4(B)} - 12\mathrm{k}^2\mu\nu^2\ln^2(B) + k^2\nu - \mathrm{a}\nu}{2\nu},$$

$$\mathrm{b}_1 = -6\left(\mathrm{k}^2\lambda\nu\ln^2(B)+\sqrt{\mathrm{k}^4\nu^2\ln^4(B)(\lambda^2-4\mu\nu)}\right),$$

$$\mathrm{b}_2 = -6\mathrm{k}^2\nu^2\ln^2(B), \mathrm{b} = -\frac{5\sqrt{\mathrm{k}^4\nu^2\left(\lambda^2-4\mu\nu\right)\ln^4(B)}}{\mathrm{k}\nu\ln(B)}, \mathrm{c}=0, \mathrm{d}=0.$$

Family-2.

$$b_0 = \frac{1}{2}\left(k^2\ln^2(B)(-\left(\lambda^2+8\mu\nu\right))+k^2-a\right),$$

$$b_1 = -6k^2\lambda\nu\ln^2(B), b_2 = -6k^2\nu^2\ln^2(B), b=0, c=0, d=0.$$

The solutions to Eq. (1) for the family 2 can be summarized as follows.

1. For the cases $\lambda^2 - 4\mu\nu < 0$ *and* $\nu \neq 0$, the following pair of outcomes that we find for both trigonometric as well as mixed-trigonometric solutions.

$$\phi_1(x,t) = \frac{1}{2\nu}\times\left\{G_1\right)\Bigg\} + \frac{1}{2\nu}\left\{\nu\left(3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+k^2-\mathrm{a}\right)+\right\} \frac{1}{2\nu}\left\{3\mathrm{k}^2\nu\ln^2(B)\left(\lambda^2-4\mu\nu\right)tan_B^2\left(\frac{\sqrt{(4\mu\nu-\lambda^2)}}{2}\zeta\right)\right\}, \tag{47}$$

where

$$G_1 = -6\sqrt{4\mu\nu-\lambda^2}\sqrt{\mathrm{k}^4\nu^2\ln^4(B)(\lambda^2-4\mu\nu)}\tan_B(\frac{\sqrt{(4\mu\nu-\lambda^2)}}{2}\zeta),$$

$$\phi_2(x,t) = \frac{1}{2\nu}\left\{\{\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]+G_2\right\} + \frac{1}{2\nu}\left\{3\mathrm{k}^2\nu\ln^2(B)\left(\lambda^2-4\mu\nu\right)\cot_B^2\left(\frac{\sqrt{4\mu\nu-\lambda^2}}{2}\zeta\right)\right\}, \tag{48}$$

where

$$G_2 = 6\sqrt{4\mu\nu-\lambda^2}\sqrt{\mathrm{k}^4\nu^2\ln^4(B)(\lambda^2-4\mu\nu)}\tan_B(\frac{\sqrt{(4\mu\nu-\lambda^2)}}{2}\zeta) \tag{49}$$

$$\phi_3(x,t) = \frac{1}{2\nu}\left\{\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]-6\sqrt{\mathrm{k}^4\nu^2\left(\lambda^2-4\mu\nu\right)\ln^4(B)}\sqrt{4\mu\nu-\lambda^2}\right. \left(\tan_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\pm\sqrt{pq}\sec_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\right)+3k^2\nu\ln^2(B)\left(\lambda^2-4\mu\nu\right) \left.\times\left(\tan_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\pm\sqrt{pq}\sec_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\right)^2\right\}, \tag{50}$$

$$\phi_4(x,t)=\frac{1}{2\nu}\Big\{\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]+6\sqrt{\mathrm{k}^4\nu^2\left(\lambda^2-4\mu\nu\right)\ln^4(B)}\sqrt{4\mu\nu-\lambda^2}$$

$$\left(\cot_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\pm\sqrt{pq}\csc_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\right)+3\mathrm{k}^2\nu\ln^2(B)\left(\lambda^2-4\mu\nu\right)$$

$$\times\left(\left(\cot_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\pm\sqrt{pq}\csc_B\left(\sqrt{4\mu\nu-\lambda^2}\,\zeta\right)\right)\right)^2\Big\}, \tag{51}$$

$$\phi_5(x,t)=\frac{1}{8\nu}\Big\{4\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]-3\sqrt{\mathrm{k}^4\nu^2\ln^4(B)\left(\lambda^2-4\mu\nu\right)}$$

$$4\sqrt{4\mu\nu-\lambda^2}\tan_B\left(\frac{\sqrt{-\left(\lambda^2-4\mu\nu\right)}}{4}\,\zeta\right)-\cot_B\left(\frac{\sqrt{-\left(\lambda^2-4\mu\nu\right)}}{4}\,\zeta\right)+3\mathrm{k}^2\lambda\nu\ln^2(B)]\times$$

$$\left(\lambda^2-4\mu\nu\right)\left(\tan_B\left(\frac{\sqrt{-\left(\lambda^2-4\mu\nu\right)}}{4}\,\zeta\right)-\cot_B\left(\frac{\sqrt{-\left(\lambda^2-4\mu\nu\right)}}{4}\,\zeta\right)\right)^2\Big\}. \tag{52}$$

1. For the cases $\lambda^2-4\mu\nu>0$ *and* $\nu\neq0$, several results are discovered as follows. The way to solve the kink is expressed as

$$\phi_6(x,t)=\frac{1}{2\nu}\times\Big\{\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]+G_3$$

$$-3\mathrm{k}^2\nu\ln^2(B)\times\left(\lambda^2-4\mu\nu\right)tanh_B^2\left(\frac{\sqrt{\left(\lambda^2-4\mu\nu\right)}}{2}\zeta\right)\Big\}, \tag{53}$$

where

$$G_3=6\sqrt{\mathrm{k}^4\nu^2\ln^4(B)\left(\lambda^2-4\mu\nu\right)}\times\sqrt{\left(\lambda^2-4\mu\nu\right)}\tanh_B\left(\frac{\sqrt{\left(\lambda^2-4\mu\nu\right)}}{2}\zeta\right),$$

$$\phi_7(x,t)=\frac{1}{2\nu}\times\Big\{\nu\left[3\mathrm{k}^2\ln^2(B)\left(\lambda^2-4\mu\nu\right)+\mathrm{k}^2-a\right]+G_4$$

$$-3\mathrm{k}^2\nu\ln^2(B)\times\left(\lambda^2-4\mu\nu\right)coth_B^2\left(\frac{\sqrt{\left(\lambda^2-4\mu\nu\right)}}{2}\zeta\right)\Big\}, \tag{54}$$

$$6\sqrt{\mathrm{k}^4\nu^2\ln^4(B)\left(\lambda^2-4\mu\nu\right)}\times\sqrt{\left(\lambda^2-4\mu\nu\right)}\coth_B\left(\frac{\sqrt{\left(\lambda^2-4\mu\nu\right)}}{2}\zeta\right)$$

$$\phi_8(x,t)=\frac{1}{2\nu}\times\{\nu\left[3\mathrm{k}^2\ln^2(B)(\Delta)+\mathrm{k}^2-a\right]+6\sqrt{\mathrm{k}^4\nu^2\ln^4(B)(\Delta)}\times$$

$$\sqrt{(\Delta)}\left(\tanh_B\left(\sqrt{(\Delta)}\,\zeta\right)\pm i\sqrt{pq}\operatorname{sech}_B\left(\sqrt{(\Delta)}\,\zeta\right)\right)-3\mathrm{k}^2\nu\ln^2(B)$$

$$\times (\Delta) \left(\tanh_B \left(\sqrt{(\Delta)}\, \zeta\right) \pm i\sqrt{pq}\mathrm{sech}_B \left(\sqrt{(\lambda^2 - 4\mu\nu)}\, \zeta\right)\right)^2 \}, \tag{55}$$

$$\phi_9 (x,t) = \frac{1}{2\nu} \times \{\nu \left[3\mathrm{k}^2 \ln^2 (B) (\Delta) + \mathrm{k}^2 - a\right] + 6\sqrt{\mathrm{k}^4 \nu^2 \ln^4 (B) (\Delta)} \times$$
$$\sqrt{(-\Delta)} \left(\coth_B \left(\sqrt{(\Delta)}\, \zeta\right) \pm \sqrt{pq}\mathrm{csch}_B \left(\sqrt{(\Delta)}\, \zeta\right)\right) + 3\mathrm{k}^2 \nu \ln^2 (B)$$
$$\left(\lambda^2 - 4\mu\nu\right) \left(\left(\coth_B \left(\sqrt{(\Delta)}\, \zeta\right) \pm \sqrt{pq}\mathrm{csch}_B \left(\sqrt{(\Delta)}\, \zeta\right)\right)\right)^2 \}, \tag{56}$$

$$\phi_{10} (x,t) = \frac{1}{8\nu} \times \{4\nu \left[3\mathrm{k}^2 \ln^2 (B) (\Delta) + \mathrm{k}^2 - a\right] - 3\sqrt{\mathrm{k}^4 \nu^2 \ln^4 (B) (\Delta)} \times$$
$$4\sqrt{(4\mu\nu - \lambda^2)} \left(\tanh_B \left(\frac{\sqrt{(\Delta)}}{4}\, \zeta\right) + \coth_B \left(\frac{\sqrt{(\Delta)}}{4}\, \zeta\right)\right) + 3k^2 \nu \ln^2 (B)$$
$$\left(\lambda^2 - 4\mu\nu\right) \left(\left(\tanh_B \left(\frac{\sqrt{(\lambda^2 - 4\mu\nu)}}{4}\, \zeta\right) + \coth_B \left(\frac{\sqrt{(\lambda^2 - 4\mu\nu)}}{4}\, \zeta\right)\right)\right)^2 \}, \tag{57}$$

where

$$\Delta = \lambda^2 - \mu\nu, \text{ for the Eq.s } (55), (56), (57). \tag{58}$$

2. For $\mu\nu > 0$ *and* $\lambda = 0$,

$$\phi_{11} (x,t) = \{-\frac{12\sqrt{\mu}\sqrt{-\mathrm{k}^4 \mu\nu^3 \ln^4 (B)}tan_B \left(\sqrt{\mu\nu}\, \zeta\right)}{\sqrt{\nu}} - 6k^2 \mu\nu \ln^2 (B) \left(tan_B^2 \left(\sqrt{\mu\nu}\, \zeta\right) + 1\right)$$
$$+ \frac{1}{2} \left(k^2 - a\right) \}, \tag{59}$$

$$\phi_{12} (x,t) = \frac{12\sqrt{\mu}\sqrt{-\mathrm{k}^4 \mu\nu^3 \ln^4 (B)}cot_B \left(\sqrt{\mu\nu}\, \zeta\right)}{\sqrt{\nu}} -$$
$$6k^2 \mu\nu \ln^2 (B) \left(cot_B^2 \left(\sqrt{\mu\nu}\, \zeta\right) + 1\right) + \frac{1}{2} \left(k^2 - a\right). \tag{60}$$

$$\phi_{13} (x,t) = \{\frac{-12\sqrt{\mu}\sqrt{-\mathrm{k}^4 \mu\nu^3 \ln^4 (B)} \left(tan_B \left(2\sqrt{\mu\nu}\, \zeta\right) \pm \sqrt{pq}sec_B \left(2\sqrt{\mu\nu}\, \zeta\right)\right)}{\sqrt{\nu}} -$$
$$6\mathrm{k}^2 \mu\nu \ln^2 (B) \left(\left(\left(tan_B \left(2\sqrt{\mu\nu}\, \zeta\right) \pm \sqrt{pq}sec_B \left(2\sqrt{\mu\nu}\, \zeta\right)\right)\right)^2 + 1\right) + \frac{1}{2} \left(k^2 - a\right) \}, \tag{61}$$

$$\phi_{14} (x,t) = \{\frac{12\sqrt{\mu}\sqrt{-\mathrm{k}^4 \mu\nu^3 \ln^4 (B)} \left(cot_B \left(2\sqrt{\mu\nu}\, \zeta\right) \pm \sqrt{pq}csc_B \left(2\sqrt{\mu\nu}\, \zeta\right)\right)}{\sqrt{\nu}} -$$
$$6\mathrm{k}^2 \mu\nu \ln^2 (B) \left(\left(cot_B \left(2\sqrt{\mu\nu}\, \zeta\right) \pm \sqrt{pq}csc_B \left(2\sqrt{\mu\nu}\, \zeta\right)\right)^2 + 1\right) + \frac{1}{2} \left(k^2 - a\right) \}, \tag{62}$$

$$\phi_{15}(x,t)=\frac{1}{2\nu}\times\{\frac{12\nu\sqrt{\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)}\left(cot_B\left(\frac{\sqrt{\mu\nu}}{2}\zeta\right)-\ tan_B\left(\frac{\sqrt{\mu\nu}}{2}\zeta\right)\right)}{\sqrt{\nu}}+$$

$$\nu\left[-12k^2\mu\nu^2\ln^2(B)+k^2-a\right]-3k^2\mu\nu^2\ln^2(B)\left(tan_B\left(\frac{\sqrt{\mu\nu}}{2}\zeta\right)-\ cot_B\left(\frac{\sqrt{\mu\nu}}{2}\zeta\right)\right)^2\}. \tag{63}$$

1. For $\mu\nu<0$ *and* $\lambda=0$,

$$\phi_{16}(x,t)=\{G_4+6k^2\mu\nu\ln^2(B)\left(tanh_B^2\left(\sqrt{-\mu\nu}\,\zeta\right)-1\right)+\frac{1}{2}\left(k^2-a\right), \tag{64}$$

where

$$G_4=\frac{12\sqrt{-\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)}tanh_B\left(\sqrt{-\mu\nu}\,\zeta\right)}{\sqrt{\nu}},$$

$$\phi_{17}(x,t)=G_5+6k^2\mu\nu\ln^2(B)\left(coth_B^2\left(\sqrt{-\mu\nu}\,\zeta\right)-1\right)+\frac{1}{2}\left(k^2-a\right), \tag{65}$$

where

$$G_5=\{\frac{12\sqrt{-\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)}coth_B\left(\sqrt{-\mu\nu}\,\zeta\right)}{\sqrt{\nu}},$$

$$\phi_{18}(x,t)=\{\frac{G_6\left(tanh_B\left(2\sqrt{-\mu\nu}\,\zeta\right)\pm i\sqrt{pq}sech_B\left(2\sqrt{-\mu\nu}\,\zeta\right)\right)}{\sqrt{\nu}}+$$

$$6\mathrm{k}^2\mu\nu\ln^2(B)\left(\left(tanh_B\left(2\sqrt{-\mu\nu}\,\zeta\right)\pm i\sqrt{pq}sech_B\left(2\sqrt{-\mu\nu}\,\zeta\right)\right)^2-1\right)+\frac{1}{2}\left(k^2-a\right)\}, \tag{66}$$

where

$$G_6=12\sqrt{-\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)},$$

$$\phi_{19}(x,t)=\{\frac{12\sqrt{-\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)}(\Omega)}{\sqrt{\nu}}+$$
$$6\mathrm{k}^2\mu\nu\ln^2(B)\left((\Omega)^2-1\right)+\frac{1}{2}\left(k^2-a\right)\}, \tag{67}$$

where

$$\Omega=coth_B\left(2\sqrt{-\mu\nu}\,\zeta\right)\pm\sqrt{pq}csch_B\left(2\sqrt{-\mu\nu}\,\zeta\right),$$

$$\phi_{20}(x,t)=\frac{1}{2\nu}\times\{\frac{G_7\left(tanh_B\left(\frac{\sqrt{-\mu\nu}}{2}\zeta\right)-\ coth_B\left(\frac{\sqrt{-\mu\nu}}{2}\zeta\right)\right)}{\sqrt{\nu}}+$$

$$\nu\left[-12k^2\mu\nu^2\ln^2(B)+k^2-a\right]+3k^2\mu\nu^2\ln^2(B)\,G_8\}. \tag{68}$$

where

$$\begin{aligned} G_7 &= 12\nu\sqrt{-\mu}\sqrt{-\mathrm{k}^4\mu\nu^3\ln^4(B)},\\ G_8 &= \left(tanh_B\left(\frac{\sqrt{-\mu\nu}}{2}\,\zeta\right)-\ coth_B\left(\frac{\sqrt{-\mu\nu}}{2}\,\zeta\right)\right)^2, \end{aligned}$$

2. For $\lambda=0$ *and* $\nu=\mu$,

$$\phi_{21}(x,t)=\left\{G_9-\frac{1}{2}k^2\left[12\mu^2\ln^2(B)+12\mu^2\ln^2(B)\,tan_B^2(\mu\zeta)-1\right]-\frac{a}{2}\right\}, \tag{69}$$

where

$$G_9=-12\sqrt{-\mathrm{k}^4\mu^4\ln^4(B)}tan_B(\mu\zeta),$$

$$\phi_{22}(x,t)=\left\{-\frac{1}{2}k^2\left[12\mu^2\ln^2(B)+12\mu^2\ln^2(B)\,cot_B^2(\mu\zeta)-1\right]-\frac{a}{2}\right\}, \tag{70}$$

where

$$G_{10}=12\sqrt{-\mathrm{k}^4\mu^4\ln^4(B)}cot_B(\mu\zeta), \tag{71}$$

$$\begin{aligned} &\phi_{23}(x,t)=\{-12\sqrt{-\mathrm{k}^4\mu^4\ln^4(B)}\ (tan_B(2\mu\zeta)\pm\sqrt{pq}sec_B(2\mu\zeta))\\ &-\frac{1}{2}k^2[12\mu^2\ln^2(B)+12\mu^2\ln^2(B)\,(tan_B(2\mu\zeta)\pm\sqrt{pq}sec_B(2\mu\zeta))^2-1]-\frac{a}{2}\}, \end{aligned} \tag{72}$$

$$\begin{aligned} &\phi_{24}(x,t)=\{12\sqrt{-\mathrm{k}^4\mu^4\ln^4(B)}\ (cot_B(2\mu\zeta)\pm\sqrt{pq}csc_B(2\mu\zeta))\\ &-\frac{1}{2}k^2[12\mu^2\ln^2(B)+12\mu^2\ln^2(B)\,(cot_B(2\mu\zeta)\pm\sqrt{pq}csc_B(2\mu\zeta))^2-1]-\frac{a}{2}\}, \end{aligned} \tag{73}$$

$$\begin{aligned} &\phi_{25}(x,t)=\frac{1}{2}\times\{-12\sqrt{-\mathrm{k}^4\mu^4\ln^4(B)}\left(tan_B\left(\frac{\mu}{2}\,\zeta\right)-\ cot_B\left(\frac{\mu}{2}\,\zeta\right)\right)\\ &-12k^2\mu^2\ln^2(B)+k^2-a-3k^2\mu^2\ln^2(B)\left(tan_B\left(\frac{\mu}{2}\,\zeta\right)-\ cot_B\left(\frac{\mu}{2}\,\zeta\right)\right)^2\}. \end{aligned} \tag{74}$$

1. For $\lambda=0$ *and* $\nu=-\mu$,

$$\begin{aligned} \phi_{26}(x,t)=12\sqrt{\mathrm{k}^4\mu^4\ln^4(B)}tanh_B(\mu\zeta)+6k^2\mu^2\ln^2(B)-\\ 6k^2\mu^2\ln^2(B)\,tanh_B^2(\mu\zeta)\,z+\frac{1}{2}(k^2-a), \end{aligned} \tag{75}$$

$$\phi_{27}(x,t) = 12\sqrt{\mathrm{k}^4\mu^4\ln^4(B)}coth_B(\mu\zeta) + 6k^2\mu^2\ln^2(B) - 6k^2\mu^2\ln^2(B)\,coth_B^2(\mu\zeta) + \frac{1}{2}(k^2-a), \quad (76)$$

$$\phi_{28}(x,t) = -12\sqrt{\mathrm{k}^4\mu^4\ln^4(B)}\left(-tanh_B(2\mu\zeta) \pm i\sqrt{pq}sech_B(2\mu\zeta)\right) + 6k^2\mu^2\ln^2(B) - 6k^2\mu^2\ln^2(B)\bigg(-tanh_B(2\mu\zeta) \pm i\sqrt{pq}sech_B(2\mu\zeta)\bigg)^2 + \frac{1}{2}(k^2-a), \quad (77)$$

$$\phi_{29}(x,t) = -12\sqrt{\mathrm{k}^4\mu^4\ln^4(B)}\left(-coth_B(2\mu\zeta) \pm \sqrt{pq}csch_B(2\mu\zeta)\right) + 6k^2\mu^2\ln^2(B) - 6k^2\mu^2\ln^2(B)\bigg(-coth_B(2\mu\zeta) \pm \sqrt{pq}csch_B(2\mu\zeta)\bigg)^2 + \frac{1}{2}(k^2-a), \quad (78)$$

$$\phi_{30}(x,t) = \frac{1}{2}\times\{12\sqrt{\mathrm{k}^4\mu^4\ln^4(B)}\left(tanh_B\left(\frac{\mu}{2}\zeta\right) + cot_B\left(\frac{\mu}{2}\zeta\right)\right) + 12k^2\mu^2\ln^2(B) + k^2 - a - 3k^2\mu^2\ln^2(B)\left(tanh_B\left(\frac{\mu}{2}\zeta\right) + cot_B\left(\frac{\mu}{2}\zeta\right)\right)^2\}. \quad (79)$$

2. For $\lambda^2 = 4\mu\nu$,

$$\phi_{31}(x,t) = \frac{1}{\lambda^2}\bigg(6k^2\mu\nu\ln B(\lambda^2 - 4\mu\nu)(\ln(B) + 4)\bigg) + \frac{1}{2}(k^2-a)\lambda^4\zeta^2 - \frac{1}{\lambda^4\zeta^2}\bigg(96k^2\mu^2\nu^2 bigg). \quad (80)$$

3. For $\lambda = \chi,\ \mu = r\chi\,(r \neq 0)$and $\nu = 0$,

$$\phi_{32}(x,t) = 0. \quad (81)$$

4. For $\lambda = \nu = 0$,

$$\phi_{33}(x,t) = 0. \quad (82)$$

5. For $\lambda = \mu = 0$.

$$\phi_{34}(x,t) = \left\{\frac{1}{2}\left(k^2-a\right) - \frac{6k^2}{\zeta^2}\right\}. \quad (83)$$

6. For $\lambda \neq 0,\ \mu = 0$, we found the hyperbolic function solutions

$$\phi_{35}(x,t) = \frac{3\lambda\sqrt{\mathrm{k}^4\nu^2\lambda^2\ln^4(B)}\,(-\cosh_B(\lambda\zeta) - \sinh_B(\lambda\zeta) + p)}{\nu\Big(\cosh_B(\lambda\zeta) - \sinh_B(\lambda\zeta) + p\Big)} + \frac{1}{2}(k^2 - a) + \frac{6p\lambda^2k^2ln^2(B)\Big(cosh_B(\lambda\zeta) - sinh_B(\lambda\zeta)\Big)}{\Big(cosh_B(\lambda\zeta) - sinh_B(\lambda\zeta) + p\Big)^2}, \tag{84}$$

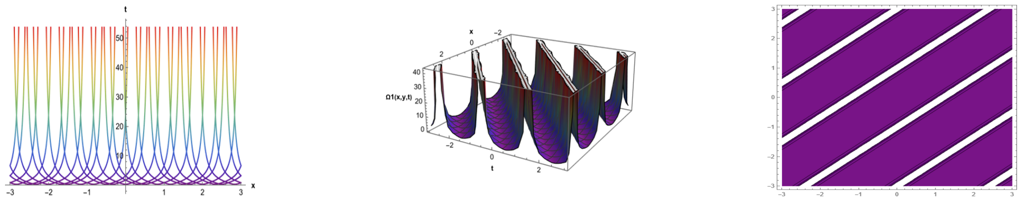
$$\phi_{36}(x,t) = \frac{1}{2}\Big(3\lambda^2k^2\ln^2(B) + k^2 - a\Big) - \frac{3\lambda^2k^2\ln^2(B)\Big(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta)\Big)^2}{\Big(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta) + q\Big)^2} + \frac{42k^2\nu\lambda^2\ln^2(B)\Big(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta)\Big)}{\nu\Big(\sinh_B(\lambda\zeta) + \cosh_B(\lambda\zeta) + q\Big)}, \tag{85}$$

$$\phi_{37}(x,t) = \frac{1}{2}\left[\frac{6\sqrt{\mathrm{k}^4\nu^2\chi^4\ln^4(B)}\Big(r(2p-q)B^{\chi\zeta} + p\Big)}{r\Big(rqB^{\chi\zeta} - p\Big)} + k^2 - a\right] - \frac{6k^2pr\chi^2\ln^2(B)B^{\chi\zeta}\Big(r(p-q)B^{\chi\zeta} + p\Big)}{\Big(p - rqB^{\chi\zeta}\Big)^2}, \tag{86}$$

for all solutions $\zeta = x - ct$.

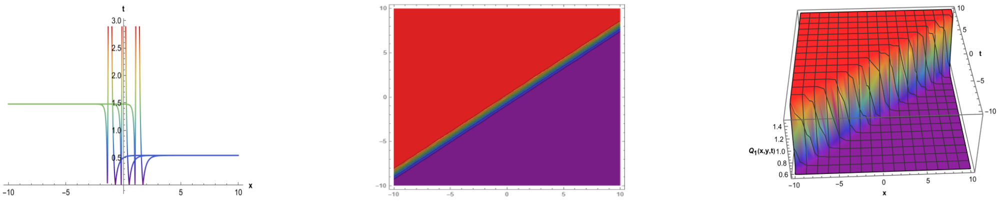
2. Graphical Representation

The traveling wave solutions are presented in Figure 1-9 in three types of diagrams, a contour plot on an arbitrary constants range, a two-dimensional plotline as well and a three-dimensional plotline by using Mathematica. Graphical representations are important because they may help us better understand solutions and analyze complex data. In soliton theory, wave propagation for the GIB model plays a significant role.



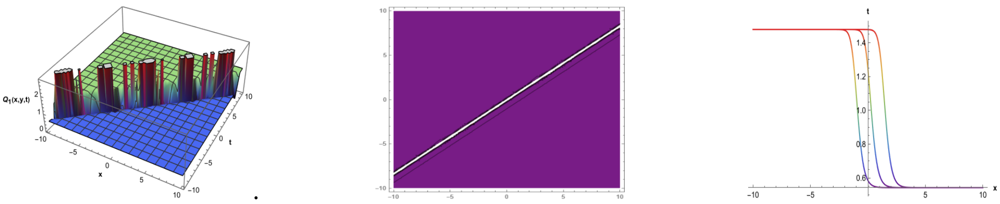
(a) (b) (c)

Figure 1: Visual representation of the Eq. 47, with the suitable parameters$\mu = 1.2, \nu = 1.5, \lambda = 0.6, a = 0.5, k = 1.2$ *and* $B = 0.9$.



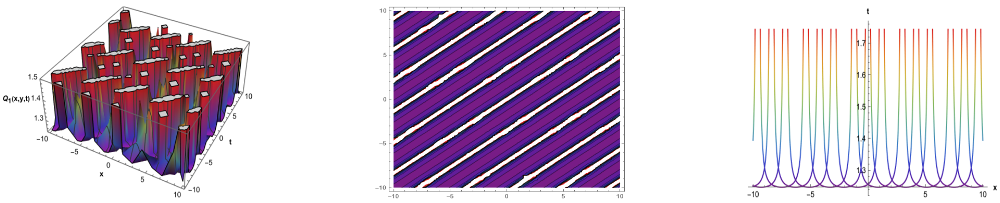
(a) (b) (c)

Figure 2: Graphical illustration of the Eq. 48, with the suitable parameters $\mu = 1.2, \nu = -1.5, \lambda = 1.6, a = 0.5, k = 1.2$ and $B = 0.9$



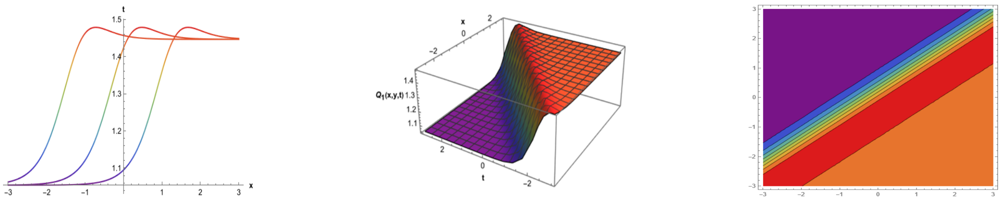
(a) (b) (c)

Figure 3: Visual representation of the Eq. 53, with the suitable parameters $\mu = 1.2, \nu = -1.5, \lambda = 1.6, a = 0.5, k = 1.2$ and $B = 0.9$.



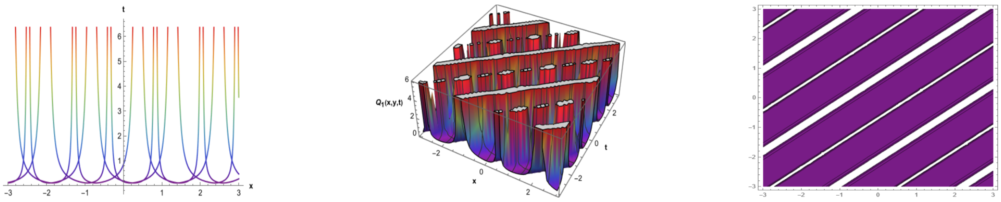
(a) (b) (c)

Figure 4: Visual representation of the Eq. 56, with the suitable parameters $\mu = 1.2,\ \nu = 1.5,\quad \lambda = 1.6,\quad a = 0.5,\ k = 1.2$ and $B = 0.9$.



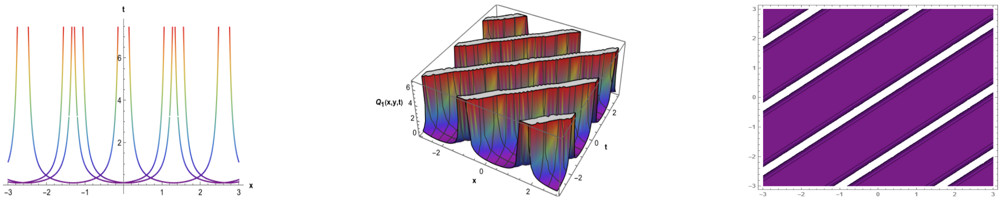
(a) (b) (c)

Figure 5: Visual representation of the Eq. 58, with the suitable parameters $\mu = 1.2,\ \nu = 1.5,\ \lambda = 0,\ a = 0.5,\ k = 1.2$ and $B = 0.9$.



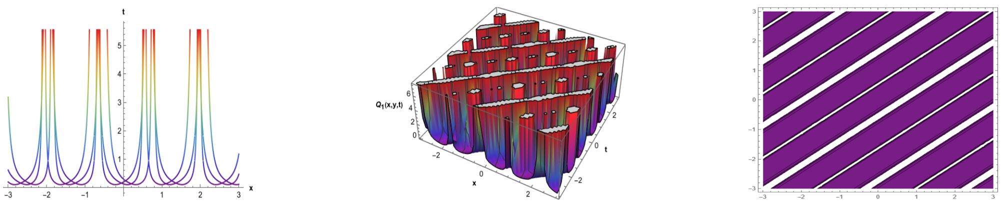
(a) (b) (c)

Figure 6: Visual representation of the Eq. 60, with the suitable parameters$\mu = 1.2,\ \nu = 1.5,\ \lambda = 0,\ a = 0.5,\ k = 1.2,\ B = 0.9,\ p = 0.6$ and $q = 0.4$.



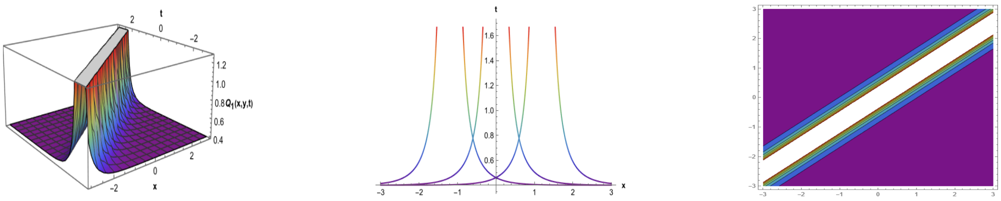
(a) (b) (c)

Figure 7: Visual representation of the Eq. 68, with the suitable parameters$\mu = 1.2, \nu = 1.5,\ \lambda = 0,\ a = 0.5, k = 1.2, B = 0.9, p = 0.6$ and $q = 0.4$.



(a) (b) (c)

Figure 8: Visual representation of the Eq. 69, with the suitable parameters$\mu = 1.2, \nu = 1.5, \lambda = 0, a = 0.5, k = 1.2, B = 0.9, p = 0.6$ and $q = 0.4$.



(a) (b) (c)

Figure 9: Graphical illustration of the Eq. 73, with the suitable parameters$\mu = 1.2, \nu = 1.5, \lambda = 0, a = 0.5, k = 1.2, B = 0.9, p = 0.6$ and $q = 0.4$.

3. Conclusions

In conclusion, the new extended direct algebraic method has proven to be a potent tool for solving nonlinear evolution equations, offering valuable insights into their complex dynamics. By applying this method, researchers can unlock new exact solutions, shedding light on the intricate behavior of nonlinear systems. Future studies can further harness the potential of this method, exploring its applicability to diverse nonlinear equations and refining its capabilities to drive innovation in relevant fields.

This work solved the generalized improved Boussinesq equation (GIBE), which is a mathematical model that describes nonlinear processes, by applying the new extended direct algebraic method (NEDAM). Using creative methods and systematic investigation, a wide variety of soliton solutions were found, providing insight into the behavior of nonlinear systems. The solutions were classified as bright, dark, complicated and combination forms, as well as single forms like kink, anti-kink and solitary wave solutions among others. The derivation procedure revealed new families of periodic, hyperbolic and exponential wave solutions with arbitrary parameters. For certain parameter values, the results were displayed using contour plots, 2D line plots and 3D surface plots. These findings significantly advance the field and enhance our understanding of complicated nonlinear systems.The kink wave solutions, bright and dark soliton solutions, and periodic solutions that are obtained in this research have vast applications in various realistic physical systems. Some of the fields of application are nonlinear optics, fluid dynamics, quantum systems, etc. The soliton obtained in this study describes the pulse propagation in an optical fiber under the conditions that balance both the nonlinearity and dispersion. Shallow water waves in fluid dynamics. Also, the dynamics of particle density distribution is modeled by the soliton solutions of the quantum system (say), Bose-Einstein condensation. We can conclude that the exact solution generated in this study portrays both the experimental and theoretical analyses across their domains.

The investigation of soliton dynamics using both analytical and graphical techniques creates new research opportunities and has implications for several scientific and technical fields. In addition to expanding theoretical knowledge, this work establishes a strong basis for future research on nonlinear processes.

References

[1] G. W. Wang and T. Z. Xu. Invariant analysis and exact solutions of nonlinear time fractional Sharma–Tasso–Olver equation by Lie group analysis. *Nonlinear Dynamics*, 76:571–580, 2014.

[2] Y. H. Yin, S. J. Chen, and X. Lü. Localized characteristics of lump and interaction solutions to two extended Jimbo–Miwa equations. *Chinese Physics B*, 29(12):120502, 2020.

[3] X. Lü, Y. F. Hua, S. J. Chen, and X. F. Tang. Integrability characteristics of a novel (2+1)-dimensional nonlinear model: Painlevé analysis, soliton solutions, Bäcklund transformation, Lax pair, and infinitely many conservation laws. *Communications in Nonlinear Science and Numerical Simulation*, 95:105612, 2021.

[4] G. Adomian. A new approach to nonlinear partial differential equations. *Journal of Mathematical Analysis and Applications*, 102(2):420–434, 1984.

[5] M. M. A. Khater, S. H. Alfalqi, J. F. Alzaidi, and R. A. M. Attia. Analytically and numerically, dispersive, weakly nonlinear wave packets are presented in a quasi–monochromatic medium. *Results in Physics*, 46:106312, 2023.

[6] J. S. Russell. Report on Waves: Made to the Meetings of the British Association in 1842–43, 1845.

[7] A. A. Soliman. The modified extended direct algebraic method for solving nonlinear partial differential equations. *International Journal of Nonlinear Sciences*, 6(2):136–144, 2008.

[8] R. L. Horne. A (very) brief introduction to soliton theory in a class of nonlinear PDEs, 2002.

[9] A. Özkan, N. Özdemir, and E. M. Özkan. A view of optical soliton solution of the coupled Schrödinger equation with a different approach. *Physica Scripta*, 99(8):085223, 2024.

[10] S. Kumar and A. Kukkar. Dynamics of several optical soliton solutions of a (3+1)-dimensional nonlinear Schrödinger equation with parabolic law in optical fibers. *Modern Physics Letters B*, 39(09):2450453, 2025.

[11] J. Zhang, Y. Leng, C. Wu, and C. Su. Data-driven soliton solution implementation based on nonlinear adaptive physics-informed neural networks. *Nonlinear Dynamics*, 113(2):1467–1488, 2025.

[12] W. Zhang, Y. Guo, and X. Zhang. Periodic wave solutions and solitary wave solutions of Kundu equation. *Physica Scripta*, 2025.

[13] D. Kumar, A. Saharan, and A. Kumar. Exploring soliton patterns and dynamical analysis of the solitary wave form solutions of the (3+1)-dimensional Wazwaz–Benjamin–Bona–Mahony equation. *Modern Physics Letters B*, page 2550102, 2025.

[14] Z. G. Liu, Y. S. Wang, and G. Huang. Solitary waves in a granular chain of elastic spheres: multiple solitary solutions and their stabilities. *Physical Review E*, 99(6):062904, 2019.

[15] A. Kumar, S. Kumar, N. Bohra, G. Pillai, R. Kapoor, and J. Rao. Exploring soliton solutions and interesting wave-form patterns of the (1+1)-dimensional longitudinal

wave equation in a magnetic-electro-elastic circular rod. *Optical and Quantum Electronics*, 56(6):1029, 2024.

[16] W. O. Apeanti, D. Yaro, and S. W. Akuamoah. Solitary Wave Solutions and Modulation Instability of First-and Second-Order Benjamin–Ono Fluid Dynamics Models With Applications. *Journal of Applied Mathematics*, 2025(1):8610610, 2025.

[17] L. Bengel. Stability of solitary wave solutions in the Lugiato–Lefever equation. *Zeitschrift für angewandte Mathematik und Physik*, 75(4):130, 2024.

[18] K. J. Wang. The Perturbed Chen–Lee–Liu Equation: Diverse Optical Soliton Solutions and Other Wave Solutions. *Advances in Mathematical Physics*, 2024(1):4990396, 2024.

[19] R. Li, J. Manafian, H. A. Lafta, H. A. Kareem, K. F. Uktamov, and M. Abotaleb. The nonlinear vibration and dispersive wave systems with cross-kink and solitary wave solutions. *International Journal of Geometric Methods in Modern Physics*, 19(10):2250151, 2022.

[20] S. Y. Hong, W. G. Zhang, and X. Li. Asymptotic Stability of Monotone Increasing Kinked Solitary Wave Solution for the Generalized Boussinesq Equation with Dissipative Term. *Journal of Nonlinear Mathematical Physics*, 32(1):16, 2025.

[21] L. Yang and B. Gao. Solitary wave solution, traveling wave solution and other solutions of variable-coefficients Chiral Schrödinger equation. *International Journal of Geometric Methods in Modern Physics*, 21(05):2450100, 2024.

[22] W. Lu, J. Ahmad, S. Akram, and K. A. Aldwoah. Soliton solutions and sensitive analysis to nonlinear wave model arising in optics. *Physica Scripta*, 99(8):085230, 2024.

[23] W. Lu, J. Ahmad, S. Akram, and K. A. Aldwoah. Soliton solutions and sensitive analysis to nonlinear wave model arising in optics. *Physica Scripta*, 99(8):085230, 2024.

[24] Y. Shang. Abundant explicit non-traveling wave solutions for the (2+1)-dimensional breaking soliton equation. *Applied Mathematics Letters*, 131:108029, 2022.

[25] B. A. Babajanov, S. O. Sadullaev, and M. M. Ruzmetov. Integration of the Kaup–Boussinesq system via inverse scattering method. *Partial Differential Equations in Applied Mathematics*, 11:100813, 2024.

[26] J. Satsuma. Hirota bilinear method for nonlinear evolution equations. In *Direct and Inverse Methods in Nonlinear Evolution Equations: Lectures Given at the CIME Summer School Held in Cetraro, Italy, September 5–12, 1999*, pages 171–222. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[27] E. H. Zahran and M. M. Khater. Modified extended tanh-function method and its applications to the Bogoyavlenskii equation. *Applied Mathematical Modelling*, 40(3):1769–1775, 2016.

[28] L. Li, J. Zhang, and M. Wang. Application of Simplified Homogeneous Balance Method to Multiple Solutions for (2+1)-Dimensional Burgers' Equations. *Mathematics*, 10(18):3402, 2022.

[29] A. M. Wazwaz. A sine-cosine method for handling nonlinear wave equations. *Mathematical and Computer Modelling*, 40(5–6):499–508, 2004.

[30] D. J. Evans and K. R. Raslan. The tanh function method for solving some important non-linear partial differential equations. *International Journal of Computer Mathematics*, 82(7):897–905, 2005.

[31] A. K. Hussain, F. S. Fadhel, N. Rusli, and Z. R. Yahya. Iteration variational method for solving two-dimensional partial integro-differential equations. In *Journal of Physics: Conference Series*, volume 1591, page 012091. IOP Publishing, July 2020.

[32] A. K. S. Hossain, M. A. Akbar, and M. I. Hossain. Modified simple equation technique for first-extended fifth-order nonlinear equation, medium equal width equation and Caudrey–Dodd–Gibbon equation. *Journal of Umm Al-Qura University for Applied Sciences*, pages 1–10, 2024.

[33] P. N. Ryabov, D. I. Sinelshchikov, and M. B. Kochanov. Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations. *Applied Mathematics and Computation*, 218(7):3965–3972, 2011.

[34] A. M. Shahoot, K. A. Alurrfi, M. O. Elmrid, A. M. Almsiri, and A. M. Arwiniya. The (G'/G)-expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines. *Journal of Taibah University for Science*, 13(1):63–70, 2019.

[35] N. K. Vitanov. Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity. *Communications in Nonlinear Science and Numerical Simulation*, 15(8):2050–2060, 2010.

[36] J. Vahidi, A. Zabihi, H. Rezazadeh, and R. Ansari. New extended direct algebraic method for the resonant nonlinear Schrödinger equation with Kerr law nonlinearity. *Optik*, 227:165936, 2021.

[37] S. U. Rehman, M. Bilal, and J. Ahmad. Highly dispersive optical and other soliton solutions to fiber Bragg gratings with the application of different mechanisms. *International Journal of Modern Physics B*, 36(28):2250193, 2022.

[38] S. U. Rehman and J. Ahmad. Investigation of exact soliton solutions to Chen–Lee–Liu equation in birefringent fibers and stability analysis. *Journal of Ocean Engineering and Science*, 2022.

[39] T. A. Sulaiman, U. Younas, M. Younis, J. Ahmad, S. U. Rehman, M. Bilal, and A. Yusuf. Modulation instability analysis, optical solitons and other solutions to the (2+1)-dimensional hyperbolic nonlinear Schrodinger's equation. *Computational Methods for Differential Equations*, 10(1):179–190, 2022.

[40] J. Ahmad. Dynamics of optical and other soliton solutions in fiber Bragg gratings with Kerr law and stability analysis. *Arabian Journal for Science and Engineering*, 48(1):803–819, 2023.

[41] Shafqat ur Rehman and J. Ahmad. Stability analysis and novel optical pulses to Kundu–Mukherjee–Naskar model in birefringent fibers. *International Journal of Modern Physics B*, 38(15):2450192, 2024.

[42] M. Shahzad, R. Anjum, N. Ahmed, M. Z. Baber, and N. Shahid. On the closed form exact solitary wave solutions for the time-fractional order nonlinear unsteady convection diffusion system with unique-existence analysis. *International Journal of*

Geometric Methods in Modern Physics, page 2550128, 2025.

[43] A. Akgül, N. Ahmed, M. Shahzad, M. Z. Baber, M. S. Iqbal, and C. K. Chan. Regularity and wave study of an advection–diffusion–reaction equation. *Scientific Reports*, 14(1):20776, 2024.

[44] M. Jawaz, M. Shahzad, N. Ahmed, M. Z. Baber, M. Iqbal, and A. Akgül. Analytical dynamics to the interactions of a diffusive mussel–algae model. *Partial Differential Equations in Applied Mathematics*, 14:101151, 2025.

[45] M. Shahzad, N. Ahmed, M. S. Iqbal, M. Inc, M. Z. Baber, and R. Anjum. Classical Regularity and Wave Structures of Fractional Order Selkov-Schnakenberg System. *International Journal of Theoretical Physics*, 63(4):95, 2024.

[46] M. Shahzad, N. Ahmed, M. S. Iqbal, M. Inc, M. Z. Baber, R. Anjum, and N. Shahid. Application of fixed point theory and solitary wave solutions for the time-fractional nonlinear unsteady convection-diffusion system. *International Journal of Theoretical Physics*, 62(12):263, 2023.

[47] Muhammad Bilal Riaz, Faiza Naseer, Muhammad Abbas, Magda Abd El-Rahman, Tahir Nazir, and Choon Kit Chan. Solitary wave solutions of Sawada-Kotera equation using two efficient analytical methods. *AIMS Mathematics*, 8(12):31268–31292, 2023.